Team Decision Theory for Linear Continuous-Time Systems

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Abstract—**This paper develops a team decision theory for linear-quadratic (LQ) continuous-time systems. First, a counterpart of the well-known result of Radner on quadratic static teams is obtained for two-member continuous-time LQ static team problems when the statistics of the random variables involved are not necessarily Gaussian. An iterative convergent scheme is developed, which in the limit yields the optimal team strategies. For the special case of Gaussian distributions, the team-optimal solution is affine in the information available to each DM, and for the further special case when the team cost function does not penalize the intermediate values of state, the optimal strategies can be obtained by solving a Liapunov type time-invariant matrix equation. This static theory is then extended to LQG continuous-time dynamic teams with sampled observations under the one-step-delay observation sharing pattern. The unique solution is again affine in the information available to each DM, and further, it features a certainty-equivalence property.**

I. Introduction

Team theory, originally developed by Radner and Marschak [1], [2], has penetrated the control literature through the works of Ho and Chu [3], [4]. In particular, a result of Radner in [2] has attracted attention in the control literature, which states that a static, strictly convex linear-quadratic Gaussian (LQG) team problem (with decision variables taken as vectors in appropriate dimensional Euclidean spaces) admits a unique team-optimal solution that is affine in the observation of each decision maker (DM). This result, the so-called Radner's theorem, has found recent applications in the decentralized control of dynamic discrete-time LQG team problems under the one-step-delay information sharing pattern [5]–[7] or equivalently under the one-step-delay observation sharing pattern [8, Remark 6]. By repeated application of Radner's theorem at each stage, it can be shown that such decentralized control problems admit affine solutions which also exhibit some kind of a separation property.

In the present paper, we develop an analogous theory for continuous-time systems. We first derive a counterpart of Radner's theorem for two-member continuous-time linear-quadratic static team problems when the statistics of the random variables involved are not necessarily Gaussian. Existence and uniqueness of the solution is established, and it is shown that the team-optimal solution satisfies a pair of integral equations which can be solved as the limit of a convergent iterative scheme. For the special case of Gaussian distributions, however, the team-optimal solution of the static team problem is affine in the observation of each DM, with the coefficients involved satisfying a pair of linear integral equations. It is further shown that the solution of these integral equations are related to the solution of a time-invariant matrix equation of Liapunov type when the cost function assigns only terminal quadratic cost to the state variables. Finally, the static theory is extended to LQG continuous-time dynamic team problems with discrete observations, under the quasi-classical one-step-delay observation sharing pattern. The unique team-optimal solution is obtained explicitly, which is again affine in the information available to each DM.

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II. General Formulation of the Dynamic Team Problem

Let $\{x_t, t\geq t_0\}$ denote an n-dimensional stochastic process satisfying the Ito differential equation

$$dx_t=[A(t)x_t+B^1(t)u_t^1+B^2(t)u_t^2]\,dt+F(t)\,dw_t$$
$$t\geq t_0,\ x_{t_0}=x_0, \tag{1}$$

and whose sample paths are continuous. Here, x_0 is a Gaussian random vector with mean $\bar{x}_0$ and covariance Σ_0, and $\{w_t, t\geq t_0\}$ is an n-dimensional standard Wiener process. $A(\cdot)$, $B^1(\cdot)$, $B^2(\cdot)$, and $F(\cdot)$ are appropriate dimensional matrices with continuous entries on $[t_0,t_f]$. $\{u_t^1, t\geq t_0\}$ and $\{u_t^2, t\geq t_0\}$ are, respectively, r_1- and r_2-dimensional stochastic processes denoting the controls of DM1 and DM2, respectively.

The decision makers make independent sampled noisy measurements of the state. Specifically, it is assumed that an m_i-dimensional observation

$$y_j^i=C_j^i x_{t_j}+v_j^i,\qquad i=1,2 \tag{2}$$

is available to DMi at the sampled time instant t_j where $j=0,1,\cdots,N-1$, and $t_0<t_1<\cdots<t_{N-1}<t_N=t_f$. Let us denote the index set of time samples by $\theta=\{0,1,\cdots,N-1\}$. Then, the random vectors $\{v_j^i, j\in\theta, i=1,2\}$ are assumed to have independent Gaussian statistics, and with $v_j^i\sim N(0,R_j^i)$, $R_j^i>0$, $j\in\theta$, $i=1,2$. Their statistics are also taken to be independent of the Wiener process $\{w_t, t\geq t_0\}$ and the Gaussian vector x_0. C_j^i is an observation matrix of appropriate dimensions.

We now adopt a *quasi-classical* information pattern for this decision problem. Specifically, it is assumed that the decisionmakers exchange their independent sampled observations with a delay of one sampling interval. Such an information pattern is known as the *one-step-delay observation sharing pattern* [8].

Mathematically speaking, the information available to DMi in the time interval

$$[t_j,t_{j+1})\text{ is }\eta_j^i\qquad\text{where }\eta_j^i=\{y_j^i,\delta_{j-1}\}, \tag{3a}$$

and δ_{j-1} denotes the common information available to the decision makers in the same sampling interval, i.e.,

$$\delta_{j-1}=\{y_{j-1}^1,y_{j-1}^2,\cdots,y_0^1,y_0^2\}. \tag{3b}$$

Let σ_j^i denote the sigma-algebra generated by the information set η_j^i. Further, let H_N^i denote the class of second-order stochastic processes $\{u_t^i, t\geq t_0\}$, which satisfy the requirement that their restriction to the interval $[t_j,t_{j+1})$ is σ_j^i-measurable, for all $j\in\theta$. Then, a permissible decision law (strategy) for DMi is a mapping γ^i: $[t_0,t_f]\times\mathbb{R}^{(m_1+m_2)N}\to\mathbb{R}^{r_i}$, such that $\gamma^i(\cdot,\eta)\in H_N^i$. Denote the class of all such strategies for DMi by Γ_N^i. It should be noted that, for each pair of elements in $H_N^1\times H_N^2$, the stochastic differential equation (1) admits a unique solution whose sample paths are continuous [9].

For each $\{\gamma^1\in\Gamma_N^1,\gamma^2\in\Gamma_N^2\}$, we now define the quadratic strictly convex cost function for the team (comprised of these two decision makers) as

$$J(\gamma^1,\gamma^2)=E\left\{x_{t_f}'Q_f x_{t_f}+\int_{t_0}^{t_f}\left(x_t'Q(t)x_t+u_t^{1\prime}u_t^1+u_t^{2\prime}u_t^2\right)dt\,\middle|\,u_t^i=\gamma^i(t,\eta^i),\ i=1,2\right\} \tag{4}$$

where $Q_f\geq 0$, $Q(\cdot)\geq 0$, the latter has continuous entries on $[t_0,t_f]$, and the expectation operation is taken over the underlying statistics.

Then, an optimal solution for this continuous-time dynamic team problem is a pair $\{\gamma^{1^0}\in\Gamma_N^1,\gamma^{2^0}\in\Gamma_N^2\}$, such that

$$\inf_{\Gamma_N^1}\inf_{\Gamma_N^2}J(\gamma^1,\gamma^2)=J(\gamma^{1^0},\gamma^{2^0}). \tag{5}$$

Before obtaining the solution of this dynamic team problem, we first consider its static version (obtained by setting $N=1$) in the next section, and then we turn to the more general version in Section IV.

III. Static Team Problem

A. A More General Formulation

In the static version of the dynamic team problem formulated in the previous section, the decision makers make noisy linear observations of the random initial state x_0, and do not acquire any further information as the decision process proceeds. Hence, the static version can be recovered from the previous general formulation by simply setting $N=1$. In this section, we actually first treat a more general version of this static problem, in which x_0 is a second-order random vector with known (but not necessarily Gaussian) statistics, and the static observation y^i of DMi is not related to x_0 necessarily in a linear fashion. In fact, we only assume that the conditional joint probability distribution of (y^1, y^2) given x_0 is *a priori* known, but this distribution need not be Gaussian.

When the information structure of each DM is static in nature, it is not necessary to differentiate between a strategy and its realized value (control), and hence, hereafter in this section, we will only consider the controls $\{u_t^1, t>t_0\}$ and $\{u_t^2, t>t_0\}$ as the decision variables of interest. Consistent with this adoption, we will investigate the minimizing solution in the product space $H_1^1\times H_1^2$ instead of in $\Gamma_1^1\times\Gamma_1^2$. Here, H_1^i stands, by abuse of notation, for the modified version of H_N^i (introduced in Section II) with $N=1$, that also accounts for the more general (not necessarily Gaussian) statistics introduced above. The same statement applies to Γ_1^i, too. We now introduce an inner product $\langle\cdots\rangle_i$ on H_1^i through the relation

$$\langle u, v\rangle_i = E\left\{\int_{t_0}^{t_f} u_t(\omega)' v_t(\omega)\, dt\right\} \tag{6a}$$

for each pair $\{u\in H_1^i, v\in H_1^i\}$ where $\omega\in\Omega$, with $(\Omega, \mathfrak{B}, \mathfrak{P})$ denoting the underlying probability space. Together with this inner product, and for each $i=1,2$, H_1^i becomes a Hilbert space which we simply denote by H^i.

To complete the formulation of the static team problem, we let $L_{2f}([t_0,t_f]\times\Omega)$ denote the space of functions from $[t_0,t_f]\times\Omega$ into $\mathbf{R}^n$, under the inner product

$$\langle x, z\rangle_f = E\left\{x_{t_f}(\omega)' z_{t_f}(\omega) + \int_{t_0}^{t_f} x_t(\omega)' z_t(\omega)\, dt\right\}. \tag{6b}$$

Further, let Q be an operator mapping $L_{2f}([t_0,t_f]\times\Omega)$ into itself, defined for each $x\in L_{2f}([t_0,t_f]\times\Omega)$ by

$$(Qx)_t(\omega) = \begin{cases} Q(t)x_t(\omega), & t_0<t<t_f \\ Q_f x_{t_f}(\omega), & t=t_f \end{cases}. \tag{7}$$

Then, the static continuous-time quadratic team problem under consideration in this section is the following.

Static Team Problem: Determine a pair $\{u^{1^o}\in H^1, u^{2^o}\in H^2\}$ that minimizes

$$J(u^1,u^2) = \langle Qx, x\rangle_f + \langle u^1, u^1\rangle_1 + \langle u^2, u^2\rangle_2, \tag{8}$$

subject to the constraint (1).

B. Existence of a Unique Team-Optimal Solution

The Hilbert space setting formulation given above leads to a rather simple proof of existence and uniqueness of the minimizing solution $\{u^{1^o}, u^{2^o}\}$, as well as to a set of two coupled linear equations that the desired solution satisfies. Let us first define a Volterra operator (see [10] for a definition) $\mathfrak{L}_i$: $H^i\to L_{2f}([t_0,t_f]\times\Omega)$ by

$$(\mathfrak{L}_i u)_t(\omega) = \int_{t_0}^{t}\Phi(t,s)B^i(s)u_s(\omega)\, ds, \qquad i=1,2 \tag{9a}$$

where $\Phi(t,s)$ is a state transition matrix function satisfying

$$\frac{d\Phi(t,s)}{dt} = A(t)\Phi(t,s); \qquad \Phi(s,s)=I. \tag{9b}$$

Further, let $r\in L_{2f}([t_0,t_f]\times\Omega)$ be defined as

$$r_t(\omega) = \Phi(t,t_0)x_0(\omega) + \int_{t_0}^{t}\Phi(t,s)F(s)\, dw_s(\omega). \tag{9c}$$

Then, it is easy to see that, for each $\{u^1\in H^1, u^2\in H^2\}$, the unique solution of (1) can be written as

$$x_t = (\mathfrak{L}_1u^1)_t(\omega) + (\mathfrak{L}_2u^2)_t(\omega) + r_t(\omega), \qquad t>t_0, \tag{10}$$

which, when substituted in (8), yields the following equivalent expression for J:

$$J(u^1,u^2) = \langle Q(\mathfrak{L}_1u^1+\mathfrak{L}_2u^2+r), \mathfrak{L}_1u^1+\mathfrak{L}_2u^2+r\rangle_f + \langle u^1,u^1\rangle_1 + \langle u^2,u^2\rangle_2. \tag{11}$$

We now have

Lemma 1: A pair $\{u^{1^o}\in H^1, u^{2^o}\in H^2\}$ is a minimizing solution if, and only if, it satisfies the pair of equations

$$\left.\begin{aligned} (I+\mathfrak{L}_1^*Q\mathfrak{L}_1)u^{1^o} + \mathfrak{L}_1^*Q\mathfrak{L}_2u^{2^o} &= -\mathfrak{L}_1^*Qr \\ \mathfrak{L}_2^*Q\mathfrak{L}_1u^{1^o} + (I+\mathfrak{L}_2^*Q\mathfrak{L}_2)u^{2^o} &= -\mathfrak{L}_2^*Qr \end{aligned}\right\} \tag{12}$$

where $\mathfrak{L}_i^*$ denotes the adjoint of $\mathfrak{L}_i$.

Proof: Since J is continuously differentiable and strictly convex on $H^1\times H^2$, every person-by-person optimal solution is also team-optimal, and furthermore, the first-order conditions are also sufficient. Hence, the result follows by taking the Gateaux variations of J separately with respect to u^1 and u^2, and by setting them equal to zero. □

The following lemma now proves existence and uniqueness of the minimizing solution.

Lemma 2: The pair of equations (12) admits a unique solution.

Proof: Let us first note that (12) can also be written as

$$(I+\mathfrak{L}^*Q\mathfrak{L})\begin{bmatrix} u^{1^o} \\ u^{2^o}\end{bmatrix} = -\mathfrak{L}^*Qr \tag{13}$$

where $\mathfrak{L}$: $H^1\times H^2\to L_{2f}([t_0,t_f]\times\Omega)$ is defined as $\mathfrak{L}=(\mathfrak{L}_1,\mathfrak{L}_2)$, and I is the identity operator mapping $H^1\times H^2$ into itself. Then, it readily follows that (12) admits a unique solution if, and only if, $(I+\mathfrak{L}^*Q\mathfrak{L})$ is invertible. Now, since $\mathfrak{L}_i$ is a Volterra operator, it is completely continuous (i.e., compact) [10], and so is its adjoint $\mathfrak{L}_i^*$. This implies that $\mathfrak{L}$ and $\mathfrak{L}^*$ are also compact. Furthermore, Q is a bounded operator. Since the product of compact and bounded operators is compact [10], it now follows that $\mathfrak{L}^*Q\mathfrak{L}$ is compact, which is also self-adjoint and nonnegative. This, then, implies that the operator $(I+\mathfrak{L}^*Q\mathfrak{L})$ is indeed invertible, since it is the sum of an identity operator (with is strongly positive) and a *nonnegative* self-adjoint compact operator. □

C. Functional Equations for the Team-Optimal Solution

We now seek to obtain, as an equivalent counterpart of (12), a set of functional (integral) equations that the team-optimal solution should satisfy. To this end, we start with the operator form (12) and first rewrite those equations as

$$\left.\begin{aligned} u^{1^o} &= -\mathfrak{L}_1^*Q(\mathfrak{L}_1u^{1^o}+\mathfrak{L}_2u^{2^o}+r) = -\mathfrak{L}_1^*Qx^0 \\ u^{2^o} &= -\mathfrak{L}_2^*Qx^0 \end{aligned}\right\} \tag{14}$$

where x^0 denotes the optimal team trajectory. By utilizing certain standard properties of adjoint operators, we can show (see [22, Appendix I]) that $\mathfrak{L}_i^*$ has the functional form

$$\mathfrak{L}_i^* z^i = v^i, \qquad i=1,2,$$

where

$$v_t^i(\omega)=\int_t^{t_f}B^i(t)'\Phi(s,t)'E\left[z_s^i(\omega)|\sigma^i\right]ds+B^i(t)'\Phi(t_f,t)'E\left[z_{t_f}^i(\omega)|\sigma^i\right] \tag{15}$$

and σ^i stands for the sigma-algebra generated by the information set of DMi. This result, together with some routine, but cumbersome, manipulations applied to (14), leads to the following result, whose proof can be found in [22].

Theorem 1: The unique optimal solution $\{u^{1^0}\in H^1, u^{2^0}\in H^2\}$ of the static team problem satisfies, and is the unique solution of, the following pair of coupled integral equations:

$$u_t^{1^0}=B^1(t)'\left\{S^1(t)\int_{t_0}^t\Psi^1(t,s)B^1(s)B^1(s)'l_s^1\,ds-l_t^1\right\} \tag{16a}$$

$$u_t^{2^0}=B^2(t)'\left\{S^2(t)\int_{t_0}^t\Psi^2(t,s)B^2(s)B^2(s)'l_s^2\,ds-l_t^2\right\} \tag{16b}$$

where $S^i(t)$, $i=1,2$ are the unique nonnegative definite matrix function solutions of the Riccati equations

$$\dot{S}^i(t)=-A(t)'S^i(t)-S^i(t)A(t)-Q(t)+S^i(t)B^i(t)B^i(t)'S^i(t),$$
$$S^i(t_f)=Q_f; \tag{17a}$$

$\Psi^i(t,s)$, $i=1,2$, are the state transition matrices for the systems

$$\dot{x}=(A(t)-B^i(t)B^i(t)'S^i(t))x;$$

and

$$l_t^i=S^i(t)\left\{\Phi(t,t_0)E[x_0(\omega)|\sigma^i]\right.$$
$$\left.+\int_{t_0}^t\Phi(t,s)B^j(s)E\left[u_s^{j^0}(\omega)|\sigma^i\right]ds\right\}+k_t^i, \tag{17b}$$
$$i\neq j;\ i,j=1,2$$

where k_t^i, $i=1,2$, satisfy

$$\dot{k}_t^i(\omega)=-(A(t)'-S^i(t)B^i(t)B^i(t)')k_t^i(\omega)-S^i(t)B^j(t)E\left[u_t^{j^0}(\omega)|\sigma^i\right]$$
$$k_{t_f}^i(\omega)=0;\qquad j\neq i,\ j\in\{1,2\}. \tag{17c}$$

□

When the underlying statistics are not Gaussian, it is, in general, quite difficult to solve the pair of equations (16) mainly because of the presence of conditional expectations. One can, however, obtain the solution as the limit of a convergent sequence of iterations, by means of what is known as the infinite second guessing technique [11], [12]:

The Infinite Second Guessing Algorithm:

1) Start with any $u^2\in H^2$, substitute this in (16a), and solve the resulting equation for the corresponding $u^1\in H^1$.

2) Substitute this $u^1\in H^1$ into (16b) and solve for the corresponding $u^2\in H^2$.

3) Use the solution of (16b) obtained at step 2 to replace the starting choice at step 1, and reiterate.

Proposition 1:

1) In the preceding algorithm, the corresponding linear integral equations at each step admit a unique solution.

2) Regardless of the initial choice, the infinite second guessing algorithm converges to the unique optimal solution.

Proof: First note that, for each $u^2\in H^2$, (16a) constitutes a necessary and sufficient condition for $u^1\in H^1$ to minimize $J(u^1,u^2)$ over H^1. Likewise, (16b) provides a necessary and sufficient condition for minimization of $J(u^1,u^2)$ over H^2, for each fixed $u^1\in H^1$. Hence, the proposition readily follows, since J is continuously differentiable and strictly convex on $H^1\times H^2$ and it has a unique minimum (Lemmas 1 and 2). □

D. The Special Case of Gaussian Statistics

When the underlying statistics are Gaussian, it is possible to determine the structure of the team-optimal controls explicitly. To this end, and in view of the formulation of the general problem in Section II, let $x_0\sim N(\bar{x}_0,\Sigma_0)$ and the observation y^i of DMi be given as

$$y^i=C^ix_0+v^i,\qquad i=1,2 \tag{18}$$

where $v_i\sim N(0,R^i)$, $R^i>0$, and these three random vectors are statistically independent. Now, consider the iterative algorithm of the previous subsection, starting at step 1 with $u^2\equiv 0$ a.s. Then, the resulting expression for u_t^1 is

$$u_t^1=-B^1(t)'S^1(t)\psi^1(t,t_0)E[x_0|\sigma^1] \tag{19}$$

so that u_t^1 is really a linear function of $E[x_0|\sigma^1]$, which, in turn, is affine in y^1 because of the underlying Gaussian statistics.

Now, if this functional form is substituted in (16b), at step 2, it follows through a similar argument (but this time via the solution of a linear differential equation) that the solution u_t^2 will be affine in y^2, again because of Gaussian statistics. This argument then iteratively yields (also in view of Proposition 1) the conclusion that, when the underlying statistics are Gaussian, the unique team-optimal solution is affine in the information available to each DM. We have, in fact, the following.

Theorem 2: The continuous-time two-member LQG static team problem formulated in this section admits the unique solution

$$u_t^{1^0}=P^1(t)[y^1-C^1\bar{x}_0]-B^1(t)'S(t)\Psi(t,t_0)\bar{x}_0 \tag{20a}$$

$$u_t^{2^0}=P^2(t)[y^2-C^2\bar{x}_0]-B^2(t)'S(t)\Psi(t,t_0)\bar{x}_0 \tag{20b}$$

where $S(t)$, $t_0<t<t_f$, is the unique nonnegative definite matrix function solution of the Riccati equation

$$\dot{S}(t)+A(t)'S(t)+S(t)A(t)-S(t)[B^1(t)B^1(t)'+B^2(t)B^2(t)']$$
$$S(t)+Q(t)=0,\qquad S(t_f)=Q_f, \tag{21}$$

$\Psi(t,s)$ is the state transition matrix function satisfying

$$\frac{d\Psi(t,s)}{dt}=[A(t)-[B^1(t)B^1(t)'$$
$$+B^2(t)B^2(t)']S(t)]\Psi(t,s)\qquad \Psi(s,s)=I. \tag{22}$$

The pair $\{P^1(\cdot),P^2(\cdot)\}$ satisfies (and constitutes the unique solution for) the coupled set of integral equations

$$P^1(t)=B^1(t)'S^1(t)\int_{t_0}^t\Psi^1(t,s)B^1(s)B^1(s)'L^1(s)\,ds-B^1(t)'L^1(t) \tag{23a}$$

$$P^2(t)=B^2(t)'S^2(t)\int_{t_0}^t\Psi^2(t,s)B^2(s)B^2(s)'L^2(s)\,ds-B^2(t)'L^2(t) \tag{23b}$$

where

$$L^i(t)=S^i(t)\left\{\Phi(t,t_0)+\int_{t_0}^t\Phi(t,s)B^j(s)P^j(s)\,ds\,C^j\right\}\Sigma^i+K^i(t),$$
$$i\neq j\ \ i,j=1,2 \tag{24a}$$

and

$$K^i(t)=-(A(t)'-S^i(t)B^i(t)B^i(t)')K^i(t)-S^i(t)B^j(t)P^j(t)C^j\Sigma^i,$$
$$i\neq j;\ i,j=1,2 \tag{24b}$$

and

$$\Sigma^i=\Sigma_0C^{i\prime}(C^i\Sigma_0C^{i\prime}+R^i)^{-1},\qquad i=1,2. \tag{25}$$

Proof: A proof of this theorem is provided in the Appendix. □

We obtain extremely simple expressions for the optimal $u_t^{i^0}$, $i=1,2$, in the special case when $Q(t)\equiv 0$, i.e., if no cost is assigned to the intermediate values of the state vector.

Corollary: For the continuous-time two-member LQG static team problem formulated in this section, and with $Q(\cdot)$ taken to vanish identically on the interval $[t_0, t_f]$, the unique minimizing solution is

$$u_t^{1^0} = -B^1(t)'K(t)\Phi(t,t_0)D^{1^0}[y^1 - C^1\bar{x}_0] - B^1(t)'S(t)\Psi(t,t_0)\bar{x}_0 \tag{26a}$$

$$u_t^{2^0} = -B^2(t)'K(t)\Phi(t,t_0)D^{2^0}[y^2 - C^2\bar{x}_0] - B^2(t)'S(t)\Psi(t,t_0)\bar{x}_0, \tag{26b}$$

where D^{1^0} uniquely satisfies the Lyapunov-type matrix equation

$$(I+M^1)D^{1^0} - M^2(I+M^2)^{-1}M^1D^{1^0}C^1\Sigma^2 = \Sigma^1 - M^2(I+M^2)^{-1}\Sigma^2C^2\Sigma^1 \tag{27a}$$

and D^{2^0} is given by

$$D^{2^0} = (I+M^2)^{-1}(\Sigma^2 - M^1D^{1^0}C^1\Sigma^2). \tag{27b}$$

Here, M^i is a constant matrix defined by

$$M^i = \int_{t_0}^{t_f}\Phi(t_0,s)B^i(s)B^i(s)'K(s)\Phi(s,t_0)\,ds, \qquad i=1,2 \tag{27c}$$

Proof: Details are given in [22]. □

E. Minimum Team Cost Under Gaussian Statistics

We now obtain an expression for the minimum value of the cost function of the static LQG team problem solved in the previous subsection. The results to be derived in the sequel will especially be useful in the derivation of the optimal solution of the dynamic LQG team problem in Section IV.

Let $\|\cdot\|$ denote the standard norm on an appropriate dimensional Euclidean space. Then, we first have

Lemma 3: The cost function J, defined by (4), can equivalently be written as

$$J(u^1,u^2) = \int_{t_0}^{t_f} E\{\|u_t^1 + B^1(t)'S(t)x_t\|^2 + \|u_t^2 + B^2(t)'S(t)x_t\|^2\}\,dt + \bar{x}_0'S(0)\bar{x}_0 + J_r \tag{28a}$$

where $S(\cdot)$ is defined by (22) and J_r is given by

$$J_r = Tr(\Sigma_0 S(0)) + Tr\left(\int_{t_0}^{t_f} S(t)F(t)F(t)'\,dt\right) \tag{28b}$$

and is independent of the controls.

Proof: This result follows from the standard "completing the square" argument of LQ stochastic control [13] by appropriate decomposition. □

Now, to obtain an expression for the minimum team cost, it will be sufficient to substitute the optimal team solution given in Theorem 2 into J. If this is done, then the integral term in (28a) reads

$$\int_{t_0}^{t_f} E\{\|P^1(t)[y^1 - C^1\bar{x}_0] + B^1(t)'S(t)[x_t^0 - \Psi(t,t_0)\bar{x}_0]\|^2 + \|P^2(t)[y^2 - C^2\bar{x}_0] + B^2(t)'S(t)[x_t^0 - \Psi(t,t_0)\bar{x}_0]\|^2\}\,dt \tag{29}$$

where x_t^0, $t > t_0$, denotes the optimal team trajectory and is determined as the unique solution of

$$dx_t^0 = \left[A(t)x_t^0 + B^1(t)u_t^{1^0} + B^2(t)u_t^{2^0}\right]dt + F(t)\,dw_t, \qquad x_{t_0} = x_0. \tag{30}$$

Let us now decompose x_t^0 into two parts and write it as $x_t^0 = m_t + \xi_t$ where m_t and ξ_t, respectively, satisfy

$$\frac{dm_t}{dt} = A(t)m_t - [B^1(t)B^1(t)' + B^2(t)B^2(t)']S(t)\Psi(t,t_0)m_0, \qquad m_0 = \bar{x}_0 \tag{31a}$$

$$d\xi_t = A(t)\xi_t\,dt - \left(B^1(t)P^1(t)[y^1 - C^1\bar{x}_0] + B^2(t)P^2(t)[y^2 - C^2\bar{x}_0]\right)dt + F(t)\,dw_t, \qquad \xi_0 = x_0 - \bar{x}_0. \tag{31b}$$

The solution of (31a) can readily be obtained as

$$m_t = \psi(t,t_0)\bar{x}_0$$

where $\psi(t,s)$ is defined by (22). Then, if the decomposition $x_t^0 = \psi(t,t_0)\bar{x}_0 + \xi_t$ is used in (29), the resulting expression becomes only a function of the stochastic process $\{\xi_t, t > t_0\}$, i.e., it can be written as

$$\int_{t_0}^{t_f} E\{\|P^1(t)[C^1\xi_0 + v^1] + B^1(t)'S(t)\xi_t\|^2 + \|P^2(t)[C^2\xi_0 + v^2] + B^2(t)'S(t)\xi_t\|^2\}\,dt. \tag{32}$$

This expression can further be simplified by making use of the solution of (31b) and the statistical independence property of ξ_0, v^1 and v^2. The final form is given below as the second term of expression (34) and the result is summarized in Lemma 4. The details of the manipulations involved to arrive at expression (34) will not be given here since they are rather straightforward (although cumbersome) and not that interesting for our purposes. What is important to note is the structure of the minimum value (as a function of $\bar{x}_0$) given in Lemma 4.

Preliminary Notation for Lemma 4:

Define the appropriate dimensional matrix functions $\Lambda_0^1(\cdot)$, $\Lambda_0^2(\cdot)$, $\Lambda_1^1(\cdot)$, $\Lambda_1^2(\cdot)$, $\Lambda_2^1(\cdot)$, $\Lambda_2^2(\cdot)$ on $[t_0, t_f]$ as

$$\Lambda_0^i(t) = P^i(t)C^i + B^i(t)'S(t)\Phi(t,t_0) - B^i(t)'S(t)\int_{t_0}^{t}\Phi(t,s)B^i(s)P^i(s)\,ds\,C^i - B^i(t)'S(t)\int_{t_0}^{t}\Phi(t,s)B^j(s)P^j(s)\,ds\,C^j, \qquad i\neq j,\ i,j=1,2. \tag{33a}$$

$$\Lambda_i^i(t) = P^i(t) + B^i(t)'S(t)\int_{t_0}^{t}\Phi(t,s)B^i(s)P^i(s)\,ds; \qquad i=1,2, \tag{33b}$$

$$\Lambda_i^j(t) = B^j(t)'S(t)\int_{t_0}^{t}\Phi(t,s)B^i(s)P^i(s)\,ds; \qquad i\neq j,\ i,j=1,2. \tag{33c}$$

Further, let

$$J_m = J_r + Tr\int_{t_0}^{t_f}\Big[(\Lambda_0^1(t)'\Lambda_0^1(t) + \Lambda_0^2(t)'\Lambda_0^2(t))\Sigma_0 + (\Lambda_1^1(t)'\Lambda_1^1(t) + \Lambda_1^2(t)'\Lambda_1^2(t))R^1 + (\Lambda_2^1(t)'\Lambda_2^1(t) + \Lambda_2^2(t)'\Lambda_2^2(t))R^2 + S(t)'(B^1(t)B^1(t)' + B^2(t)B^2(t)')S(t)\cdot\int_{t_0}^{t}\Phi(t,\tau)F(\tau)F(\tau)'\Phi(t,\tau)'\,d\tau\Big]\,dt. \tag{34}$$

Lemma 4: The minimum value of the cost function of the static LQG team problem under consideration is

$$J_0 \triangleq J(u^{1^0}, u^{2^0}) = \bar{x}_0'S(0)\bar{x}_0 + J_m \tag{35}$$

where J_m is given by (34) and is independent of $\bar{x}_0$. □

IV. Solution of the Dynamic Team Problem

The solution of the dynamic LQG team problem formulated in Section II can now be obtained by making use of the static theory developed in the previous section. The derivation basically involves a dynamic programming type of argument, and one has to utilize Theorem 2 and Lemma 4 at every sampling time interval. In the sequel, this will be achieved by first enlarging the strategy spaces of the decisionmakers so as to formulate a new team problem whose team-optimal solution can be obtained more readily, and then by relating the solution of the original team problem to the one obtained for the auxiliary one. Such an indirect derivation seems to be inevitable, since otherwise the analysis gets quite cumbersome.

The only difference between the new dynamic team problem to be introduced and the original one lies in the information patterns. Specifically, the new one is defined by replacing η_j^i and δ_{j-1}, given by (3), by $\tilde{\eta}_j^i$ and $\tilde{\delta}_{j-1}$, respectively, where

$$\tilde{\eta}_j^i=\{y_j^i,\tilde{\delta}_{j-1}\} \tag{36a}$$

$$\tilde{\delta}_{j-1}=\{\delta_{j-1};u_t^1,u_t^2,t<t_j\}. \tag{36b}$$

Under this new information pattern, the decision makers have also access to each other's control values used during all past sampling intervals. It should be noted that this information pattern is not the continuous-time counterpart of the one-step-delay information sharing pattern [14] and, the way it stands, it is not of much practical importance. It, however, provides mathematical convenience in obtaining the solution of the original team problem, as it will become clear later.

Under this new information pattern, let us replace the strategy spaces Γ_N^1 and Γ_N^2 by $\tilde{\Gamma}_N^1$ and $\tilde{\Gamma}_N^2$, respectively, where the latter are defined in an analogous way, but under the new information pattern. Since the new strategy spaces are larger, we immediately have the inequality

$$\min_{\Gamma_N^1}\min_{\Gamma_N^2}J(\gamma^1,\gamma^2)\geq\min_{\tilde{\Gamma}_N^1}\min_{\tilde{\Gamma}_N^2}J(\gamma^1,\gamma^2), \tag{37}$$

i.e., the minimum cost of the new team problem provides a lower bound for the minimum cost of the original one. The following lemma now says that, in fact, they have to be the same.

Lemma 5:

1) To every pair $\{\tilde{\gamma}^1\in\tilde{\Gamma}_N^1,\tilde{\gamma}^2\in\tilde{\Gamma}_N^2\}$, there corresponds a unique pair $\{\gamma^1\in\Gamma_N^1,\gamma^2\in\Gamma_N^2\}$ so that $J(\tilde{\gamma}^1,\tilde{\gamma}^2)=J(\gamma^1,\gamma^2)$.

2) The inequality in (37) is, in fact, an equality.

Proof: 1) For each $\{\tilde{\gamma}^1\in\tilde{\Gamma}_N^1,\tilde{\gamma}^2\in\tilde{\Gamma}_N^2\}$, the implicit equations[1]

$$\left.\begin{aligned}\tilde{\gamma}_j^1(\tilde{\eta}_j^1)&=u_j^1(\omega)\\ \tilde{\gamma}_j^2(\tilde{\eta}_j^2)&=u_j^2(\omega)\end{aligned}\right\}\quad j=N-1,\cdots,0 \tag{38}$$

can be solved recursively for $\{u_j^i(\omega),j=N-1,\cdots,0;i=1,2\}$ as functions of $\{\eta_j^i,j=N-1,\cdots,0;i=1,2\}$ because of the nature of the information pattern. Then, the resulting functional relations provide a pair in $\Gamma_N^1\times\Gamma_N^2$, and a unique one since the stochastic differential equation (1) admits a unique solution in each sampling interval.

2) This result follows readily from 1). □

Remark 1: There, in fact, exist uncountably many pairs in $\tilde{\Gamma}_N^1\times\tilde{\Gamma}_N^2$ corresponding to a given pair in $\Gamma_N^1\times\Gamma_N^2$; equivalently, a pair of strategies under the original information structure has several *representations* [15] under the new (enlarged) information pattern. Hence, Lemma 5 also says that all representations of a minimizing solution pair for the original team problem, and those only solve the new team problem. In the sequel, we will obtain one such representation which is, in fact, the simplest one to derive; and then we solve implicit equations of the type (38) to obtain the desired optimal team solution. □

[1] Here, $\tilde{\gamma}_j^i$ and u_j^i denote, respectively, the restrictions of $\tilde{\gamma}^i$ and u^i to the sampling interval $[t_j,t_{j+1})$.

An Auxiliary Result: In the derivation of optimal solution for the new team problem, we will need the expressions for $\hat{x}_{t_j}\triangleq E[x_{t_j}|\tilde{\delta}_{j-1}]$ and $\operatorname{cov}(\hat{x}_{t_j},\hat{x}_{t_j})$, which we first obtain. To this end, let us first introduce a stochastic process $\{z_t,t\geq t_0\}$ and a deterministic matrix function $\Sigma(\cdot)$ on $[t_0,t_f]$ by

$$\left.\begin{aligned}&\frac{dz_t}{dt}=A(t)z_t+B^1(t)u_t^1+B^2(t)u_t^2,\qquad z_{t_0}=\bar{x}_0\\ &t_{j-1}\leq t<t_j,\qquad j=1,\cdots,N.\\ &z_{t_j}=z_{t_j^-}+K_j[y_j-C_jz_{t_j^-}]\end{aligned}\right\} \tag{39}$$

$$\left.\begin{aligned}&\frac{d\Sigma(t)}{dt}=A(t)\Sigma(t)+\Sigma(t)A(t)'+F(t)F(t)'\\ &\Sigma(t_0)=\Sigma_0,\qquad t_{j-1}\leq t<t_j,\ j=1,\cdots,N\\ &\Sigma(t_j)=\Sigma(t_j^-)-K_jC_j\Sigma(t_j^-)\end{aligned}\right\} \tag{40}$$

where

$$K_j=\Sigma(t_j^-)C_j'\left[C_j\Sigma(t_j^-)C_j'+R_j\right]^{-1} \tag{41a}$$

$$R_j=\operatorname{diag}(R_j^1,R_j^2) \tag{41b}$$

$$y_j=(y_j^{1'},y_j^{2'})'. \tag{41c}$$

$$C_j=(C_j^{1'},C_j^{2'})'. \tag{41d}$$

It should be noted that the matrix $\Sigma(\cdot)$, as well as the sample paths of $\{z_t,t\geq t_0\}$, have discontinuities at the sampling points $t_1,\cdots,t_{N-1}$. Now, we have the following result.

Lemma 6:

$$\hat{x}_{t_j}\triangleq E\left[x_{t_j}|\tilde{\delta}_{j-1}\right]=z_{t_j^-},\qquad j=1,\cdots,N-1. \tag{42a}$$

$$\operatorname{cov}(\hat{x}_{t_j},\hat{x}_{t_j})=\Sigma(t_j^-),\qquad j=1,\cdots,N-1. \tag{42b}$$

Proof: Relations (39)–(40) (without the term $B^1u_t^1+B^2u_t^2$) constitute the filtering equations for linear continuous-time systems with discrete (sampled) observations [16]. Inclusion of the term $B^1u_t^1+B^2u_t^2$ in (39) is possible (as in the standard LQG control theory) because of the nature of the information pattern involved. Specifically, $z_t=E[x_t|\delta_{j-1};u_s^1,u_s^2,s<t]$ for $t_{j-1}\leq t<t_j$, and $z_{t_j}=E[x_{t_j}|\delta_j]$, and $\Sigma(t)=\operatorname{cov}(z_t,z_t)$. Equations (42a) and (42b) then readily follow from these relations in light of (36b). □

Derivation of an Optimal Solution for the New Team Problem:

We now seek to obtain a solution for the optimization problem

$$\min_{\tilde{\Gamma}_N^1}\min_{\tilde{\Gamma}_N^2}J(\tilde{\gamma}^1,\tilde{\gamma}^2).$$

To this end let us first attempt to obtain the restriction of the sought pair of strategies $\{\tilde{\gamma}^{1^0},\tilde{\gamma}^{2^0}\}$ to the sampling interval $[t_{N-1},t_N)$, which we denote by $\{\tilde{\gamma}_{N-1}^{1^0},\tilde{\gamma}_{N-1}^{2^0}\}$ where $t_N=t_f$. Defining

$$g_t(u_t^1,u_t^2,x_t)\triangleq x_t'Q(t)x_t+u_t^{1'}u_t^1+u_t^{2'}u_t^2 \tag{43a}$$

and

$$J_{N-j}\triangleq\sum_{k=1}^{N-j}\int_{t_{k-1}}^{t_k}E\{g_t(u_t^1,u_t^2,x_t)|(\tilde{\gamma}_{k-1}^1,\tilde{\gamma}_{k-1}^2)\}\,dt^2\qquad 1\leq j\leq N-1, \tag{43b}$$

we note that J can be written in the equivalent form

[2] This is not a conditional expectation, but simply expectation with u_t^i determined by the control strategy $\tilde{\gamma}_{k-1}^i$ in the interval $[t_{k-1},t_k)$.

$$J(\tilde{\gamma}^1,\tilde{\gamma}^2)=E\left\{\left(x_{t_f}'S(t_f)x_{t_f}+\int_{t_{N-1}}^{t_f}g_t(u_t^1,u_t^2,x_t)\,dt\right)\Big|(\tilde{\gamma}_{N-1}^1,\tilde{\gamma}_{N-1}^2)\right\}+J_{N-1} \quad (43c)$$

where the second term does not depend on the restriction of $\{\tilde{\gamma}^1,\tilde{\gamma}^2\}$ to the sampling interval $[t_{N-1},t_N)$. Hence, to determine $\{\tilde{\gamma}_{N-1}^{1^0},\tilde{\gamma}_{N-1}^{2^0}\}$, we can confine our attention to the first term of (43c), which we may denote by J^{N-1} where we define, in general,

$$J^j=E\left\{E\left\{x_{t_f}'S(t_f)x_{t_f}+\sum_{k=j+1}^{N}\int_{t_{k-1}}^{t_k}g_t(u_t^1,u_t^2,x_t)\,dt\,|(\tilde{\gamma}_j^1,\tilde{\gamma}_j^2),\tilde{\delta}_j\right\}\right\}$$
$$j=N-1,\cdots,0. \quad (44)$$

That is, first conditioned on the common information $\tilde{\delta}_j$ available to the decisionmakers in the sampling interval $[t_j,t_{j+1})$, and then full expectation. Now, note that the probability distribution of $x_{t_{N-1}}$ conditioned on $\tilde{\delta}_{N-1}$ is Gaussian (because of linear state dynamics and linear observation equations) and furthermore, it has mean $\hat{x}_{t_{N-1}}$ and covariance $\Sigma(t_{N-1}^-)$ by Lemma 6. Then, the team problem defined by J^{N-1} becomes equivalent to the static LQG team problem of Section III-D, with only t_0 replaced by t_{N-1}, $\bar{x}_0$ by $\hat{x}_{t_{N-1}}$, Σ_0 by $\Sigma(t_{N-1}^-)$ and C^i by C_{N-1}^i. Consequently, the result of Theorem 2 directly applies here, implying that the solution will be given by

$$\tilde{\gamma}^{1^0}(\tilde{\eta}^1)=\tilde{P}^1(t)\left[y_{N-1}^1-C_{N-1}^1\hat{x}_{t_{N-1}}\right]-B^{1'}(t)S(t)\psi(t,t_{N-1})\hat{x}_{t_{N-1}} \quad (45a)$$

$$\tilde{\gamma}^{2^0}(\tilde{\eta}^2)=\tilde{P}^2(t)\left[y_{N-1}^2-C_{N-1}^2\hat{x}_{t_{N-1}}\right]-B^{2'}(t)S(t)\psi(t,t_{N-1})\hat{x}_{t_{N-1}} \quad (45b)$$

where $\{\tilde{P}^1(\cdot),\tilde{P}^2(\cdot)\}$ will be given by a pair of equations, which is a counterpart of (23).

Now, if this solution is substituted into J^{N-1}, we know from Lemma 4 that it will have the functional form

$$J^{N-1^0}=E\{\hat{x}_{t_{N-1}}'S(t_{N-1})\hat{x}_{t_{N-1}}\}+J_m^{N-1} \quad (46)$$

where the second term is given as a counterpart of (34) and does not depend on the past controls (or strategies). Furthermore, replacing $\hat{x}_{t_{N-1}}$ in the preceding expression by $\hat{x}_{t_{N-1}}-x_{t_{N-1}}+x_{t_{N-1}}$, we can express its first term as

$$E\{x_{t_{N-1}}'S(t_{N-1})x_{t_{N-1}}+2x_{t_{N-1}}'S(t_{N-1})(\hat{x}_{t_{N-1}}-x_{t_{N-1}})\}$$
$$+\mathrm{tr}[S(t_{N-1})\Sigma(t_{N-1}^-)]$$

which is equivalent to

$$E\{x_{t_{N-1}}'S(t_{N-1})x_{t_{N-1}}\}-\mathrm{tr}[S(t_{N-1})\Sigma(t_{N-1}^-)] \quad (47)$$

using standard properties of conditional mean. Hence, while determining the pair $\{\tilde{\gamma}_{N-2}^1,\tilde{\gamma}_{N-2}^2\}$, the expression of interest is [also from (43c)]

$$J^{N-2}=E\left\{E\left\{x_{t_{N-1}}'S(t_{N-1})x_{t_{N-1}}+\int_{t_{N-2}}^{t_{N-1}}g_t(u_t^1,u_t^2,x_t)\,dt\,|(\tilde{\gamma}_{N-2}^1,\tilde{\gamma}_{N-2}^2),\tilde{\delta}_{N-2}\right\}\right\} \quad (48)$$

since $\Sigma(\cdot)$ is independent of the controls. But, this team problem is analogous to the one considered on the sampling interval $[t_{N-1},t_N)$, thereby admitting a solution in the structural form (45). Proceeding in this manner, we obtain, by induction, the following proposition.

Preliminary notation for Proposition 2:

Let $\tilde{\Sigma}_j^i$ be appropriate dimensional matrices defined by

$$\tilde{\Sigma}_j^i=\Sigma(t_j)C_j^{i'}\left(C_j^i\Sigma(t_j)C_j^{i'}+R_j^i\right)^{-1}, \quad i=1,2, \quad j\in\theta. \quad (49)$$

Let $\tilde{P}^1(\cdot),\tilde{P}^2(\cdot)$ be piecewise continuous functions on $[t_0,t_f]$, which satisfy the coupled set of linear integral equations

$$\tilde{P}^1(t)=B^1(t)'\tilde{S}^1(t)\int_{t_j}^{t}\tilde{\Psi}_j^1(t,s)B^1(s)B^1(s)'\tilde{L}_j^1(s)\,ds-B^1(t)'\tilde{L}_j^1(t) \quad (50a)$$

$$\tilde{P}^2(t)=B^2(t)'\tilde{S}^2(t)\int_{t_j}^{t}\tilde{\Psi}_j^2(t,s)B^2(s)B^2(s)'\tilde{L}_j^2(s)\,ds-B^2(t)'\tilde{L}_j^2(t)$$
$$\cdot t_j<t<t_{j+1},\ j=0,1,\cdots,N-1 \quad (50b)$$

where

$$\tilde{L}_j^i(t)=\tilde{S}^i(t)\left\{\Phi(t,t_j)+\int_{t_j}^{t}\Phi(t,s)B^k(s)\tilde{P}^k(s)\,ds\,C_j^k\right\}\tilde{\Sigma}_j^i+\tilde{K}_j^i(t) \quad (51a)$$

and

$$\dot{\tilde{K}}_j^i(t)=-\left(A(t)'-\tilde{S}^i(t)B^i(t)B^i(t)'\right)\tilde{K}_j^i(t)-\tilde{S}^i(t)B^k(t)\tilde{P}^k(t)C_j^k\tilde{\Sigma}_j^i$$
$$\tilde{K}_j^i(t_{j+1})=0, \qquad i\neq k,\ i,k=1,2,\ j\in\theta \quad (51b)$$

where $\tilde{S}^i(t)$ satisfies

$$\dot{\tilde{S}}^i(t)=-A(t)'\tilde{S}^i(t)-\tilde{S}^i(t)A(t)-Q(t)+\tilde{S}^i(t)B^i(t)B^i(t)'\tilde{S}^i(t),$$
$$t_{j-1}<t<t_j,\ \tilde{S}^i(t_j)=S(t_j), \qquad i=1,2,\ j=N,\cdots,1, \quad (51c)$$

while

$\tilde{\Psi}_j^i(t,s)$ is the state transition matrix of the system

$$\dot{x}=\left(A(t)-B^i(t)B^i(t)'\tilde{S}^i(t)\right)x, \qquad t\in[t_j,t_{j+1}),\ i=1,2,\ j\in\theta.$$

Proposition 2: 1) The set of equations (50) admits a unique solution pair $\{\tilde{P}^1(\cdot),\tilde{P}^2(\cdot)\}$. 2) The LQG dynamic team problem of Section II, and under the amended information structure $(\tilde{\eta}^1,\tilde{\eta}^2)$, admits an optimal solution whose restriction to the sampling interval $[t_j,t_{j+1})$ is

$$\tilde{\gamma}^{1^0}(t,\tilde{\eta}^1)=\tilde{P}^1(t)\left[y_j^1-C_j^1\hat{x}_{t_j}\right]-B^1(t)'S(t)\psi(t,t_j)\hat{x}_{t_j} \quad (52a)$$

$$\tilde{\gamma}^{2^0}(t,\tilde{\eta}^2)=\tilde{P}^2(t)\left[y_j^2-C_j^2\hat{x}_{t_j}\right]-B^2(t)'S(t)\psi(t,t_j)\hat{x}_{t_j} \quad (52b)$$
$$t_j<t<t_{j+1}, \qquad j=0,\cdots,N-1.$$

Proof: 1) This result readily follows from Theorem 2, since the pair of equations (50) on each sampling interval is analogous to the pair (23). 2) The inductive argument for derivation of this optimal solution has already been outlined prior to the statement of this proposition. □

It should be noted that we cannot claim uniqueness of the solution presented above, in view of Remark 1. In fact, all pairs of strategies in $\tilde{\Gamma}_N^1\times\tilde{\Gamma}_N^2$ that provide the same minimum value for J will be different representations of $\{\tilde{\gamma}^{1^0},\tilde{\gamma}^{2^0}\}$. One such representation will, however, lie in $\Gamma_N^1\times\Gamma_N^2$, which will constitute the unique solution of the original team problem by Lemma 5. This particular representation is given below in Theorem 3.

Theorem 3: The LQG dynamic team problem of Section II, and under the one-step-delay observation sharing pattern as formulated there, admits the unique solution (in $\Gamma_N^1\times\Gamma_N^2$) whose restriction to the sampling interval $[t_j,t_{j+1})$ is given as

$$\gamma^{1^0}(t,\eta^1)=\tilde{P}^1(t)\left[y_j^1-C_j^1\hat{\xi}_j\right]-B^1(t)'S(t)\psi(t,t_j)\hat{\xi}_j \quad (53a)$$

$$\gamma^{2^0}(t,\eta^2)=\tilde{P}^2(t)\left[y_j^2-C_j^2\hat{\xi}_j\right]-B^2(t)'S(t)\psi(t,t_j)\hat{\xi}_j, \quad (53b)$$
$$t_j<t<t_{j+1}, \qquad j=0,\cdots,N-1$$

where $\hat{\xi}_j$ is defined as

$$\hat{\xi}_j = z^0(t_j^-), \qquad j \in \theta \tag{54}$$

where $z^0(t)$ is the solution of (39) with u_t^i replaced by $\gamma^{i^0}(t, \eta^i)$. □

Proof: This result is an immediate consequence of Proposition 2 and Lemma 5, since (53) indeed exists by recursively solving the differential equation (39), and it is clearly an element of $\Gamma_N^1 \times \Gamma_N^2$. □

Remark 2: Since $\psi(\cdot,\cdot)$ is the fundamental matrix corresponding to the feedback system of the deterministic version of the problem, it should be clear from (53) that the optimal solution of the dynamic team problem features a certainty-equivalence property. The second terms in (53) yield exactly the solution of the deterministic version of the problem if $\hat{\xi}_j$ is replaced by $x(t_j)$. The first terms, on the other hand, reflect the contribution of the nonshared part of the information to each DM's strategy, i.e., they are the innovation terms. □

V. Conclusions

In this paper we have presented a complete solution to the LQG continuous-time two-member team problem, in which the decision-makers make independent noisy measurements of the state at sampled instants of time, and exchange this information with a delay of one sampling interval. The optimal team solution is affine in the information available to each DM, and the coefficient terms involved are determined recursively and by solving a pair of integral equations at each step. It is shown that, under certain conditions, the solutions of these integral equations can be obtained by solving Liapunov type time-invariant matrix equations.

The delayed observation sharing pattern considered in this paper within the context of continuous-time dynamic teams seems to be a natural counterpart of the discrete-time one-step delayed observation sharing pattern [8]. Other types of delayed information sharing patterns for continuous-time systems have recently been considered in the literature, notably in [18] and [19]. The former article is devoted to decentralized control of Gauss-Poisson processes, in which case each DM instantaneously observes the jumps occurring in his own system dynamics but transmits this information to the other DM's with a certain amount of delay.

The latter reference formulates a general decentralized LQ team problem and also makes use of techniques of functional analysis (in particular, properties of Volterra operators) in arriving at certain general conclusions. The general model of [19] is, however, restricted considerably by the assumption that the state and information variables are unaffected by the controls of the DM's; hence, it is basically static in character. Our static team problem considered in Section III can definitely be viewed as a special case of this general model, but [19] does not contain the explicit optimal team solution presented in this paper and, furthermore, it does not discuss the existence and uniqueness questions thoroughly investigated in this paper.

One natural (although not straightforward) extension of the results of this paper would be to obtain Nash equilibria of similarly structured stochastic nonzero-sum differential game problems. For a counterpart of Lemma 2 to be valid in that case, one has to impose certain additional restrictions on the parameters of the problem. This has actually been done in [20] where authors obtain a sufficient condition for the LQG nonzero-sum differential game to admit a unique Nash equilibrium solution under static information. For the dynamic continuous-time LQG nonzero-sum differential game, and under the one-step-delay observation sharing pattern of this paper, Nash equilibria will again be unique whenever it exists, and the equilibrium strategies of the decisionmakers (players) will be affine in their information, i.e., a direct counterpart of the result of [21] will hold true for the continuous-time problem also. A verification of this result, however, will require an analysis quite different from the one employed in Section IV of this paper, since Lemma 5 has no counterpart in a game situation. Details of this analysis, as well as the expressions for the unique equilibrium strategies, will be presented in a forthcoming paper.

Appendix

Proof of Theorem 2: We first assume that $\bar{x}_0 = 0$. With the Gaussian assumption,

$$E[x_0 | \sigma^i] = E[x_0 y^{i'}](E[y^i y^{i'}])^{-1} y^i = \Sigma^i y^i, \qquad i = 1,2$$

where Σ^i is given by expression (25). From the discussion prior to the statement of Theorem 2, we see that, when $\bar{x}_0 = 0$, the optimal $u_t^{i^0}$ are linear functions of y^i, $i=1,2$. Thus, we can write $u_t^{i^0} = P^i(t) y^i$ where $P^i(t)$ are to be determined. Using this form in (16a) and (16b), we get (23a) and (23b). An obvious iterations scheme is to start with $P^2(t) \equiv 0$, determine $P^1(t)$, and iterate. The iteration converges on account of the convergence of the second guessing scheme. We now want to extend this result to the case when $\bar{x}_0 \neq 0$. For this, we first take another look at the criterion. The system state evolves according to (1) and the criterion to be minimized is (4). Let

$$B(t) = [B^1(t), B^2(t)],$$

$u_t = \begin{bmatrix} u_t^1 \\ u_t^2 \end{bmatrix}$ and $S(t)$ be the unique solution of the Riccati equation

$$\dot{S}(t) + A(t)'S(t) + S(t)A(t) + Q(t) - S(t)B(t)B(t)'S(t) = 0,$$
$$S(t_f) = Q_f.$$

Then J can be expressed in an alternate form

$$J = E\left[\int_{t_0}^{t_f} \| u_t + B(t)'S(t)x_t \|^2 \, dt + \int_{t_0}^{t_f} \operatorname{tr} S(t)F(t)F(t)' \, dt + x_0' S(0) x_0\right].$$

This follows from the standard completion of squares argument [13] (see Lemma 3). Since only the first term depends on u_t, the solution obtained is also the solution for the team problem with the criterion

$$\tilde{J} = E\int_{t_0}^{t_f} \| u_t + B(t)'S(t)x_t \|^2 \, dt.$$

We now study the case when $\bar{x}_0 \neq 0$. The observations $y^i = C^i x_0 + v^i$ may be converted into zero mean quantities by defining

$$\tilde{y}^i = y^i - C^i \bar{x}_0 = C^i(x_0 - \bar{x}_0) + v^i, \qquad i = 1,2.$$

Let $\bar{x}_t$ be the unique solution of

$$\dot{\bar{x}}_t = A(t)\bar{x}_t + B(t)u_t, \qquad \bar{x}_{t_0} = \bar{x}_0.$$

Then,

$$u_t + B(t)'S(t)\bar{x}_t = u_t + B(t)'S(t)\int_{t_0}^{t} \Phi(t,s)B(s)u_s \, ds + B(t)'S(t)\Phi(t,t_0)\bar{x}_0.$$

Let

$\tilde{u}_t = u_t - m(t)$ where $m(t)$ is yet to be chosen. Then,

$$u_t + B(t)'S(t)\bar{x}_t = \tilde{u}_t + m(t) + B(t)'S(t)\int_{t_0}^{t} \Phi(t,s)B(s)[\tilde{u}_s + m(s)] \, ds + B(t)'S(t)\Phi(t,t_0)\bar{x}_0.$$

Choose $m(t)$ so that

$$m(t) + B(t)'S(t)\int_{t_0}^{t} \Phi(t,s)B(s)m(s) \, ds = -B(t)'S(t)\Phi(t,t_0)\bar{x}_0.$$

We can solve this integral equation in $m(t)$ to obtain

$$m(t) = -B(t)'S(t)\left[\Phi(t,t_0) - \int_{t_0}^{t} \Psi(t,s)B(s)B(s)'S(s)\Phi(s,t_0) \, ds\right]\bar{x}_0$$

where $\Psi(t,s)$ is the fundamental matrix for the system

$$\dot{x}=(A(t)-B(t)B(t)'S(t))x.$$

Then, clearly

$$\Phi(t,t_0)-\int_{t_0}^{t}\Psi(t,s)B(s)B(s)'S(s)\Phi(s,t_0)\,ds=\Psi(t,t_0)$$

and

$$m(t)=-B(t)'S(t)\Psi(t,t_0)\bar{x}_0.$$

With $\tilde{u}_t$ as the decision vector, we may write

$$\tilde{J}=E\left\{\int_{t_0}^{t_f}\|u_t+B(t)'S(t)\bar{x}_t+B(t)'S(t)(x_t-\bar{x}_t)\|^2\,dt\right\}$$
$$=E\left\{\int_{t_0}^{t_f}\|\tilde{u}_t+B(t)'S(t)z_t\|^2\,dt\right\}$$

where

$$z_t=x_t-\bar{x}_t+\int_{t_0}^{t}\Phi(t,s)B^1(s)\tilde{u}_s^1\,ds+\int_{t_0}^{t}\Phi(t,s)B^2(s)\tilde{u}_s^2\,ds,$$

and

$$\tilde{u}=\begin{bmatrix}\tilde{u}_t^1\\ \tilde{u}_t^2\end{bmatrix},\text{ so that}$$

$$dz_t=A(t)z_t\,dt+B^1(t)\tilde{u}_t^1\,dt+B^2(t)\tilde{u}_t^2\,dt+F(t)\,dw_t,$$
$$z_{t_0}=x_0-\bar{x}_0 \text{ with } E[z_{t_0}]=0.$$

But this is the minimization problem with zero mean initial condition, which has already been solved. In fact, the optimal solution is

$$\tilde{u}_t^i=P^i(t)\tilde{y}^i,\qquad i=1,2$$

where $P^i(t)$, $i=1,2$, satisfy the pair (23a), (23b). This yields

$$u_t^i=P^i(t)[y^i-C^i\bar{x}_0]+m^i(t)$$

where we have split $m(t)$ as $\begin{bmatrix}m^1(t)\\ m^2(t)\end{bmatrix}$. We have, then,

$$u_t^i=P^i(t)[y^i-C^i\bar{x}_0]-B^i(t)'S(t)\Psi(t,t_0)\bar{x}_0;\qquad i=1,2.$$

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An Adaptive *d*-Step Ahead Predictor Based on Least Squares

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Abstract—**This paper examines the asymptotic properties of a least squares algorithm for adaptively calculating a *d*-step ahead prediction of a time series. It is shown that, with probability one, the sample mean-square difference between the recursive prediction and the optimal linear prediction converges to zero. Relatively weak assumptions are required regarding the underlying model of the time series.**

I. Introduction

There is a growing literature on the question of convergence of recursive algorithms for parameter estimation in time series models; see, for example, [4]–[8]. In most of this work, the emphasis has been on establishing consistency and other asymptotic properties for the estimated parameters.

In [1] an alternative approach was described in which emphasis was placed on the performance of a predictor designed using the estimated parameters rather than the properties of the estimated parameters themselves. The advantages of this approach are that it is not necessary to consider the predictor performance as a separate issue and it is possible to weaken the assumptions on the model and experimental conditions.

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