Online Appendix for *Team Contests with Multiple Pairwise Battles:*
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Proof of Theorem 3

Parts (b) and (c) of Theorem 3 can be shown following the same arguments for their counterparts in Theorem 2. We now focus on part (a). To establish part (a), we show that the ex ante expected prize spread in battle t is $E(V_t) = \pi_t + \theta(n|\,2n)|_{-t}$, which is independent of the temporal structure.

In this extended setting, we continue to denote by (k_A, k_B) the state of the contest. The tuple indicates the number of victories each team has secured before battles in a cluster z are carried out.

We first illustrate that for any battle t within any arbitrary cluster, the prize spreads of the two teams must be symmetric regardless of the prevailing state (k_A, k_B). The case of $\max\{k_A, k_B\} \geq n+1$ is trivial. In this case, one team has won. The prize spread is simply π_t, which is symmetric. Without loss of generality, hereafter we focus on the case of $k_A, k_B < n+1$.

Let $n(z)$ be the number of battles included in a cluster z. Let $\mathcal{T}_z$ denote the set of battles in a cluster z. We use t to index a battle in a cluster z with state (k_A, k_B). A player $A(t)$ receives π_t if he wins, and the contest may enter any state $(k_A+1+l, k_B+n(z)-1-l)$, with $l \in \{0, 1, \ldots, n(z)-1\}$, after all the $n(z)$ battles in z are fought. Note that $(k_A+l, k_B+n(z)-1-l)$ can be used to denote a *stochastic contest state* facing battle t. If he loses, then the contest may enter any state $(k_A + l, k_B + n(z) - l)$, with $l \in \{0, \ldots, n(z) - 1\}$, after all the battles in z are fought.

Suppose that after z the contest is in state $(\tilde{k}_A, \tilde{k}_B)$. Let $\tilde{v}_i(\tilde{k}_A, \tilde{k}_B)$ denote team i's conditional winning probability. Clearly, $\tilde{v}_i(\tilde{k}_A, \tilde{k}_B) = 1$ and $\tilde{v}_j(\tilde{k}_A, \tilde{k}_B) = 0$ when $\tilde{k}_i \geq n+1$.

As a result, player $A(t)$'s effective spread amounts to

$$V_t^A(k_A, k_B) = \pi_t + \Delta v_t^A(k_A, k_B), \forall t \in \mathcal{T}_z.$$

Here, $$\Delta v_t^A(k_A, k_B) = \sum_{l=0}^{n(z)-1} \left\{ \begin{array}{c} [\tilde{v}_A(k_A + (1+l), k_B + n(z) - (1+l)) \\ -\tilde{v}_A(k_A + l, k_B + n(z) - l)] \cdot \tilde{\theta}_A(l|\,n(z)-1)\Big|_{\mathcal{T}_z \setminus \{t\}} \end{array} \right\},$$

where $\tilde{\theta}_A(l|\,n(z)-1)\Big|_{\mathcal{T}_z \setminus \{t\}}$ is the probability that team A wins l out of the $n(z)-1$ simultaneous (nontrivial) battles in cluster z, excluding battle t.

Similarly, the effective prize spread for player $B(t)$ is $V_t^B(k_A, k_B) = \pi_t + \Delta v_t^B(k_A, k_B), \forall t \in$

$\mathcal{T}_z$, where

$$\Delta v_t^B(k_A,k_B) = \sum_{l=0}^{n(z)-1}\left\{\begin{array}{c}[\tilde{v}_B(k_A+l,k_B+n(z)-l)\\ -\tilde{v}_B(k_A+(1+l),k_B+n(z)-(1+l))]\\ \cdot\,\tilde{\theta}_A(l|\,n(z)-1)\Big|_{\mathcal{T}_z\backslash\{t\}})\end{array}\right\}.$$

Note that $\sum_{i\in\{A,B\}}\tilde{v}_i(k_A+l,k_B+n(z)-l)=1$ and $\sum_{i\in\{A,B\}}\tilde{v}_i(k_A+(1+l),k_B+n(z)-(1+l))=1$ hold for every $l\in\{0,...,n(z)-1\}$. We thus obtain $V_t^A(k_A,k_B)=V_t^B(k_A,k_B)$. Hence, each battle in z is symmetrically valued, and each nontrivial battle t in any cluster z has a stochastic outcome $(\mu_{A(t)},\mu_{B(t)})$, which is solely determined by matched players' cost distributions by Theorem 1.

We then consider $V_t^A(k_A,k_B)$ to further pin down the symmetric prize spread $V_t(k_A,k_B)$ for a battle t in any cluster. The case of $\max\{k_A,k_B\}\geq n+1$ is trivial. One team has won, thus the prize spread is simply π_t. We now focus on the case of $k_A,k_B\leq n$. For the unclustered battles after the last clustered battles, the results of Lemma 1(a) apparently hold. We now consider the last cluster that contains more than one battle, which is denoted by z_1. We assume without loss of generality that there are unclustered battles following this cluster.[1] Lemma 1(a) applies to all (unclustered) battles that follow cluster z_1. Recall that $\Delta v(k_A,k_B)$ is defined in Observation 1 as $v_A(k_A+1,k_B)-v_A(k_A,k_B+1)$ or equivalently $v_B(k_A,k_B+1)-v_B(k_A+1,k_B)$. Note that from Lemma 1(a), we have $\Delta v(k_A,k_B)=\theta_i(n-k_i|\,2n-k_A-k_B)|_{t+1}^{2n+1}$.

For a stochastic state $(k_A+l,k_B+n(z_1)-1-l)$, let

$$\begin{aligned}&\xi_t^A(k_A+l,k_B+n(z_1)-1-l)\\ =\;&\tilde{v}_A(k_A+(1+l),k_B+n(z_1)-(1+l))-\tilde{v}_A(k_A+l,k_B+n(z_1)-l).\end{aligned}$$

We have

$$\begin{aligned}&\xi_t^A(k_A+l,k_B+n(z_1)-1-l)\\ =\;&v_A(k_A+(1+l),k_B+n(z_1)-(1+l))-v_A(k_A+l,k_B+n(z_1)-l)\\ =\;&\Delta v(k_A+l,k_B+n(z_1)-1-l)\\ =\;&\theta_A((n+1)-(k_A+1+l)|\,2n+1-(k_A+k_B+n(z_1)))|_{k_A+k_B+n(z_1)+1}^{2n+1},\end{aligned}$$

if $k_A+l<n+1$, and $k_B+n(z_1)-1-l<n+1$; and it boils down to zero otherwise. Hence,

$$\begin{aligned}\Delta v_t^A(k_A,k_B)\;&=\;\sum_{l=0}^{n(z_1)-1}[\xi_t^A(k_A+l,k_B+n(z_1)-1-l)\cdot\tilde{\theta}_A(l|\,n(z_1)-1)\Big|_{\mathcal{T}_{z_1}\backslash\{t\}}]\\ &=\;\sum_{l=\max\{0,k_B+n(z_1)-1-n\}}^{\min\{n(z_1)-1,n-k_A\}}[\xi_t^A(k_A{+}l,k_B{+}n(z_1)-1-l)\cdot\tilde{\theta}_A(l|\,n(z_1)-1)\Big|_{\mathcal{T}_{z_1}\backslash\{t\}}]\end{aligned}$$

[1] The case in which no more battles follow the cluster z_1 is simpler and yields the same result.

$$
\begin{aligned}
&= \sum_{l=\max\{0,k_B+n(z_1)-1-n\}}^{\min\{n(z_1)-1,n-k_A\}} \Big[\left. \begin{matrix} \theta_A((n+1)-(k_A+1+l)|\,2n+1 \\ -(k_A+k_B+n(z_1))) \end{matrix} \right|_{k_A+k_B+n(z_1)+1}^{2n+1} \\
&\quad \cdot \tilde{\theta}_A(l|\,n(z_1)-1)\Big|_{\mathcal{T}_{z_1}\backslash\{t\}} \Big] \\
&= \theta_A(n-k_A|\,2n-(k_A+k_B))|_{\{\tilde{t}|\tilde{t}\geq k_A+k_B+1,\tilde{t}\neq t\}}, \forall t \in \mathcal{T}_{z_1},
\end{aligned}
$$

where $\theta_A(n-k_A|\,2n-(k_A+k_B))|_{\{\tilde{t}|\tilde{t}\geq k_A+k_B+1,\tilde{t}\neq t\}}$ is the probability of team A's winning exactly $n-k_A$ out of the $2n-k_A-k_B$ (nontrivial) battles (excluding battle t) when the contest enters the state (k_A,k_B) after the cluster that precedes z_1 is contested.

It follows that $V_t(k_A,k_B)=V_t^A(k_A,k_B), t\in\mathcal{T}_{z_1}$ can be rewritten as

$$
\pi_t+\Delta v_t^A(k_A,k_B)=\pi_t+\theta_A(n-k_A|\,2n-(k_A+k_B))|_{\{\tilde{t}|\tilde{t}\geq k_A+k_B+1,\tilde{t}\neq t\}}.
$$

The above formula for $V_t(k_A,k_B)$ still applies for the case $\max\{k_A,k_B\}\geq n+1$. In this case, clearly $\theta_A(n-k_A|\,2n-(k_A+k_B))|_{\{\tilde{t}|\tilde{t}\geq k_A+k_B+1,\tilde{t}\neq t\}}=0$. Because $k_A+l<n+1$, and $k_B+n(z_1)-1-l<n+1$, the range of l (i.e. from $\max\{0,k_B+n(z_1)-1-n\}$ to $\min\{n(z_1)-1,n-k_A\}$) excludes all (and only) the events that the result of battle t does not make a difference in determining the winning probability of the whole contest.

We then consider the cluster that immediately precedes z_1, which is denoted by z_2. Suppose that it faces an arbitrary state (k_A,k_B). Note that we allow it to include only one battle. Recall that we can focus on the case of $k_A,k_B<n+1$. We have

$$
\begin{aligned}
&\Delta v_t^A(k_A,k_B) \\
&= \sum_{l_2=0}^{n(z_2)-1} \left\{ \begin{matrix} [\tilde{v}_A(k_A+(1+l_2),k_B+n(z_2)-(1+l_2)) \\ -\tilde{v}_A(k_A+l_2,k_B+n(z_2)-l_2)]\cdot\tilde{\theta}_A(l_2|\,n(z_2)-1)\Big|_{\mathcal{T}_{z_2}\backslash\{t\}} \end{matrix} \right\}.
\end{aligned}
$$

Suppose that after z_2 the contest is in a state $(\tilde{k}_A,\tilde{k}_B)$. Recall that $\tilde{v}_i(\tilde{k}_A,\tilde{k}_B)=1$ and $\tilde{v}_j(\tilde{k}_A,\tilde{k}_B)=0$ when $\tilde{k}_i\geq n+1$. Note that we have shown that in cluster z_1, every nontrivial battle t has winning probabilities $(\mu_{A(t)},\mu_{B(t)})$. We now calculate $\tilde{v}_i(\tilde{k}_A,\tilde{k}_B)$ for $\tilde{k}_A,\tilde{k}_B\leq n$. Note in this case, every battle in cluster z_1 is nontrivial. Thus

$$
\tilde{v}_A(\tilde{k}_A,\tilde{k}_B)=\sum_{l_1=0}^{n(z_1)}[v_A(\tilde{k}_A+l_1,\tilde{k}_B+n(z_1)-l_1)\cdot\tilde{\theta}_A(l_1|\,n(z_1))\Big|_{\mathcal{T}_{z_1}}], \text{ with } \tilde{k}_A+\tilde{k}_B=k_A+k_B+n(z_2).
$$

Hence, $\forall l_2\in\{0,1,...,n(z_2)-1\}$,

$$
\begin{aligned}
&\tilde{v}_A(k_A+(1+l_2),k_B+n(z_2)-(1+l_2))-\tilde{v}_A(k_A+l_2,k_B+n(z_2)-l_2) \\
&= \sum_{l_1=0}^{n(z_1)}[v_A(k_A+(1+l_2)+l_1,k_B+n(z_2)-(1+l_2)+n(z_1)-l_1) \\
&\quad -v_A(k_A+l_2+l_1,k_B+n(z_2)-l_2+n(z_1)-l_1)]\cdot\tilde{\theta}_A(l_1|\,n(z_1))\Big|_{\mathcal{T}_{z_1}}
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{l_1=0}^{n(z_1)} \Delta v(k_A + l_2 + l_1, k_B + n(z_2) - (1 + l_2) + n(z_1) - l_1) \cdot \tilde{\theta}_A(l_1 |\, n(z_1))\Big|_{\mathcal{T}_{z_1}} \\
&= \sum_{l_1=0}^{n(z_1)} \{ \theta_A(n - (k_A + l_2 + l_1) |\, 2n + 1 - (k_A + k_B + n(z_2) + n(z_1))|_{k_A+k_B+n(z_2)+n(z_1)+1}^{2n+1} \\
&\quad \cdot \tilde{\theta}_A(l_1 |\, n(z_1))\Big|_{\mathcal{T}_{z_1}} \} \\
&= \theta_A(n - (k_A + l_2) |\, 2n + 1 - (k_A + k_B + n(z_2))|_{k_A+k_B+n(z_2)+1}^{2n+1} . \qquad (1)
\end{aligned}
$$

Therefore, $\forall t \in \mathcal{T}_{z_2}$, we have

$$
\begin{aligned}
&\Delta v_t^A(k_A, k_B) \\
&= \sum_{l_2=0}^{n(z_2)-1} \{ \theta_A(n - (k_A + l_2) |\, 2n + 1 - (k_A + k_B + n(z_2))|_{k_A+k_B+n(z_2)+1}^{2n+1} \\
&\quad \cdot \tilde{\theta}_A(l_2 |\, n(z_2) - 1)\Big|_{\mathcal{T}_{z_2} \setminus \{t\}} \} \\
&= \theta_A(n - k_A |\, 2n - (k_A + k_B))|_{\{\tilde{t} | \tilde{t} \geq k_A + k_B + 1, \tilde{t} \neq t\}} . \qquad (2)
\end{aligned}
$$

In view of (1), we can also obtain $\Delta v_t^A(k_A, k_B), \forall t \in \mathcal{T}_{z_3}$ by considering clusters z_3 and z_2 while applying the procedure for deriving (2) by considering clusters z_2 and z_1. The following general formula can then be obtained: for any battle t in any cluster z_k,

$$
\Delta v_t^A(k_A, k_B) = \theta_A(n - k_A |\, 2n - (k_A + k_B))|_{\{\tilde{t} | \tilde{t} \geq k_A + k_B + 1, \tilde{t} \neq t\}} , \forall t \in \mathcal{T}_{z_k}, \qquad (3)
$$

where (k_A, k_B) is the contest state before cluster z_k is carried out.

By repeating the exercise in the proof of Lemma 1(b), we can conclude that each battle t has an ex ante expected prize spread of $\pi_t + \theta_A(n |\, 2n)|_{-t}$, which does not depend on how these battles are clustered or sequenced. We then complete the proof.