Technical Change, Variable Elasticity of Substitution and Economic Growth

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Abstract

We incorporate a variable elasticity of substitution production function into an overlapping generations model à la Diamond (1965). We show that a certain parameter in the production function is a source of biased technical change is a crucial determinant of the economy's growth dynamics. For positive values of this parameter, which lead to an elasticity of substitution between capital and labour which is greater than 1, the economy always reaches a unique and stable steady state which is similar to the conditional convergence in the standard Solow growth model. For negative values of this parameter, which yield an elasticity of substitution below 1, the economy could either fall into a poverty trap; or display two steady states, of which one is stable while the other is not, which could, depending on the value of the initial capital stock, potentially result in divergence towards unbounded growth. These different outcomes are consistent with the observed diversity in international growth experiences. The capital biased technical change generated by a higher value of this parameter improves productivity in steady state, but causes an exacerbation of intergenerational inequality.

Keywords: variable elasticity of substitution; biased technical change; steady state analysis; inequality

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1. Introduction

The elasticity of substitution between labour and capital is an important determinant of an economy's long run performance. Ever since de La Grandville (1989) used a constant elasticity of substitution (CES) production function to demonstrate that a country with a higher elasticity of substitution could achieve a higher rate of economic growth and a higher value of income per capita at steady state, there has been a considerable increase in the volume of research, both theoretical and empirical, on this theme (see, for instance, Karagiannis, Palivos, & Papageorgiou, 2005; Klump & De La Grandville, 2000; Klump & Preissler, 2000; Mallick, 2010, 2012; Miyagiwa & Papageorgiou, 2007; Palivos & Karagiannis, 2010; Yip & Xue, 2013; Yuhn, 1991).

In what was perhaps the earliest empirical study on the elasticity of factor substitution, Arrow, Chenery, Minhas, and Solow (1961) observed that factor substitutability can vary between industries and across countries. More recently, this view has been expounded by a number of studies that have concluded that the elasticity of substitution, typically assumed constant in standard growth models, could, instead, be an endogenous variable related to the level of economic development of a country (see, for example, Miyagiwa & Papageorgiou, 2007). For instance, Duffy and Papageorgiou (2000) use cross-country data to show that generally, richer countries have an elasticity of substitution above 1 while poorer countries have an elasticity of substitution which is below 1. The constant elasticity of substitution (CES) production function, which has been used in most extant studies that explore the way in which the elasticity of substitution can impact on the long run behaviour of an economy, cannot take such variability into account. In the CES production function, first discussed in Solow (1956), for example, the value of the elasticity of substitution remains constant across different input combinations and over time.

The variety of estimates for the elasticity of factor substitution available in the CES literature is suggestive of a possible mis-specification of the underlying production function, which creates the need to consider alternative, more flexible production functions which can capture the differences in factor substitutability between developed and developing countries and provide an explanation for their diverse growth experiences. In this regard, the variable elasticity of substitution (VES) production function, first discussed by Sato and Hoffman (1968) and Revankar (1971), is an appealing alternative to the CES production function. Much of the properties and dynamics associated with the VES production function are determined by a technology parameter labelled b. The value of this parameter is a crucial determinant of a number of key features of the economy such as capital intensity, the direction of technical change and the share of capital in output. Through the technical change it generates, this parameter relates the elasticity of substitution between labour and capital linearly to the capital stock per worker. Positive values of b result in an elasticity of substitution above 1 while negative values lead to an elasticity of substitution below 1. The elasticity of substitution is thus endogenous in that it is driven by the parameter b and the economy's capital-labour ratio. The fact that the parameter b can take a range of different values enables the VES production function to capture the differences in elasticities of factor substitution between developed and developing countries that have been observed in extant empirical studies.

The empirical relevance of the VES has been demonstrated in a number of studies which have suggested that the VES production function can be a better specification than the CES production function. The VES production function has been shown to be an effective specification for capturing the production technology in urban housing construction, which is an industry characterized by considerable variations in the ratio of land to non-land inputs, and it may therefore be reasonable to assume that the elasticity of substitution between land and non-land inputs depends on the relative land intensity (Färe & Yoon, 1981; Sirmans, Kau, & Lee, 1979). Using data for Japan covering the period 1878-1938, Bairam (1989) demonstrates that the capital-labour ratio as well as the elasticity of substitution between capital and labour increased during this time, suggesting that the VES form is the correct functional specification of the production function for the Japanese economy during this period. Kazi (1980) demonstrates using data on Indian manufacturing industries that the elasticity of factor substitution varies between industries, and that assuming a CES production function leads to an upward bias of the estimate of the elasticity of substitution, and thereby arrives at the conclusion that the VES production function is the correct functional specification for the production function in the case of Indian manufacturing industries.

Despite the appealing properties and empirical relevance of the VES production function, it has received limited attention in the theoretical growth literature. To the best of our knowledge, within the theoretical literature exploring the long run macroeconomic outcomes associated with the elasticity of substitution, there are only two studies that employ the VES production function. The first is by Karagiannis et al. (2005), who incorporate a variable elasticity of substitution (VES) production function into an otherwise standard Solow-Swan model and demonstrate that an economy can display unbounded growth if the elasticity of factor substitution exceeds a certain minimum value. The other is a paper by Brianzoni, Mammana, and Michetti (2012), who consider the VES production function in the context of a discrete time one sector Solow–Swan growth model with differential savings, and demonstrate that small values of the elasticity of substitution, resulting from negative values of the parameter b, could result in the emergence of complex dynamics.

While extant theoretical studies such as Karagiannis et al. (2005) and Brianzoni et al. (2012) apply the VES production function in the context of a Solow-Swan model, to the best of our knowledge, there have been no studies that have looked at the VES production function in the context of an overlapping generations model. The overlapping generations model, by taking into account the life-cycle aspects of a consumer's utility maximizing problem, possesses the ability to provide potentially deeper insights into long run outcomes and transitional dynamics than the Solow-Swan model. Hence, in this paper, we incorporate a VES production function into an otherwise standard Diamond (1965) overlapping generations model. In contrast to the literature on the CES production function which concentrates on the various macroeconomic effects associated with the elasticity of substitution, in this paper, we shift our focus to the effects upon the long run behaviour of the economy created by the technology parameter b and investigate a hitherto unexplored aspect of the VES production function, which is that the way in which the *technical change* generated by this parameter impacts on the *mechanisms of growth*.

We demonstrate that the technology parameter b influences the marginal products of capital and labour capital differently, with the marginal product of capital rising in the value of b while the marginal product of labour is falling in it. By influencing relative marginal products, the parameter b determines the direction of technical change in the economy. A higher value of b leads to capital biased technical change by raising the relative marginal product of capital.[1] We are also able to show that a higher value of the parameter b is associated with steeper long run expansion path, implying that it leads to a higher capital intensity in production.

Another important feature of the VES production function is that it is a flexible functional specification that can potentially provide an explanation for the movements in factor shares experienced by many countries by tracing them back to changes in the technology parameter b. In order to elaborate on this point, we present Figure 1.1 below, which shows that a decline in the labour share has been observed in developed nations such as Germany, USA and Japan, as well as in developing and transitional economies like Mexico, Turkey and the Philippines. On the other hand, Figure 1.2 shows that the labour share has risen in recent years in Australia, New Zealand and the Czech Republic.

Noticeable movements in aggregate factor shares within countries over time as well between countries, such as those seen in Figures 1.1 and 1.2, have been observed by a number of authors (see, for instance, Acemoglu, 2003; Boldrin, 1995; Durlauf, 1995; Gollin, 2002; Solow, 1997; Young, 2010). However, despite evidence of such diversity in the movements of labour shares, recent research such as Karabarbounis and Neiman (2014), Rodriguez and Jayadev (2010), ILO (2013) and OECD (2012), that cover a large number of countries, suggest

[1] As per the definitions presented in Acemoglu (2002), capital biased technical change is identified with a rise in the marginal product of capital to relative to the marginal product of labour. Ceteris paribus, this would alter the optimal input mix towards the utilization of more capital. In equilibrium, however, outcomes depend not only on this ratio (which is the marginal rate of technical substitution), but also the ratio of prices of these two inputs.

that there has been a global decline in labour share.[2] [3] Adding to such observations, Piketty and Goldhammer (2014, p. 223) warns of the extent of the potential increases in the share of capital of income by noting that, "It is quite possible that capital's share will increase in coming decades to the level it reached at the beginning of the nineteenth century." In the VES production function, a higher value of the parameter b is associated with a declining share of labour in output. Hence, the VES form provides a possible explanation for the decline in labour share experienced by many countries by attributing it a rise in the value of the parameter b.[4]

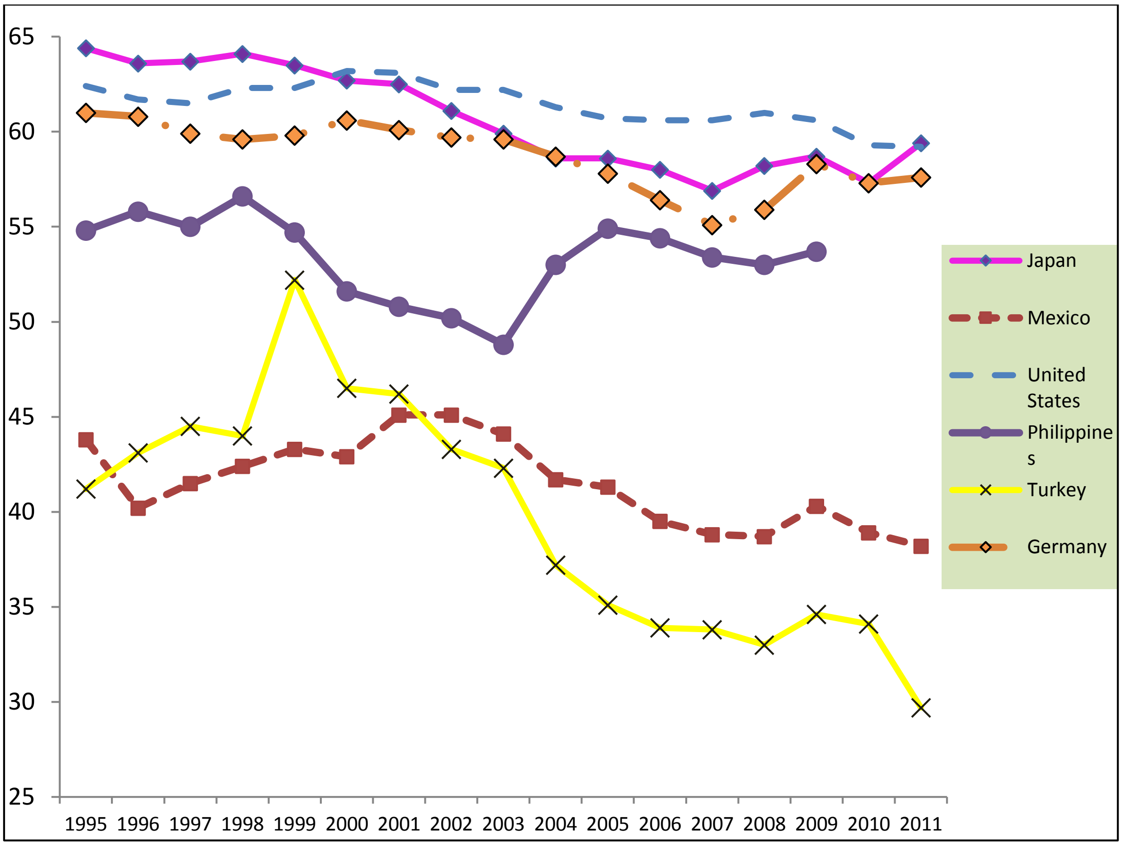

Figure 1.1: Movements in labour share in GDP (adjusted) for six countries (*Source:* ILOSTAT database)

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[2] For a current and comprehensive discussion about the declining trend in the global labour share, we direct the reader's attention to pages 67-74 of Karabarbounis and Neiman (2014).

[3] The findings of these recent studies are in contrast to the observations in Acemoglu (2003), who uses US and French data for the periods 1929-1997 and 1913-1997 and argues that, despite considerable movements, factor shares have remained fairly constant in these countries. The assumptions of a Cobb-Douglas production function and labour augmenting technical change are both consistent with Acemoglu's observations. However, the more recent evidence presented in the studies cited above motivates a further exploration of these issues.

[4] The literature referenced in the paragraph attributes the decline in labour share to technological change, increased globalization, developments in the financial sector, and the deterioration of labour market institutions and the welfare state. However, the assumption of a VES production function explains the falling share solely in terms of the capital biased technical change captured by a higher value of the parameter b.

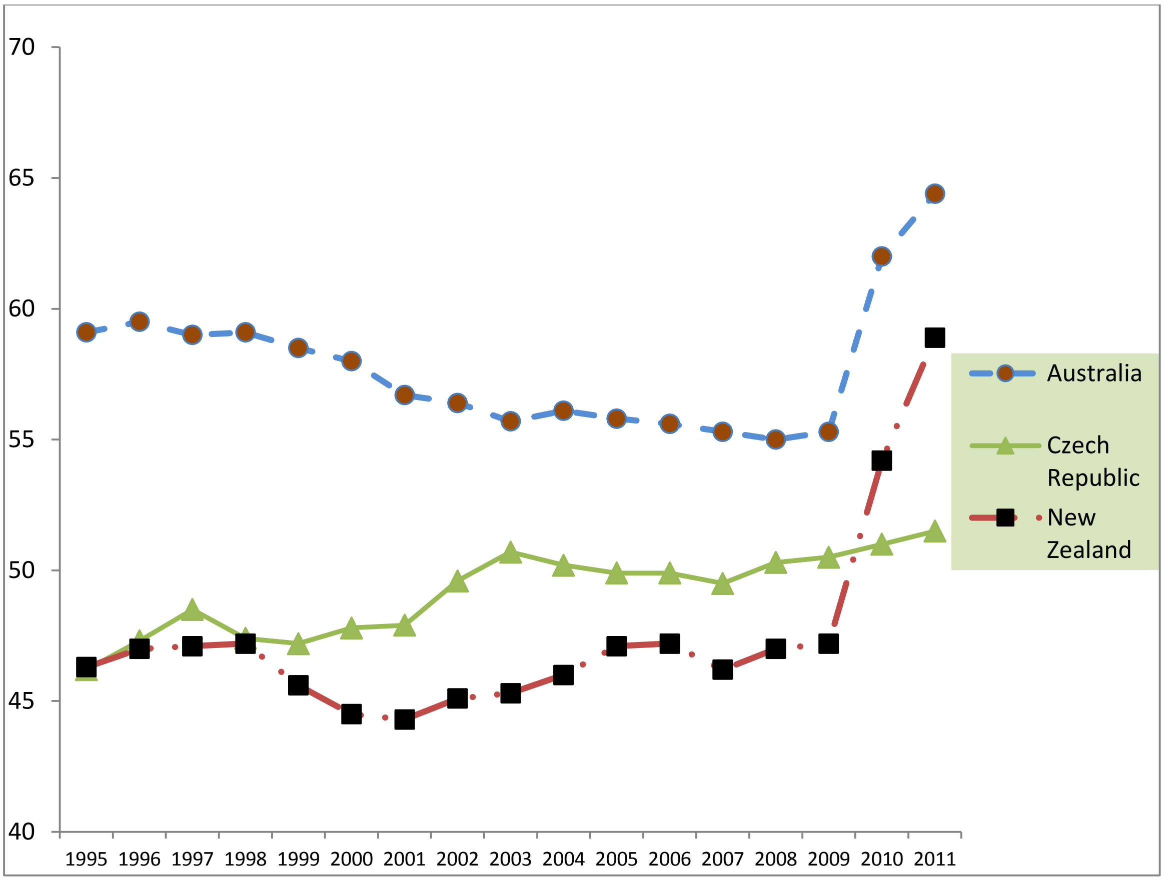

Figure 1.2: Movements in labour share in GDP for Australia, Czech Republic and New Zealand (*Source:* ILOSTAT database)

Upon discussing the properties of the VES production function, we incorporate it into an otherwise standard Diamond model. We are able to show analytically that as long as the technology parameter b is positive, which leads to an elasticity of substitution which is greater than 1, the economy always reaches a unique and stable steady state. When $b<0$ and the elasticity of substitution is between 0 and 1, the economy could either be characterized by no non-trivial steady state, or two steady states, of which one is stable and the other is not. Hence, under the assumption of a VES production function, the Diamond model could display three possible growth trajectories: monotonic convergence towards a stable steady state, the emergence of a poverty trap or unbounded growth [5] Therefore, in our model, conditional convergence towards a unique steady state like that observed in the context of a standard

[5] Miyagiwa and Papageorgiou (2003), who apply a normalized CES production function to an otherwise standard Diamond model, show a similar result. However, in their model, the dynamics are directly linked to the (constant) elasticity of substitution between capital and labour, and hence entails an interpretation that differs from our results. In our model, the dynamics are driven by the biased technological change created by the parameter b.

Solow-Swan model is not inevitable. This suggests that there is scope for the VES framework to be consistent with the observed diversity of growth experiences of different countries observed, among others, by Pritchett (2000).

We also prove that the steady state capital stock is falling in the technology parameter b, but that steady state output is positively related to it. Hence, a higher value of the parameter b is associated with greater productivity at steady state. This outcome demonstrates that although a higher value of b is associated with greater capital intensity, the equilibrium quantity of capital per worker used is determined by marginal products as well as market prices. Hence, capital-biased technical change does not necessarily imply that more capital will be used in equilibrium. In fact, in our model, capital biased technical change is manifested at steady state through the ability of the economy to produce a higher output with *less* capital.

In the presence of competitive factor markets, the reward for each factor is equal to its marginal product. Given that a higher value of the technology parameter b is associated with a higher marginal product of capital and a lower marginal product of labour, it leads to an exacerbation of inequality. This worsening of inequality is an artefact created by the heterogeneity intrinsic to the Diamond model, viz factor shares associated with capital and labour are owned by different agents - the old and the young respectively. The benefits of capital biased technical change generated through a higher value of b then accrue to the old, who are the owners of capital.[6]

Rather than merely exploring the link between the elasticity of factor substitution and economic growth which has been the central objective of many extant studies, our aim is to explore how the parameter b impacts on the optimal capital intensity in the economy, the rewards and relative contributions of factors, and the elasticity of substitution between labour and capital and thereby affects the economy's stability and productivity in the long run. Such

[6] According to Irmen and Klumpp (2009), a higher elasticity of substitution is likely to lead to higher intergenerational inequality in the case of the CES production function too, although the result can be more explicitly demonstrated in the context of a VES production function.

a shift in focus is necessitated by the fact that b generates technical change in the economy and the elasticity of factor substitution is an endogenous variable driven by this parameter. As such, our study is related to the idea of directed technical change put forward by Acemoglu (2002). The use of the VES further provides the advantage of greater tractability relative to CES, in addition to the flexibility implied by the endogeneity of the elasticity of substitution. Furthermore the explicit exploration of how the changes in factor substitutability stemming from biased technical change can impact on intergenerational inequality contributes to an aspect which has received sparse attention in the extant literature.

The rest of the paper is organized as follows: Section 2 provides a discussion of the VES production function and its properties, the model and analysis is provided in Section 3 and Section 4 concludes. A number of proofs and derivations are given in the Appendix.

2. The VES production function and its properties

The VES production function was introduced by Sato and Hoffman (1968) and Revankar (1971). We consider in this paper the more convenient form proposed by Karagiannis et al. (2005), which is expressed as:

$$Y = AK^{a\nu}\left(L + baK\right)^{(1-a)\nu} \qquad (2.1)$$

In (2.1) above, $\nu > 0$ is the returns to scale parameter and $A > 0$ captures Hick-neutral technological change. In the remainder of the paper, we assume that the production function displays constant returns to scale so that $\nu = 1$ and we also assume without loss of generality that $A = 1$. As we demonstrate shortly, the parameters a and b must conform to the ranges $0 \le a \le 1$ and $k \le 1/|b|$, so as to ensure that the VES form is characterized by the standard properties of a production function.

The VES form embeds three special cases. In equation (2.1), when $a = 1,$ the VES production function takes the Harrod-Domar fixed coefficients form, more commonly known as the AK form, as output is linearly related to the stock of capital. When $a = 0,$ as long as

$b > 0,$ it takes the perfect substitutes (linear) form. Finally, when $b = 0,$ the VES production function takes the Cobb Douglas form, and for this reason, it is regarded as a generalized form of the Cobb Douglas production function.[7]

The intensive form of the production function, where $y = Y/L$ and $k = K/L$, can be expressed as follows:

$$y = f(k) = k^a (1 + abk)^{1-a} \qquad (2.2)$$

In order to determine the parameter values for which the intensive form production function displays standard properties such as $f'(k) > 0$ and $f''(k) < 0$, we evaluate these derivatives below:

$$f'(k) = \frac{a(1+bk)}{k^{1-a}(1+abk)^a} \qquad (2.3)$$

and,

$$f''(k) = a(a-1)k^{a-2}(1+abk)^{-a-1} \qquad (2.4)$$

From (2.3) and (2.4) above, we can see that the first and the second order conditions are satisfied if and only if $0 \le a \le 1.$[8] Furthermore, for the first order condition to be satisfied when $b > 0,$ we need $k > -1/b$, which is always satisfied since the output in any given period must be real and positive. When $b < 0,$ the first order condition is satisfied if $k < 1/|b|$.[9]

[7]The fact that this particular form of the VES production function is a Cobb Douglas generalization adds to its analytical tractability, and is one of its most appealing properties (Bairam, 1989).

[8] Although the inequality signs should technically be strict in this instance, as explained below, we include equality in order to incorporate the special cases of perfect substitutes and the Harrod-Domar fixed co-efficients.

[9] Note that this range is different to that given in Karagiannis et al. (2005), who do not take into account the effect on the inequality created by the negative sign in front of $b.$ Revankar (1971), on the other hand gives the range correctly, although it is expressed differently from ours.

Note that (2.3) above is the marginal product of capital (MP_k). By differentiating (2.1) with respect to L, we get the marginal product of labour (MP_L) which is:

$$MP_L = \frac{\partial Y}{\partial L} = (1-a)\left(\frac{k}{1+abk}\right)^a \tag{2.5}$$

The value of b affects the marginal products of labour and capital differently. From equations (2.3) and (2.5) we have:

Lemma 1: A higher value of b is associated with a lower marginal product of labour but a higher marginal product of capital.

While it is clear from (2.5) that that MP_L is monotonically decreasing in b, we show in Appendix A that MP_k is always increasing in b. Since MP_k is rising in b and MP_L is falling in b, a higher value of b is associated with a steeper isoquant. As we will shortly demonstrate, this has important implications for biased technical change, as a higher value of the parameter b is associated with capital biased technical change. As we demonstrate in Section 3, Lemma 1 has important implications for inequality too. In a competitive market, the price of each factor will be equal to its marginal product. Thus, $MP_L = w$ and $MP_k = r$, where w and r are the wage and interest rates respectively. Therefore, a higher elasticity of substitution, caused by a higher value of b, is associated with a lower reward for labour and a higher reward for capital. As mentioned in the Introduction, this observation has important implications for intergenerational inequality in the context of a Diamond model, where the stock of capital is owned by the old generation while the young generations supply labour in return for a wage. This is a consideration we will return to in Section 3.

Given that the elasticity of substitution between capital and labour, which we denote with $\sigma,$ is given by the formula $\sigma = \frac{d\ln\left(K/L\right)}{d\ln\left(MP_K/MP_L\right)}$, the elasticity of substitution between capital and labour in the VES production function is:

$$\sigma = 1 + bk \tag{2.6}$$

From equation (2.6) above, we can see that to ensure that $\sigma \geq 0,$ the condition $k > -1/b$ must be satisfied when $b > 0;$ and when $b < 0,$ the required condition is $k < 1/|b|$. These conditions are analogous to those we identified earlier as necessary for a positive first derivative.

Note that the value of σ rises with the capital-to-labour ratio if $b > 0$ and falls if $b < 0.$[10] Therefore, in the VES production function, the value of σ changes along a given isoquant. Since the capital-labour ratio is constant along a particular ray from the origin drawn through a map of isoquants, this feature implies that in a map of isoquants drawn assuming a given value of $b,$ the elasticity of substitution will be identical only at points along a given ray from the origin. This feature is in contrast to the constant elasticity of substitution (CES) production function in which σ remains constant at all points on an isoquant map (Revankar, 1971). To illustrate these features, we present Figure 2.1 below, which plots the isoquants associated with different values of b when the value of the parameter a is 0.5, given an output of 10 units. From the intensive form of the VES production function given in equation (2.2) it is evident that output is monotonically increasing in the value of $b.$ Put differently, in terms of the isoquant, a higher value of b enables the firm to produce a particular level of output with less inputs, resulting in the isoquant shifting inwards as the value of b rises, as we can see in

[10] This strict monotonicity in the behaviour of σ with respect to capital intensity is considered to be a shortcoming of the VES production function (Bairam, 1989).

Figure 2.1.[11] The inward shift of the isoquant resulting from a higher elasticity of substitution is a characteristic the VES and the CES production functions share in common.[12]

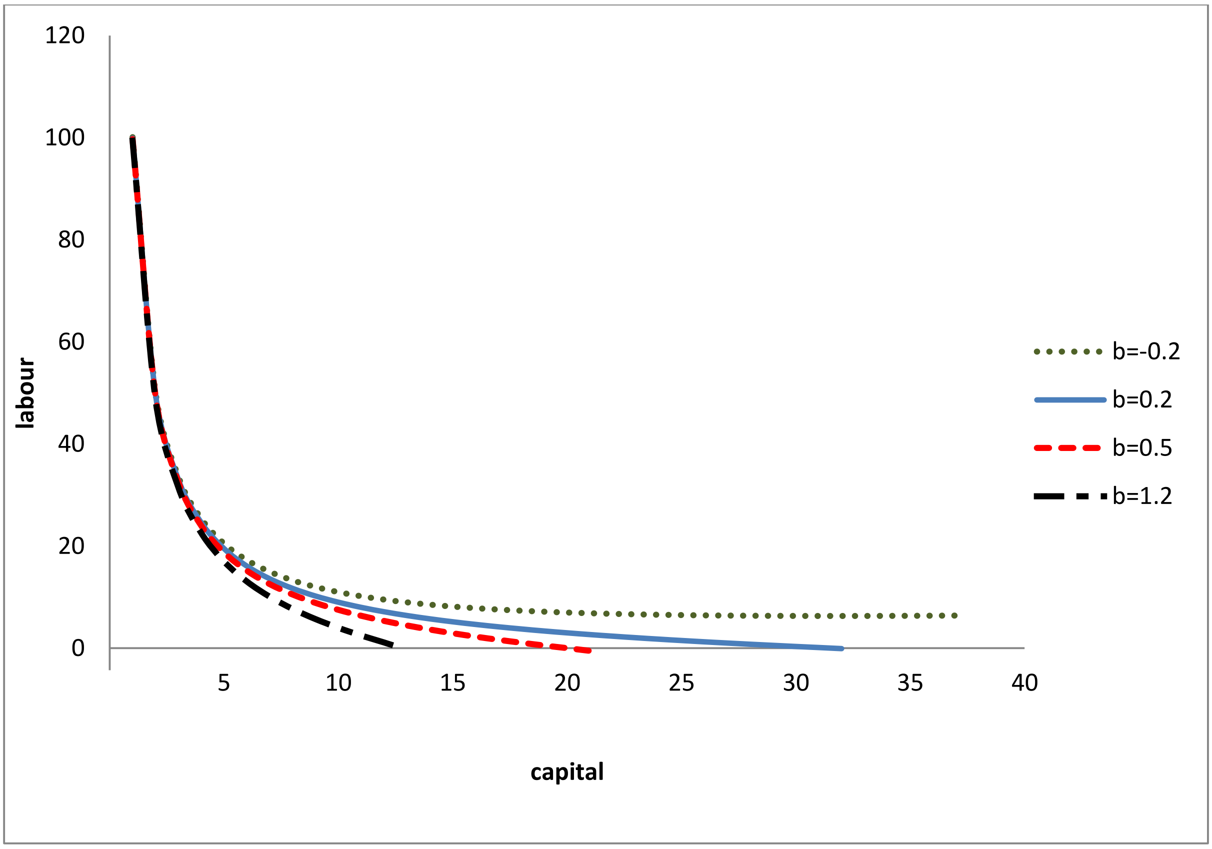

Figure 2.1: VES Isoquants associated with different values of b given $Y = 10$ and $a = 0.5$

Another interesting feature of the VES production function is that capital is the essential input in the production process, because when $K = 0$, $Y = 0$. On the other hand, when $b > 0$,

[11] In the CES literature, the technique of normalization, which was introduced by de La Grandville (1989), is usually employed to ensure that isoquants relating to different values of the elasticity of substitution share a common point of tangency, which allows one to analyse the pure effects of the elasticity of substitution upon growth. We do not carry out any normalization of the VES production function in this paper because our rather than a narrow focus on the dynamics created by the elasticity of substitution, our objective is to explore the dynamics of growth generated by the parameter b, which influences a number of features of the economy, of which the elasticity of substitution is only one.

[12] Extant theoretical studies utilizing the VES such as Karagiannis et al. (2005) and Brianzoni et al. (2012) do not use any normalization technique in their studies either. Note that the particular variant of the VES production function we consider is a Cobb-Douglas generalization. CES generalizations of the VES production function do exist (see, for instance, Kadiyala, 1972; Lu & Fletcher, 1968), although they have remained in relative obscurity in both the theoretical and empirical spheres, possibly due to their complicated nature. Normalizing such production functions is possible, and is a potentially fruitful avenue for future research in the area.

even when $L = 0,$ a positive level of output can be produced as long as a positive quantity of capital is employed in production. As is evident from Figure 2.1, this feature results in the isoquant having an intercept on the capital axis when $b > 0.$ As shown in Karagiannis et al. (2005), the limiting properties of the VES production function are as follows:

$$\lim_{k \to 0} f(k) = 0 \text{ for } b > 0 \tag{2.7}$$

$$\lim_{k \to -1/b} f(k) = \frac{(1-a)^{1-a}}{(-b)^{a}} \text{ for } b < 0 \tag{2.8}$$

Hence, the per capita capital stock in any given period is strictly bounded from below when $b < 0.$ The essential input feature of the VES production function causes one of the Inada conditions to be violated, as it means that even when labour approaches zero, the economy can produce a positive level of output. However, the assumption that capital is the essential input does not have any impact upon the key feature of the VES production function, which is its flexibility which allows us explore the implications associated with different values of b in relation to the stylized facts of growth discussed in the Introduction.[13]

We have already seen that the parameter b is instrumental in determining the elasticity of substitution between capital and labour in the VES production function. As mentioned in the Introduction, in contrast to the CES literature, where the discussions are generally centred around the elasticity of substitution $\sigma,$ in this study, we shift our attention to $b.$ In order to understand the mechanism underlying this influence the parameter b exerts on the value of the elasticity of substitution, we need to explore the manner in which b affects the direction of technical change in the economy. For this purpose, we present the shares of labour and capital, S_L and S_K, below:

$$S_L = \frac{wL}{Y} = \frac{w}{y} = \frac{(1-a)}{1+abk} \quad S_L = \frac{wL}{Y} = \frac{w}{y} = \frac{(1-a)}{1+abk} \tag{2.9}$$

[13] We have also explored the dynamics of growth associated with an alternative form of the VES production function where labour is the essential input. These results are available upon request.

$$S_K = \frac{rK}{Y} = \frac{rk}{y} = \frac{a + abk}{1 + abk} \tag{2.10}$$

It is evident from equation (2.9) that a higher value of the technology parameter b causes the share of labour to fall. On the other hand, from equation (2.10), the derivative of the capital share with respect to the parameter b is given by $\frac{ak(1-a)}{(1+abk)^2}$, which is always positive. Recall from the introduction that recent growth literature suggests a global decline in the share of labour in total output. In terms of the VES production function, then, such a decline in the share of labour, and a concurrent increase in the share of capital, can be attributed to an exogenous rise in the value of the technology parameter b.

After having looked at how factor shares are affected by b, we next note that capital intensity is not fixed; rather, it is optimally chosen by a firm or economy conditional on input prices and the production plan. To look at how the value of b influences capital intensity, we obtain the economy's long run expansion path by minimizing the cost function associated with producing a certain level of output Y. Thus, for given factor prices w and r, we minimize total cost:

$$C = wL + rK \tag{2.11}$$

Subject to the constraint:

$$Y = K^a (L + abK)^{1-a} \tag{2.12}$$

Solving this problem yields the following long run expansion path:

$$\frac{K}{L} = \frac{\frac{w}{r}}{\frac{1-a}{a} - \frac{w}{r} b} \tag{2.13}$$

We can see from (2.13) that a higher value of the parameter b is associated with a higher capital to labour ratio and thereby a steeper long run expansion path. In other words, a higher elasticity of substitution yields greater capital intensity in production. This result can be expected because we saw earlier that a higher value of b causes MP_K to rise and MP_L to fall. Hence, a higher value of b naturally encourages firms to substitute labour with capital, which is the relatively more productive input. Furthermore, as we noted before, capital is the essential input in the production process. Therefore, we observe that as the value of b rises, firms prefer to raise the quantity of the essential input needed in production.

We have seen that the optimal capital-to-labour ratio/capital intensity in this economy is endogenous, as it is determined by the value of our technology parameter b. Therefore, we can observe that the parameter b influences the value of the elasticity of substitution in two ways: directly, by appearing on the right hand side of equation (2.6) above, as well as indirectly, by acting as a determinant of the capital per worker/capital intensity k. Hence, a higher value of b, while leading to a higher elasticity of substitution in its own right, also exerts a secondary incremental effect on the elasticity of substitution by contributing towards greater capital intensity.

It is also worthwhile commenting on the fact that higher value of the parameter a is also associated with a higher capital to labour ratio. Given this observation, it is apt to supply an interpretation to the parameter a at this juncture of our discussion. As it is the exponent of k in the intensive form production function given by equation (2.2), and also because an increase in its value leads to greater capital intensification, we interpret it as the "pure" share of capital in the production function.

Acemoglu (2002) suggests that under the assumption of a CES production function, even in the presence of an abundant supply of capital, as long as the elasticity of factor substitution is above a threshold value which falls between 1 and 2, the rewards to capital would rise, leading to capital-biased technical change. To explore what connotations the VES form holds for the direction of technical change, we define the factor bias as per Acemoglu (2002), who defines it as the ratio of marginal products. Hence, for the VES form, the capital bias, which we will refer to as γ is given by:

$$\gamma = \frac{MP_K}{MP_L} = \frac{a}{(1-a)}\left(\frac{1}{k} + b\right) \qquad (2.14)$$

Hence, it is clear that γ is falling in capital intensity k, as would be expected due to the substitution effect, which yields a negatively sloped relative demand curve for capital (Acemoglu, 2002), but it is rising in the parameter b. Hence, equation (2.14) demonstrates that the parameter b leads to capital biased technical change, which can be interpreted as a situation where an exogenous increase in the value of b enables an economy to produce more output with the same amount of capital. This has some interesting and profound implications upon the long run behaviour of the economy, which we explore in the next section, b where we present our model and discuss its dynamics.

Many extant studies assume that the elasticity of substitution is an exogenous determinant of the technical change in the economy (Mallick, 2012), and only a few studies such as Miyagiwa and Papageorgiou (2007) and R. W. Jones (2008) explore the issue of endogeneity of the elasticity of factor substitution. The VES production function connects with the latter view of endogeneity as the elasticity of substitution is a *result* of the directed technical change created by the value of the parameter b. Further to this point, it often concurred that distinguishing between the impacts associated with the elasticity of substitution and those resulting from non-neutral technical change may be difficult, particularly when the elasticity of substitution is different from 1 (see León-Ledesma, McAdam, & Willman, 2010 and references therein). However, assuming that the economy is characterized by a VES production function removes this fuzziness by presenting a clear cause and effect relationship between non-neutral/biased technical change and the elasticity of substitution- capital biased technical change emanating from a higher value of b leads to a rise in the elasticity of substitution.

The connection between capital deepening, the elasticity of substitution, and the share of capital created by the parameter b is of empirical relevance. For instance, Rodrik (1997) observes that East Asia experienced capital deepening and a simultaneous rise of the share of capital in total production over the decades of fast economic growth the region experienced, and points out that the growth literature often attributes these effects to either the higher elasticity of substitution between labour and capital or labour saving technical progress.

However, such observations are typically made under the assumption that the underlying production function takes the CES form, which, as mentioned in the Introduction, cannot capture the endogeneity and variability of the elasticity of substitution highlighted in many studies. If, according to authors like Kazi (1980) and Bairam (1989), the VES form is indeed the correct functional specification of the production function, the contention of the CES literature- that capital deepening and a higher capital share are consequences of a higher elasticity of substitution- may be spurious and should therefore be viewed with caution. The alternative interpretation provided by the VES production function is that capital deepening, a higher share of capital in output as well as a higher value of the elasticity of substitution are *all* the results of a higher value of the parameter b.

Having seen that b, which we hitherto referred to simply as 'a technology parameter' is a crucial determinant of many features of the economy, our discussion is incomplete without inquiring into what this parameter might represent. Essentially, it is a catchall parameter which encapsulates a number of factors that might affect the direction of technical change in the economy. Mallick (2012, p. 685) provides a comprehensive treatment of the determinants of the elasticity of substitution between capital and labour in the context of a CES production function, noting that "…the possible determinants of σ are technological progress, innovations, financial and other institutions, openness to trade, degree of unionization, and the country's inclination towards socialist ideas." While the factors that that determine the value of σ in the CES production function are likely to be analogous to the characteristics the parameter b represents, the fact that the parameter b affects factor shares, factor intensity and the direction of technical change warrants some further discussion on it.

C. I. Jones (2005) notes that a country can achieve a high elasticity of substitution only if it has access to the appropriate technology to support such a choice. Hence, a higher value of b primarily captures a technology which enables greater substitution between capital and labour. However, as a higher value of b leads to capital biased technical change, government policy relating to investment is likely to be an important determinant of the value of b. Government policy could sometimes distort the direction of investment, creating an artificial bias towards one factor. For example, Yuhn (1991) points out that the South Korean government's tax policy led to massive investments in capital, which raised the capital per worker. In the context of the VES production function, this would lead to a higher elasticity of substitution. Typically, the ideals of the ruling party may affect the value of b and thereby the direction of technical change in the economy. Usually, conservative political parties might encourage capital-biased technical change by encouraging investments that raise the relative price of capital while more liberal political parties that favour equity will not advocate a decline in the labour share and would thereby discourage capital-biased technical change. Moreover, Macpherson (1990) shows that the size and density of trade unions is positively related to the share of labour. Hence, trade unions could affect the value of b adversely, as their activities could exert pressure on governments and employers to maintain the share of labour at an artificially high level, thereby preventing a decline in wages created by a falling share and marginal product of labour.

Having discussed the properties of the VES production function, and explored the role and importance of the parameter b in determining a number of key features of the production side of the economy, we proceed to apply the VES production function to a Diamond model in the next section.

3. The model

We consider a standard two period overlapping generations model where the agent works during youth and spends her old age in retirement. Population in each period is normalized to 1. Time is discrete and is given by $t = 1,2....$ The utility of an agent born at time t is given by:

$$U_t = u(c_t) + \beta u(c_{t+1}), \qquad 0 < \beta < 1 \tag{3.1}$$

In (3.1) above, the consumer derives utility from youthful consumption c_t and old age consumption c_{t+1}. The parameter β is the discount factor.

In youth the agent supplies one unit of labour in youth and earns a wage w_t which she utilizes on consumption c_t and savings s_t. The agent finances old age consumption c_{t+1} with her savings which accumulate a gross interest of R_{t+1}. She faces the following budget constraints in youth and old age respectively:

$$c_t + s_t = w_t \tag{3.2}$$

$$c_{t+1} = R_{t+1} s_t \tag{3.3}$$

The agent's utility maximization problem yields the following standard FOC:

$$u^{'}(c_t) = \beta R_{t+1} u^{'}(c_{t+1}) \tag{3.4}$$

If we assume log utility such that $u(c_i) = \ln(c_i)$, for $i = t, t+1$; we get the following optimal solutions for consumption in both periods and savings:

$$c_t = \frac{w_t}{1+\beta} \tag{3.5}$$

$$c_{t+1} = \frac{\beta R_{t+1} w_t}{1+\beta} \tag{3.6}$$

$$s_t = \frac{\beta w_t}{1+\beta} \tag{3.7}$$

The intensive form production function for this economy is given by:

$$y_t = k_t^a (1 + abk_t)^{1-a} \tag{3.8}$$

During youth the agent supplies a single unit of labour within a competitive labour market and receives a wage w_t such that:

$$w_t = f(k_t) - k_t f^{'}(k_t) = \frac{(1-a) y_t}{1 + abk_t} \tag{3.9}$$

The gross rate of return on capital is given by:

$$R_{t+1} = 1 + f^{'}(k_{t+1}) = 1 + \frac{a y_t}{k_t} + \frac{(1-a) aby_t}{1 + abk_t} \tag{3.10}$$

Using the fact that $k_{t+1} = s_t$, we can now derive the evolution of the per capita capital stock for this economy which is given by:

$$k_{t+1} = \frac{\beta(1-a)}{(1+\beta)}\left(\frac{k_t}{1+abk_t}\right)^a \tag{3.11}$$

At steady state, $k_{t+1} = k_t = \bar{k}.$ Therefore, the steady state capital stock is defined by the following implicit equation:

$$\bar{k}^{1-a}\left(1+ab\bar{k}\right)^a - \frac{\beta(1-a)}{(1+\beta)} = 0 \tag{3.12}$$

By carrying out stability analysis by linearizing (3.11) and analysing (3.12), details of which are provided in Appendix B, we have:

Proposition 1: When $b > 0,$ the steady state is always unique and stable.
When $b < 0,$ there are two possibilities: there is either no non-trivial steady state; or there could be two steady states, of which one is stable and the other is not. In the latter case, when $a > 1/2$, at least one of these steady states satisfies the feasibility condition $\sigma > 0.$ When $a < 1/2$, one, or both of these steady states may violate this condition.

Proposition 1 is similar to the result obtained by Miyagiwa and Papageorgiou (2003), who demonstrate in the context of a standard two period Diamond model with a normalized CES production function, that for $\sigma > 1,$ there is a unique steady state capital stock while for $\sigma < 1$ there can be two or no steady states. Like in Miyagiwa and Papageorgiou (2003), in this case, a positive value of $b,$ which is associated with $\sigma > 1$ is always associated with a unique and stable steady state.

On the other hand when $b < 0,$ if a non-trivial steady state does not exist, it implies that the economy converges to the trivial steady state at $\bar{k} = 0,$ which implies that the economy falls into a poverty trap in the long run. Hence a negative value of $b,$ which leads to $\sigma < 1,$ could have a detrimental impact on the economy by causing it to regress towards the trivial steady state. Furthermore, when $b < 0,$ if there are two steady states, the feasibility of each steady state - i.e. whether it leads to a steady state elasticity of substitution which satisfies the condition

$\sigma > 0$ - crucially depends on the value of the parameter $a,$ which we interpreted earlier as the pure share of capital in the production function. In this instance, a pure share parameter which is above ½ guarantees the economy's ability to reach at least one feasible steady state. Hence, the value of a is also a critical determinant of the existence of a steady state when $b < 0.$ Even if two feasible steady states do indeed exist, we demonstrate in Appendix B that only one of them is stable. Thus, if the economy starts off with an initial capital stock which is above the value of the capital stock associated with the unstable steady state, it can achieve unbounded growth. So, in summary, a negative value of b presents three possible trajectories for the economy: it could fall into a poverty trap, converge towards a stable steady state or achieve unbounded growth. Empirical evidence suggests that countries do not always converge towards a unique and stable steady state as implied by the Solow growth model. Instead, as Pritchett (2000)asserts, in practice countries could experience poverty traps or unbounded growth. Hence, the VES production function can, depending on the value of $b,$ capture a number of empirically observed growth trajectories

In relation to extant literature, our results are different from those of Klump and Preissler (2000), who demonstrate that when the CES production function is applied to a Solow-Swan model, the resulting steady state is always unique and stable. Furthermore, our results also differ from the findings of Karagiannis et al. (2005) who show that when the VES production function is applied to a standard Solow-Swan model, the economy could display unbounded endogenous growth for the case $b > 0$ if the condition $sA(ba)^{1-a} > n$ is satisfied, where s is the savings rate, A is the technological parameter and n is the rate of population growth. These authors note that when the elasticity of factor substitution is greater than 1, and hence the marginal product of capital is bounded from below, it is possible for the economy to display unbounded endogenous growth. On the other hand, when $b < 0,$ Karagiannis et al. (2005) show that the economy could either reach a unique steady state when $sA(ba)^{1-a} < n$ or reach a corner solution where $k = -1/b$ when $sA(ba)^{1-a} > n.$[14] A possible reason for the

[14] An observation which was overlooked by Karagiannis et al. (2005) is that when $b < 0,$ the quantity $sA(ba)^{1-a}$ takes a complex value, and the economy may therefore be incapable of achieving a steady state at all. An observation which was overlooked by Karagiannis et al. (2005) is that when $b < 0,$ the quantity $sA(ba)^{1-a}$ takes a complex value, and the economy may therefore be incapable of achieving a steady state at all.

differences in the outcomes may emerge from the use of different frameworks. While in the Solow-Swan model, a fixed proportion of output per capita is saved, in the Diamond model with logarithmic preferences, the agent saves a fixed proportion of wage income. Although labour is the non-essential input in the VES production function, within an overlapping generations framework with logarithmic preferences, the accumulation of capital -which is the essential input- entirely depends on the wage received by the agent. Hence, these dynamics could introduce instability into the system.

Next we present another proposition, the proof of which is supplied in Appendix C:

Proposition 2: For any stable steady state, the steady state per capita capital stock $\bar{k}$ is falling in the value of the parameter b. However, for any feasible steady state, regardless of whether it is stable or not, steady state output $\bar{y}$ is rising in b.

Proposition 2 shows us that there is a pronounced benefit accruing to an economy due to a higher value of b, as it enables the economy to achieve a higher output per worker at steady state with a lower capital stock. This result ties up with, and in fact accentuates, the capital biased technical change created by a higher value of b, which was discussed in detail in Section 2. While capital biased technical change would enable the economy to produce more output with the same amount of capital as before, at steady state, the capital bias caused by a higher value of the parameter b actually allows the firm to produce more output using *less* capital per worker. Proposition 2 is similar in spirit to Klump and De La Grandville (2000), who demonstrate that under the assumption of a CES production function, a higher elasticity of substitution enables the economy to produce a higher level of output per capita. This result also supports the idea put forward by Kazi (1980) who argues that a low elasticity of substitution might be a reason for slow growth and development in many developing economies. However, as discussed in the Introduction, the elasticity of factor substitution can vary between industries belonging to a particular industry too. Given such heterogeneity within a particular economy, Kazi (1980) suggests that the economy's productivity will rise if investments are geared towards industries with a higher value of factor substitutability.

However, it is worthwhile noting the contradiction Proposition 2 creates, because, in Section 2, we demonstrated that a higher value of the factor substitutability parameter results in greater capital intensification, while at the same time, it results in an inward shift of the isoquant, implying that an economy characterized by higher value b could produce a given level of output with less capital and labour, thereby enjoying greater productivity. The former effect can be referred to as the intensification effect while the latter can be termed the productivity effect. At steady state, the fact that a higher value of the technology parameter b results in the economy producing a higher output with a lower steady state capital stock implies that the productivity effect dominates. Indeed, the productivity effect is so strong at steady state that the economy can actually produce more output with a lower capital stock per capita.

Another interesting observation in our study relates to intergenerational inequality. As we saw in Section 2, in the VES production function, MP_K is increasing in the factor substitutability parameter while MP_L is falling in the factor substitutability parameter. In the presence of competitive factor markets, factor rewards are $r = MP_K$ and $w = MP_L$ respectively. This has important implications for inter-generational inequality in the context of a standard Diamond model, where the young agents supply labour in return for a wage and the old agents are the owners of capital. Essentially, this feature results in old agents being persistently richer than the young agents. In fact, from Lemma 1 and equations (3.5) and (3.6) we can easily show the following:

Proposition 3: Intergenerational inequality is rising in the value of b.

Hence, in this economy, the higher output per capita comes at the cost of higher inequality. In fact Proposition 3 provides a potential explanation for the possible divergence in inequality created by increases in the returns to capital suggested by Piketty and Goldhammer (2014). In the context of our model, the capital biased technical change created by a higher value of b leads to a redistribution of relative factor rewards in favour of the owners of capital, thereby exacerbating inequality. Delving into the mechanism behind this outcome, generally, a change in the elasticity of substitution leads to three effects: an efficiency effect, whereby a higher elasticity of substitution raises the productivity of factor inputs, resulting in a higher output; a distribution effect, which occurs because a change in the elasticity of substitution causes a redistribution of factor incomes; and finally an acceleration effect, which captures the manner in which a change in the elasticity of substitution affects capital intensity which is a consequence of the efficiency and distribution effects (Irmen & Klump, 2009). In the case of the Diamond model, Irmen and Klump (2009) note with reference to the discrete time Diamond model with CES technology by Miyagiwa and Papageorgiou (2003) and the continuous time version by Irmen (2003), that there could be a negative acceleration effect resulting from the negative distribution effect dominating the positive efficiency effect. In our Diamond-type economy with VES technology too, we see that there is always a negative distribution effect and a positive efficiency effect. As the capital stock per worker is falling in the parameter b, and thereby the elasticity of substitution, we can infer that a negative acceleration effect is also present.

4. Conclusion

In this paper, we incorporate a VES production function into an otherwise standard Diamond model. We explore how the value of an exogenous parameter, labelled b, which affects the direction of technical change in the economy and is a determinant of the elasticity of substitution between labour and capital, affects the dynamics of growth. We observe that for positive values of the parameter b, which are associated with an elasticity of substitution between labour and capital that is greater than 1, the economy can always reach a unique and stable and steady state which is feasible with the condition that the associated elasticity of substitution is positive. As such, when the value of b is positive, the behaviour of the economy, both during transition and at steady state, is identical to that associated with the standard Solow growth model. On the other hand, for negative values of b, for which the elasticity of substitution is between 0 and 1, we show that the economy could either fall into a poverty trap where the capital stock per worker is zero, or there could be two steady states of which one is stable and the other is not, implying that the economy might be able to achieve unbounded growth. These different growth paths that emerge capture a number of scenarios which are consistent with cross-country growth experiences documented in the extant economic growth literature.

We also observe that a higher elasticity of substitution enables the economy to produce a higher level of output at steady state with a lower per capita capital stock. These results lead to the conclusion that a positive value of the technology parameter b yields multiple benefits upon the economy by ensuring a stable and efficient steady state. However, a higher value of the technology parameter b entails an efficiency-equity trade-off, as intergenerational inequality is rising in the value of this parameter. Hence, whether the goal of raising the value of the technology parameter b should be pursued by policymakers depends on how they weigh the benefits accruing from a high elasticity of substitution in the form of efficiency and stability against the loss inflicted upon the economy due to the exacerbation of inequality.

A potential direction for future research would be obtaining cross country estimates for the value of b using non-linear regression techniques. Such estimates for individual countries could provide new cross-country estimates for the extent of capital-biased technical change experienced by countries as well as estimates for the values of the elasticity of substitution for different countries, and supply rich insights into the heterogeneity in factor shares and rewards observed in practice. Some of the potential theoretical offshoots of this study include considering an alternative functional form of the VES production function in which labour is the essential input, and exploring the possibility of the political economy determination of an 'optimal' value of b.

Appendices

Appendix A: Proof of Lemma 1

From (2.3), the derivative of MP_K with respect to b is given by:

$$\frac{\partial MP_K}{\partial b} = \frac{ak^a(1-a)(1+a+abk)}{(1+abk)^{a+1}} \tag{A1}$$

From (A1) above, we can see that the interest rate is increasing in b when $b > 0$.

When $b < 0$, the interest rate is increasing in b if $k < \frac{(1+a)}{a|b|}$. This condition is always satisfied in the range $k < \frac{1}{|b|}$ which we established earlier.

From (2.5), it is clear that a higher elasticity of substitution, captured by a higher value of the parameter b, yields a lower MP_L and thereby a lower wage rate.

Appendix B: Proof of Proposition 1

First, we derive the condition for stability of a given steady state.

We linearize equation (3.11) around the steady state. The linearized system can be approximated by:

$$k_{t+1} = f(\bar{k}) + f'(\bar{k})(k_t - \bar{k}) \quad \text{(B1)}$$

From (3.11), we have:

$$f'(\bar{k}) = \frac{\beta a(1-a)\bar{k}^{a-1}}{(1+\beta)(1+ab\bar{k})^{(1+a)}} \quad \text{(B2)}$$

Therefore, the linearized system can then be written as:

$$k_{t+1} = \frac{\beta a(1-a)\bar{k}^{a-1}}{(1+\beta)(1+ab\bar{k})^{(1+a)}}\left[(1+ab\bar{k})\bar{k} - a\right] + \frac{\beta a(1-a)\bar{k}^{a-1}}{(1+\beta)(1+ab\bar{k})^{(1+a)}} k_t \quad \text{(B3)}$$

For the steady state capital stock to be locally stable, the following condition must be satisfied:

$$\frac{\beta a(1-a)\bar{k}^{a-1}}{(1+\beta)(1+ab\bar{k})^{(1+a)}} < 1 \quad \text{(B4)}$$

By rewriting (3.12) as:

$$\bar{k}^{1-a}(1+ab\bar{k})^{a} = \frac{\beta(1-a)}{(1+\beta)} \quad \text{(B5)}$$

The local stability condition simplifies to:

$$1 + ab\bar{k} > a \quad \text{(B6)}$$

When $b > 0$, (B6) simplifies to:

$$\bar{k} > \frac{-(1-a)}{ab} \quad \text{(B7)}$$

The inequality given by (B7) is always satisfied because $\bar{k} \geq 0$.

When $b < 0$, we get:

$$\bar{k} < \frac{(1-a)}{a|b|} \quad \text{(B8)}$$

To find out the number of steady states associated with different parameter values, from (B5), let:

$$M = \bar{k}^{1-a}\left(1 + ab\bar{k}\right)^{a} = \frac{\beta(1-a)}{(1+\beta)}$$

The first and the second derivatives of M are given by:

$$\frac{dM}{d\bar{k}} = \frac{1-a+ab\bar{k}}{\bar{k}^{a}\left(1+ab\bar{k}\right)^{1-a}} \quad \text{(B9)}$$

$$\frac{d^2M}{d\bar{k}^2} = \frac{-a(1-a)}{\bar{k}^{a+1}\left(1+ab\bar{k}\right)^{3a+2}} \quad \text{(B10)}$$

When $b > 0$, (B9) is always positive, while for $b < 0$, the function M has a stationary point when $\bar{k} = \frac{1-a}{a|b|}$. Regardless of the value of b, the second derivative given by (B10) always positive. So, when $b > 0$, the positive first and negative second derivative implies that M is increasing and concave in $\bar{k}$, and as depicted in Figure B1, the steady state occurs at point A.

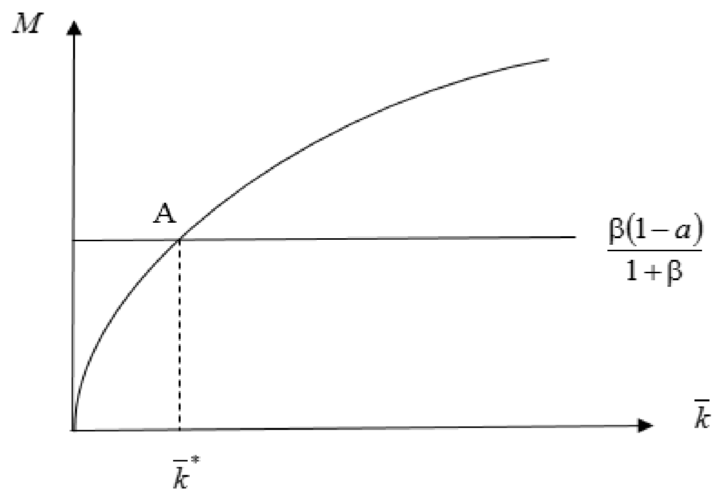

Figure B1: Depiction of the unique steady state when $b > 0$

When $b<0$, the negative second derivative implies that the stationary point is a maximum. In this instance, if $M_{max}=\left[\frac{1-a}{a|b|}\right]^{a} a^{1-a}<\frac{\beta(1-a)}{1+\beta}$, there will be no steady state. Otherwise, there are two steady states, one to the left of the stationary point and one to the right. These steady state capital stocks, denoted by $\bar{k}_1^*$ and $\bar{k}_2^*$ are depicted graphically in Figure B2 below:

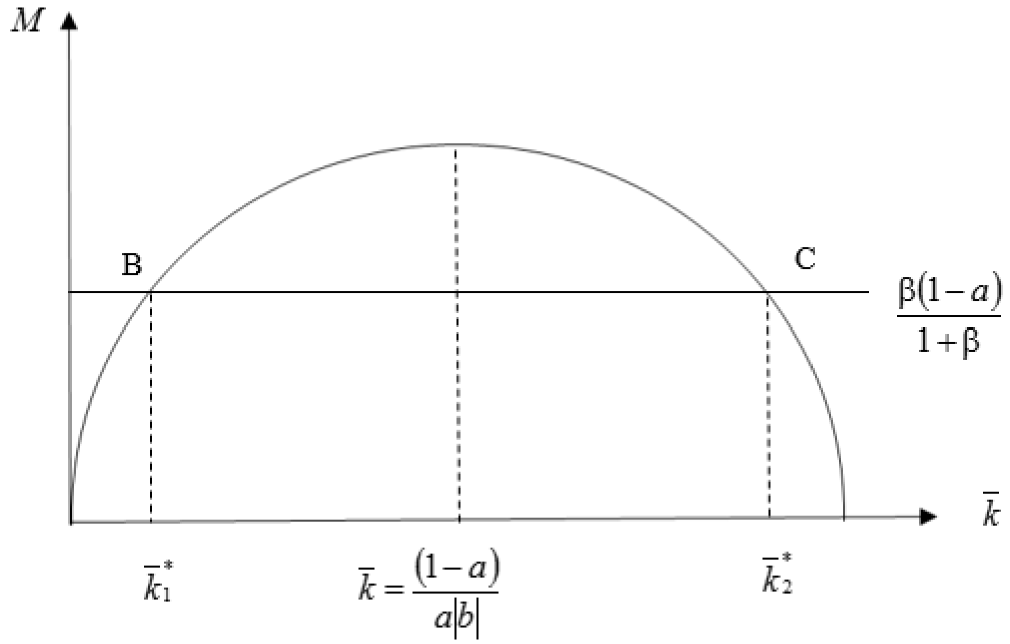

Figure B2: Depiction of the two possible steady states when $b<0$

Recall that the condition necessary for stability of the steady state when $b<0$ was given by $\bar{k}<\frac{(1-a)}{a|b|}$. So, evidently, $\bar{k}_1^*$ is a stable steady state while $\bar{k}_2^*$ is not.

A further requirement is that the steady state elasticity of substitution must satisfy the condition $\bar{\sigma}=1+b\bar{k}>0$. When $b>0$, this condition is always satisfied. When $b<0$, the condition simplifies to: $\bar{k}<1/|b|$. Accordingly, a sufficient condition for $\bar{k}_1^*$ to be a steady state is: $\frac{(1-a)}{a|b|}<\frac{1}{|b|}\Rightarrow a>\frac{1}{2}$. When $a<\frac{1}{2}$, if $\bar{k}_1^*$ falls in the range $\frac{1}{|b|}<\bar{k}_1^*<\frac{(1-a)}{a|b|}$, it is not a feasible steady state. Similarly, when $\frac{(1-a)}{a|b|}>\frac{1}{|b|}\Rightarrow a<\frac{1}{2}$, $\bar{k}_2^*$ can never be a steady state.

When $a > \frac{1}{2}$, $\bar{k}_2^*$ is a feasible steady state only if $\frac{(1-a)}{a|b|} < \bar{k}_2^* < \frac{1}{|b|}$.

Appendix C: Proof of Proposition 2

Recall from equation (3.12) that the steady state is given by the following implicit function:

$$F(\bar{k}, b) = \bar{k}^{1-a}(1 + ab\bar{k})^a - \frac{\beta(1-a)}{(1+\beta)} = 0 \tag{C1}$$

From (C1) above we have

$$F_{\bar{k}} = (1-a)\bar{k}^{-a}(1 + ab\bar{k})^a + a^2 b\bar{k}^{1-a}(1 + ab\bar{k})^{a-1} = \left(\frac{1 + ab\bar{k}}{\bar{k}}\right)^{a-1}\left(\frac{1 - a + ab\bar{k}}{\bar{k}}\right) \tag{C2}$$

$$F_b = \left(\frac{1 + ab\bar{k}}{\bar{k}}\right)^{a-1} a^2 \bar{k} \tag{C3}$$

From the implicit differentiation theorem, we have:

$$\frac{\partial \bar{k}}{\partial b} = -\frac{F_b}{F_{\bar{k}}} = \frac{-a^2\bar{k}^2}{(1 - a + ab\bar{k})} \tag{C4}$$

It is clear that, when $b > 0$, $\frac{\partial \bar{k}}{\partial b} < 0$. Recall that when $b > 0$, the unique steady state that emerges is always stable.

When $b < 0$, $\frac{\partial \bar{k}}{\partial b} < 0$ if $\bar{k} < \frac{(1-a)}{a|b|}$. From Proposition 1, this upper bound on $\bar{k}$ is also the condition necessary for a stable steady state. So when $b < 0$, for any stable steady state, there is a negative relationship between $\bar{k}$ and b.

So, when $b < 0$, the conditions under which $\frac{\partial \bar{k}}{\partial b} < 0$ are analogous to the conditions under which the steady state which emerges when $b < 0$ is locally stable. Hence, when $b < 0$, any locally stable steady state capital stock is decreasing in the parameter b.

At steady state,

$$\bar{y} = \bar{k}^a \left(1 + ab\bar{k}\right)^{1-a} \tag{C5}$$

$$\frac{d\bar{y}}{db} = a\left(\frac{1+ab\bar{k}}{\bar{k}}\right)^{1-a}\frac{d\bar{k}}{db} + (1-a)\left(\frac{1+ab\bar{k}}{\bar{k}}\right)^{-a}\left(a\bar{k} + ab\frac{d\bar{k}}{db}\right) = \left(\frac{1+ab\bar{k}}{\bar{k}}\right)^{-a}\left[a\frac{d\bar{k}}{db}\left(\frac{1+b\bar{k}}{\bar{k}}\right) + (1-a)a\bar{k}\right] \tag{C6}$$

The condition necessary for $\frac{d\bar{y}}{db} > 0$ is therefore

$$a\frac{d\bar{k}}{db}\left(\frac{1+b\bar{k}}{\bar{k}}\right) + (1-a)a\bar{k} > 0 \tag{C7}$$

Upon substituting for $\frac{d\bar{k}}{db}$ and simplifying we get:

$$ab(1-2a)\bar{k} > 2a - 1 \tag{C8}$$

When $b > 0$, this yields:

$\bar{k} > \frac{-1}{ab}$, which is always satisfied because $\bar{k} > 0$.

When $b < 0$, we get:

$\bar{k}\frac{1}{a|b|}$, which is always satisfied for any feasible steady state.

Hence, for any feasible steady state $\frac{d\bar{y}}{db} > 0$.

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