Operations Research

Publication details, including instructions for authors and subscription information:
http://pubsonline.informs.org

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An Exact Algorithm for the Time-Constrained Traveling Salesman Problem

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(Received July 1981; accepted July 1982)

The time-constrained traveling salesman problem is a variation of the familiar traveling salesman problem that includes time window constraints on the time a particular city, or cities, may be visited. This note presents a concise formulation of the time-constrained traveling salesman problem. The model assumes that the distances of the problem are symmetrical and that the triangle inequality holds. Additionally, the model allows the salesman to wait at a city, if necessary, for a time window to open. The dual of the formulation is shown to be a disjunctive graph model, which is well known from scheduling theory. A longest path algorithm is used to obtain bounding information for subproblems in a branch and bound solution procedure. Computational results are presented for several small to moderate size problems.

THE Time-Constrained Traveling Salesman Problem (TCTSP) is a special case of the traveling salesman problem (TSP). As in the TSP, the TCTSP assumes that there are n cities of interest. A salesman based at city 1 must visit each of the remaining $n - 1$ cities exactly once before returning to city 1. Additionally, however, the TCTSP requires that the visit to each city be made within specified time windows. That is, if t_i is the time that the salesman visits city i, then $l_i \leq t_i \leq u_i$, where l_i and u_i are lower and upper bounds of a specified time window. Several sets of distinct time windows, or no time windows, may be specified for each city. The TCTSP is to find the sequence of cities, or tour, that visits each city within an open time window and minimizes the total length of the tour.

The incorporation of time window constraints within the traveling salesman model may be found in recent work on dial-a-ride-problems. Sexton [1979] proposes a complete integer programming model for this many-origin-to-many-destination vehicle routing problem. As may be expected, the resultant model is very large even for small problems. Solutions to the model are obtained, however, using a heuristic procedure based upon Benders' decomposition. Psaraftis [1980] obtains exact so-

Subject classification: 491 exact solution time-constrained traveling salesman problem, 837 time-constrained route selection.

Operations Research
Vol. 31, No. 5, September-October 1983
0030-364X/83/3105-0938 $01.25

lutions to a small dial-a-ride problem by using a dynamic programming approach. Christofides et al. [1981] address the TCTSP directly using a dynamic programming state space relaxation procedure to compute bounding information within a branch and bound algorithm.

The dual of the TCTSP formulation proposed in this paper yields a disjunctive graph model (Roy and Sussman [1964]), which is well known from scheduling theory. This model uses a directed graph to represent precedence constraints between tasks to be scheduled. A longest path algorithm is then used to identify the critical path of activities. A survey of enumerative methods for the general job shop scheduling problem based on the disjunctive graph model is given by Lenstra [1977]. Hardgrave and Nemhauser [1962] explored the relationship between the longest path problem and the TSP. (Some additional observations are made by Pandit [1964].) The bounding procedure used in the proposed branch and bound algorithm is similar to those discussed by Greenberg [1968] and Schrage [1970] for resource constrained scheduling problems. The branching strategy in the current implementation of the algorithm employs the penalty method (discussed in Lenstra) for resolving "essential conflicts" in the ordering of disjunctive tasks.

1. A MODEL FOR THE TIME-CONSTRAINED TSP

Given n cities, the model proposed here defines a single, nonnegative variable, t_i, to be the time that the salesman visits city i. Since the salesman must return to city 1 at the end of the tour, the formulation includes an additional variable, t_{n+1}, to determine the time at which the tour is completed. The model assumes that a complete, symmetric, nonnegative distance matrix, $\| d_{ij} \|$, is known and that time is a scalar transformation of distance so that time and distance may be used interchangeably. Additionally, it assumes that the triangle inequality holds for the distance measure. The proposed model for the TCTSP may then be stated as

$$\text{minimize} \quad t_{n+1} - t_1 \tag{1}$$

subject to

$$t_i - t_1 \geq d_{1i} \qquad i = 2, 3, \cdots, n \tag{2}$$

$$|t_i - t_j| \geq d_{ij} \qquad i = 3, 4, \cdots, n;\ 2 \leq j < i \tag{3}$$

$$t_{n+1} - t_i \geq d_{i1} \qquad i = 2, 3, \cdots, n \tag{4}$$

$$t_i \geq 0 \qquad i = 1, 2, \cdots, n + 1 \tag{5}$$

$$l_i \leq t_i \leq u_i \qquad i = 2, 3, \cdots, n \tag{6}$$

where

t_i = the time that the salesman visits city i.
$|X|$ = the absolute value of X.
d_{ij} = the shortest time required to travel from city i to city j.
l_i = the lower bound on the time window for the salesman to visit city i. By assumption all $l_i \geq 0$.
u_i = the upper bound on the time window for the salesman to visit city i, $u_i \geq l_i$, for all i.

The proposed model for the TCTSP contains a model for the TSP. To establish the validity of the model, consider the following theorem.

THEOREM 1. *Given a set of n cities and a complete, symmetric, nonnegative distance matrix, $\|d_{ij}\|$, an optimal solution to the model defined by* (1)–(5) *corresponds to an optimal traveling salesman tour.*

Proof. A feasible solution to the constraints (2)–(5) is a set of values $t_1, t_2, \cdots, t_{n+1}$ where $t_i \geq 0$ for all i. Renumber the cities such that $t_1 \leq t_2 \leq t_3 \leq \cdots \leq t_{n+1}$. Constraints (2) and (4) guarantee that the smallest and largest t_i values will correspond to the departure from and return to city 1, respectively. For any consecutive intermediate cities $i < j$, the constraints (3) force $t_j - t_i \geq d_{ij}$. Hence, each city is assigned a unique arrival time such that the sequence of arrival times corresponds to an executable salesman's tour.

Without loss of generality, t_1 may be assumed to be zero. Then t_i may be interpreted as the cumulative time, or distance, from the start of a tour to city i. The minimization of $t_{n+1} - t_1$, therefore, requires the determination of the path of shortest total length that begins and ends at city 1 and visits each intermediate city exactly once.

The addition of the time window constraints (6) to the TSP model (1)–(5) completes the specification of the TCTSP formulation. Although the model assumes a single upper and lower bound for each time window, it is easily extended to include sets of distinct time windows for each city, one of which must be satisfied. Note that the model allows the salesman to wait at city i to ensure $l_i \leq t_i$.

The proposed model is very concise. It contains $n + 1$ variables, $2n - 2$ linear constraints, $((n - 1)^*(n - 2))/2$ absolute value constraints, and $n - 1$ time window constraints. The solution to the model, however, is complicated by the presence of the absolute value constraints. These constraints introduce both a nondifferentiability and a nonconvexity. Solutions may be obtained, however, through the use of a branch and bound procedure.

2. A BRANCH AND BOUND PROCEDURE

Discrete cases of the proposed TCTSP model are determined by the two possible cases of the $|t_i - t_j| \geq d_{ij}$ absolute value constraints. These constraints act as the disjunctive constraints of the graph scheduling model. Given a choice of one of the cases for each disjunctive constraint in the current enumeration, the resulting TCTSP model is a linear program. The solution to each subproblem, therefore, may be obtained by solving the associated linear program.

We initiate the branch and bound solution procedure by relaxing the absolute value constraints (3) and the time window constraints (6) of the proposed TCTSP model. The resulting relaxed problem, P, is

$$(P) \qquad \text{minimize} \quad t_{n+1} - t_1$$

$$\begin{aligned} \text{subject to:} \quad & t_i - t_1 \geq d_{i1} && i = 2, 3, \cdots, n \\ & t_{n+1} - t_i \geq d_{i1} && i = 2, 3, \cdots, n \\ & t_i \geq 0 && i = 1, 2, \cdots, n+1. \end{aligned}$$

The solution to problem P, and subsequent problems in the tree, may be obtained by noting that the dual of problem P is a longest path problem in a directed network with $n + 1$ nodes. The dual problem, D, may be stated as:

$$(D) \qquad \text{maximize} \quad \textstyle\sum_{j=2}^{n} (d_{j1} X_{j-1} + d_{j1} X_{j+n-2})$$

subject to:

$$\begin{aligned} & \textstyle\sum_{j=1}^{n-1} - X_j \leq -1 \\ & X_j - X_{j+n-1} \leq 0 && j = 1, 2, \cdots, n-1 \\ & \textstyle\sum_{j=1}^{n-1} X_{j+n-1} \leq 1 \\ & X_j, X_{j+n-1} \geq 0 && j = 1, 2, \cdots, n-1. \end{aligned}$$

As the cases of the absolute value constraints are added to the model, additional linear constraints are added to problem P. These constraints, in the primal problem, create additional columns in the dual. These columns in the dual add additional directed arcs to the dual network.

As arcs are added to the node structure of the dual network, it is important for the solution process that two assumptions are met. First, the distances in the original problem, and hence the dual network, must satisfy the triangle inequality, i.e., $d_{ik} + d_{kj} \geq d_{ij}$. This will guarantee that if there is a Hamiltonian path in the dual network, then its length will be the length of the longest path in the network. Second, a maximum cardinality path labeling convention must be adopted. This assumption is required to resolve ties in the triangle inequality which may allow paths equal in length to the longest path to contain less than n arcs.

To this point in the description of the branch and bound procedure, the time window constraints (6) of the proposed model have not been considered. These constraints, however, are easy to incorporate within the dual labeling algorithm. In the dual labeling procedure, each node in the dual network is permanently labeled when the value of the longest path from node 1 to that node becomes known. This label, or node potential, is exactly the value of the associated primal variable t_i for node i. Hence, the primal time window constraints (6) can be enforced directly within the dual solution procedure used to obtain the solution to the TSP model described in (1)–(5).

The implementation of this procedure is straightforward. Given time window constraint $l_i \leq t_i \leq u_i$ for node i, if the dual labeling algorithm determines t_i such that $t_i > u_i$, then the current problem in the enumeration is infeasible. If the dual algorithm determines t_i such that $t_i < l_i$, then t_i is set equal to l_i and the labeling continues. In this case, the salesman is required to wait at city i. In the event of multiple time windows for city i, the salesman would wait the shortest time until a window opened. If all windows have been closed, this case has no feasible solution.

3. COMPUTATIONAL RESULTS

The proposed branch and bound algorithm has been coded in FORTRAN on a UNIVAC 1100/81A computing system. List processing techniques were used in both the enumerative branching mechanism and in the longest path bounding algorithm. Branching was accomplished by choosing disjunctive constraints according to the penalty method discussed by Lenstra in an effort to settle essential scheduling conflicts early in the enumeration process. Longest paths were calculated using a labeling procedure (see Jensen and Barnes [1980] for example).

Computational results for the proposed branch and bound algorithm were obtained from variations of a tanker scheduling problem (see Table I for port-to-port distance matrix) and vehicle scheduling problems found in the literature. The source of the vehicle scheduling test problems was Appendix 9.1 of the Eilon et al. [1971] text, *Distribution Management*. We constructed time windows for the test problems by using a nearest neighbor heuristic to identify a feasible traveling salesman's tour. The time windows were then placed about each nearest neighbor visitation time such that no time windows overlapped. Each of the test cases was then solved for 5 problem instances with 100, 90, 75, and 50% of the time windows randomly chosen to be in effect. Table II presents the results of these tests.

The presence of distinct time windows in a vehicle scheduling problem allows some simple reductions to be performed that may reduce the complexity of the solution process. Two types of reduction procedures were used to preprocess the vehicle scheduling test problems. First, whenever $l_i + d_{ij} > u_j$ for any node pair (i, j), node j precedes node i. This node precedence allows the arc connecting the two nodes in question to be fixed in the dual network and eliminated from the list of reversible arcs requiring branching. Second, each node without a time window is examined for possible fit between each pair of nodes for which the time windows were enforced. If the fit is found to be infeasible, arcs may be fixed in the dual network and eliminated from the reversible arc list.

Columns (4), (5), and (6) of Table II report the computational experi-

TABLE I
PORT-TO-PORT DISTANCE MATRIX

From	To								
	1	2	3	4	5	6	7	8	9
1	0	815	1170	822	562	630	1083	1318	569
2	815	0	665	580	603	983	1244	1711	1122
3	1170	665	0	496	723	976	1001	1768	1326
4	822	580	496	0	293	526	870	1338	894
5	562	603	723	293	0	266	660	1116	632
6	630	983	976	526	266	0	561	1092	652
7	1083	1244	1001	870	660	561	0	829	780
8	1318	1711	1768	1338	1116	1092	829	0	835
9	569	1122	1326	894	632	652	780	835	0

Number	Port	Number	Port
1	Maracaibo, Venezuela	6	Haiti
2	Costa Rica	7	Bahamas
3	Guatemala	8	Bermuda
4	Cayman Island	9	Puerto Rico
5	Jamaica		

ence with the reduction procedures. Column (4) presents the average UNIVAC 1100/81A CPU time of the 5 problem instances considered for each test case. This processing time represents the combination of the time window precedence reduction, implemented as an $O(n^2)$ procedure, and the node insertion reduction which is implemented as an $O(n^3)$ procedure. Columns (5) and (6) report resulting numbers of fixed and reversible arcs. The number of fixed arcs reported includes the $2n - 2$ arcs that connect nodes 2 through n to nodes 1 and $n + 1$ in the dual network.

Columns (7) and (8) of Table II present the computational performance of the branch and bound algorithm on the test problems. Again the figures represent average number of vertices examined and average CPU seconds over the 5 instances considered for each case. In the last case reported, the algorithm was unable to identify an optimal solution in a preset maximum of 10,000 vertices examined. The average processor time required to examine 10,000 vertices in this 51-node problem was slightly greater than 80 seconds.

4. CONCLUSIONS AND POSSIBLE EXTENSIONS

This paper presents a concise formulation of the time-constrained traveling salesman problem. The solution to this model is obtained by a branch and bound procedure that utilizes a longest path algorithm to obtain bounding information. The method used here is similar to the use of the disjunctive graph model in job-shop scheduling. The proposed algorithm was shown to be effective on several small to moderate size vehicle scheduling problems where a large percentage of the demand points possessed time windows. The algorithm, at this point, does not offer improvements over existing algorithms for the general TSP.

TABLE II

COMPUTATIONAL RESULTS OF BRANCH AND BOUND ALGORITHM FOR TCTSP

(1) Problem	(2) Nodes	(3) Time Windows in Effect	(4) Reduction Average CPU Sec	(5) Average Arcs Fixed	(6) Average Arcs Reversible	(7) Average Vertices to Optimum	(8) Solution Average CPU Sec
Tanker	9	8	0.0476	23	0	1	0.0232
	9	7	0.0384	23	2	4	0.0254
	9	6	0.0406	23	5	16	0.0355
	9	4	0.0431	23	15	62	0.0733
A.9.2	13	12	0.1152	35	0	1	0.0348
	13	11	0.0872	35	2	4	0.0385
	13	9	0.0811	35	10	22	0.0549
	13	6	0.0749	34	29	131	0.1452
A.9.3	22	21	0.8296	62	0	1	0.0730
	22	19	0.6690	62	5	22	0.1067
	22	16	0.5056	62	21	486	0.6808
	22	11	0.2990	60	72	1754	3.5472
A.9.5	30	29	4.1528	86	0	1	0.1180
	30	26	3.1858	86	10	161	0.3855
	30	22	2.1551	85	40	574	2.5546
	30	15	1.0049	83	132	4601	15.1749
A.9.8	51	50	60.4594	149	0	1	0.2414
	51	45	45.8672	153	106	217	1.0812
	51	38	30.2430	149	194	3735	22.7844
	51	25	10.0924	149	379	>10,000	>80.0

The proposed model offers potential for application in a computer assisted routing and scheduling system. In such an operational environment, the presence of user interaction and physical limitations on the delivery system may provide substantial reductions in the number of disjunctive arcs in the arc list. Additionally, refinements in the preprocessor algorithm and the development of more efficient branching strategies could increase the computational efficiency of the algorithm.

ACKNOWLEDGMENT

This work was supported, in part, by a summer research grant from the School of Business Administration at the University of Miami and, in part, by Grant FL-11-0005 from the Urban Mass Transit Administration.

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