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Technical Note—Analysis of a Preference Order Traveling Salesman Problem

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GROSS, D., AND C. M. HARRIS. 1974. *Fundamentals of Queueing Theory*. John Wiley & Sons, New York.

HEYMAN, D. P. 1975. A Diffusion Model Approximation for the $G1/G/1$ Queue in Heavy Traffic. *Bell System Tech. J.* **54**, 1637–1646.

KARLIN, S., AND H. M. TAYLOR. 1975. *A First Course in Stochastic Processes*. Academic Press, New York.

NEWELL, G. F. 1971. *Applications of Queueing Theory*. Chapman & Hall Ltd., London.

Analysis of a Preference Order Traveling Salesman Problem

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Application is made to the preference order dynamic programming solution procedure proposed by Kao for a stochastic traveling salesman problem. Although the procedure is flawed from the myopic interpretation of the monotonicity condition, it may be used as a convenient heuristic tool for solving stochastic problems.

THE OBJECTIVE of this note is to report on certain difficulties the author encountered in trying to apply the preference order dynamic programming solution procedure proposed by Kao [1978] for a stochastic traveling salesman problem. The objective of Kao's procedure is to maximize the probability that the total travel time is equal to or smaller than a prespecified critical value. The analysis was motivated by the counterexample reported by Sniedovich [1979] showing that the procedure may yield nonoptimal solutions. This counterexample consists of $n = 4$ cities, the origin point 0, the critical completion time $C = 70$ and normal travel time variates whose means and variances are

$\{\mu_{ij}\}$ =

	0	1	2	3	4
0	—	99	99	99	13
1	99	—	10	15	99
2	8	99	—	20	99
3	14	99	24	—	99
4	99	8	99	99	—

$\{\sigma^2_{ij}\}$ =

	0	1	2	3	4
0	—	0	0	0	0
1	0	—	6	2	0
2	2	0	—	12	0
3	9	0	2	—	11
4	0	1	0	0	—

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By inspection we observe that the tours $t_1 = (0, 4, 1, 3, 2, 0)$ and $t_2 = (0, 4, 1, 2, 3, 0)$ dominate all the other tours so that either t_1 or t_2 must be optimal. According to Kao's procedure the optimal tour is t_1 while in fact t_2 is a better tour. We shall discuss this observation and indicate where the pitfall in Kao's procedure lies.

1. ANALYSIS

Under consideration is a problem of the general form

$$\max_{t\in A} f(t)$$

where A is the set of all the feasible tours and $f(t) := P[T(t) \le C]$, with P denoting the probability operator, C is the critical completion time, and $T(t)$ is the travel time induced by tour t. Two observations should be made with regard to this problem. First, it constitutes a "regular" optimization problem, i.e. *maximization* of a *real valued* objective function, and second, since A is finite, the problem *must have* at least one optimal solution.

In view of the first observation, it is not clear why the problem should be treated as a preference order problem for, as indicated by Mitten [1974], the preference order model was designed to handle situations where the real-valued objective function is replaced by preference relations. The fact that the objective function has the probabilistic form $P[T(t) \le C]$ does not seem to justify the application of Mitten's model.

We consider the second observation because the proposed solution procedure is not guaranteed to be well defined, i.e. the dynamic programming recursion may fail to yield a solution, whether optimal or otherwise (see Kao, p. 1035). While it is true that Kao (pp. 1035–1038) shows that the procedure yields an optimal solution in certain cases, these cases are characterized by the property that they can be treated as simple additive deterministic problems. In this sense they should not be considered "legitimate" preference order problems, the reason being the same as why the standard deterministic traveling salesman problem should not be considered a "legitimate" preference order problem.

In short, from a computational, and certainly from a methodological point of view, Kao's procedure is of no interest to us in the context of problems in which it degenerates to the regular dynamic programming recursion. This is the situation, for example, in the Poisson variates case and the normal variates case where the variances are a constant multiple of their means, i.e. $\sigma_{ij}^2 = \alpha\mu_{ij}$ for all cities i, j $(\alpha > 0)$. In these cases the problem can be formulated and solved as a regular additive dynamic programming problem (pp. 1036–1038).

2. DIFFICULTIES

In nontrivial cases the completeness of the preference ordering operator $\perp$ is quite difficult to verify as its definition involves $\mathscr{D}$, the set of all the distribution functions defined on $\mathscr{R}^+$ (Kao, p. 1035). Moreover, it is highly questionable whether the original preference ordering operator as specified by Eq. (1), p. 1035 can ever be implemented in a nontrivial problem. The procedure suggested by Kao (p. 1036) to resolve this difficulty is based upon the following argument. "The implication is that, if we are in city i at stage k, we must select the next destination from a set S_k in such a way that, whatever the city leading into i, the selection constitutes an optimal choice. This is essentially the monotonicity requirement underlying the preference-order dynamic programming."

On the basis of this argument Kao (p. 1036) proposes to replace $\mathscr{D}$ in (1) by a finite set $\mathscr{D}^*$ consisting of all the cities that can lead into i, i.e. the set $S_k{}^c = \{1, 2, \cdots, n\} - S_k - \{i\}$. Unfortunately, this is a *myopic* interpretation of the monotonicity condition. In order to comply with Mitten's monotonicity condition the set $\mathscr{D}^*$ should be extended to include all subtours from the origin point to city i through all the cities in $S_k{}^c$. If we attempt to make this correction in Kao's model we create a procedure which is essentially based on brute force enumeration, i.e. at stage $k = 1$ we have for each i the following situation: $S_1 \in \{1, 2, \cdots, n\} - \{i\}$, $S_1{}^c = \{1, 2, \cdots, n\} - \{i, s_1\}$, and consequently it is necessary to generate all the $(n - 2)!$ feasible permutation of the elements of $S_1{}^c$. Since there are n possible values for i and $n - 1$ possible values for s_1, at stage $k = 1$ we have to generate $n \cdot (n - 1) \cdot (n - 2)!$ tours. This is equal to $n!$ tours, which is the total number of feasible tours. Thus, it would not be necessary to execute the algorithm beyond stage $k = 1$, and we end up using brute force enumeration.

To see what the consequences of using Kao's myopic interpretation of the monotonicity condition might be, consider again the tours t_1 and t_2 specified above. From t_1 take the subtour $t_1' = (1, 3, 2, 0)$ and from tour t_2 the subtour $t_2' = (1, 2, 3, 0)$. In the context of the preference order procedure we are at $k = 2$, $i = 1$, $S_k = \{2, 3\}$ so that $\mathscr{D}^* = \{4\}$. The subtours t_1 and t_2' generate normal variates with parameters (44, 27) and (47, 6), respectively and by convoluting them with (8, 1), the travel time from 4 to 1, we obtain, according to Kao's procedure,

$$(70 - (44 + 8))/\sqrt{27 + 1} = 9/\sqrt{7} \leq 15/\sqrt{7} = (70 - (47 + 8))/\sqrt{6 + 1}.$$

Thus, we select t_1' as the optimal subtour. This decision finally yields t_1 as the optimal tour. Now, if we also consider the travel time from 0 to 4, we obtain

$$(70 - (44 + 8 + 13))/\sqrt{27 + 1 + 0} = 2.5/\sqrt{7} > (70 - (47 + 8 + 13))/\sqrt{6 + 1 + 0} = 2/\sqrt{7}.$$

This implies (a) that the monotonicity condition is not satisfied, and (b) that t_1 is not optimal. It should be emphasized that in this example Kao's procedure (p. 1037) is well defined yet the solution it yields is not optimal.

3. SUMMARY

For nontrivial problems it is difficult to verify the completeness of the proposed preference ordering operator, and an attempt to replace $\mathscr{D}$ by $\mathscr{D}^*$ may result in nonoptimal solutions. Therefore the procedure should be used with caution, unless it is possible to verify the monotonicity condition.

The flaw in the procedure stems from the myopic interpretation of the monotonicity condition. Unfortunately, an attempt to correct the procedure transforms it into brute force enumeration. Despite these difficulties, it has been the author's experience that, at least in the normal variates case, the procedure often yields optimal, or nearly optimal solutions. It may then be used as a convenient heuristic tool for solving stochastic problems.

REFERENCES

KAO, E. P. C. 1978. A Preference Order Dynamic Program for a Stochastic Traveling Salesman Problem. *Opns. Res.* **26**, 1033–1045.

SNIEDOVICH, M. 1979. Preference Order Stochastic Traveling Salesperson Problems: A Counter Example, RC7905 IBM Thomas J. Watson Research Center, Yorktown Heights, N.Y.

MITTEN, L. C. 1974. Preference Order Dynamic Programming. *Mgmt. Sci.* **21**, 43–46.

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