Operations Research

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A Note on the Group Theoretic Approach to Integer Programming and the 0-1 Case

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(Received June 24, 1971)

While the group-theoretic approach is in many ways useful for 0-1 problems too, in this latter case the sufficient condition given by Gomory for a solution to the group problem to solve the initial integer program is never satisfied. This underscores the need to use information about the constraints that are slack at the linear programming optimum.

GIVEN THE integer program

$$\max\{cx|Ax=b,\ x\geqq 0 \text{ integer}\}, \qquad (P)$$

the associated linear program $(\bar{P})$, and an optimal solution $\bar{x}$ to $(\bar{P})$, with basis B, Gomory has shown,[1,2] that the problem

$$\max\{cx|Bx_B+Rx_R=b;\ x_R\geqq 0;\ x_B,\ x_R \text{ integer}\}, \qquad (P')$$

obtained from (P) by relaxing the constraints $x_B\geqq 0$, is much easier to solve than (P). Indeed, restated as a group problem, (P') can be solved by shortest-path techniques. Clearly, whenever a solution to (P') is such that $x_B\geqq 0$, it solves (P). It is of course important to know under what circumstances such a situation will prevail, and Gomory has given a sufficient condition by showing that whenever the right-hand-side vector b satisfies a certain set of inequalities (belongs to a certain displaced convex polyhedral cone), any solution to the group problem solves the initial problem:

Theorem (Gomory[1,2]). *Let $K_B=\{y|B^{-1}y\geqq 0\}$, and let $K_B(d)$ be the cone of points in K_B at a Euclidean distance of d or more from the frontier of K_B. Further, let $D=|\det B|$ and let $l_{\max}$ be the (Euclidean) length of the longest nonbasic column of A. If $b\epsilon K_B[l_{\max}(D-1)]$, then any optimal solution to (P') is a feasible (hence optimal) solution to (P).*

For a general integer program this condition is often satisfied. Furthermore, the condition being sufficient but not necessary, and not a particularly tight one, an optimal solution to the group problem often solves the integer program even for right-hand sides that do not satisfy the above condition.

On the other hand, in the case of a 0–1 program, when the feasible set is a subset of the unit cube, the likelihood that some of the inequalities $x_B \geqq 0$ excluded from the group problem will be violated by an optimal solution to the latter seems intuitively much higher, since the hyperplanes defined by these inequalities are relatively close to the linear programming optimum: every nonredundant inequality either defines a facet of the cube, or cuts off some of its vertices.

These considerations naturally lead to the question: When is the above sufficient condition satisfied in the case of a 0–1 program? The answer, given below, is never.

PROPOSITION. *If (P) is a 0–1 program, $b \notin K_B(d)$ whenever $d>1$.*

Proof. If $b \in K_B(d)$, then $b' \in K_B$ for any b' at a distance of d or less from b. Let $b_i' = b_i$, $\forall i \neq k$, and $b_k' = b_k - d$, for some index k associated with a constraint of the form $x_j \leqq 1$ (or, in equation form, $x_j + s_j = 1$). Since $b_k = 1$, $b_k' < 0$ for any $d > 1$; and replacing b with b' makes the problem infeasible. In other words, $b' \notin K_B$ and therefore $b \notin K_B(d)$.

This proof, considerably shorter than the one in the initial version of this note, was kindly provided by a referee, who also pointed out that the proposition can be extended to arbitrary integer programs having a constraint of the form $\sum a_{kj} x_j = b_k$, with $a_{kj} \geqq 0$, $\forall_j$, provided that the condition $d>1$ is replaced by $d > b_k$. Indeed, any such integer program is made infeasible by replacing b_k with $b_k' = b_k - d$; hence, the above proof applies.

An immediate consequence of this Proposition is that the sufficient condition given in the Theorem is never satisfied for a 0–1 program. Indeed, if the elements of A are integer, unless the linear programming optimum is integer and all nonbasic columns are unit vectors, one always has $D \geqq 2$ and $l_{\max}(D-1) > 1$. Furthermore, $l_{\max}(D-1)$ is usually much larger than 1. Thus, even if the sufficient condition of the Theorem could be tightened so as to replace $l_{\max}(D-1)$ by some smaller number d, it would still never be satisfied in the 0–1 case, as long as $d > 1$.

All this does not mean, of course, that the group-theoretic approach is irrelevant for the 0–1 case. An optimal solution to the group problem may still turn out to be feasible for the initial 0–1 program, though this is likely to happen much less frequently than in the case of integer programs with 'large' right-hand sides. And even when this is not the case, an optimal solution to the group problem provides a useful bound on the value of the objective function. What the above Proposition does suggest, however, is that in the case of a 0–1 problem the constraints that are slack at the linear programming optimum tend to play a role in determining the integer programming optimum. Therefore one should try to develop approaches that use information about these latter constraints.

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Published in Operations Research on February 01, 1973 as DOI: 10.1287/opre.21.1.321.
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