Transportation Science

Publication details, including instructions for authors and subscription information:
http://pubsonline.informs.org

Technical Note—Conical Volume-Delay Functions

Heinz Spiess,

Conical Volume-Delay Functions

HEINZ SPIESS

Centre de recherche sur les transports, Université de Montréal, Montréal, Canada H3C 3J7, and EMME/2 Support Center, Haldenstrasse 16, CH-2558 Aegerten, Switzerland

The widely used BPR volume-delay functions have some inherent drawbacks. A set of conditions is developed which a "well behaved" volume delay function should satisfy. This leads to the definition of a new class of functions named conical volume-delay functions, *due to their geometrical interpretation as hyperbolic conical sections. It is shown that these functions satisfy all conditions set forth and, thus, constitute a viable alternative to the BPR type functions.*

INTRODUCTION

In most traffic assignment methods, the effect of road capacity on travel times is specified by means of *volume-delay functions* $t(v)$ which are used to express the travel time (or cost) on a road link as a function of the traffic volume v. Usually these functions are expressed as the product of the free flow time t_0 multiplied by a normalized congestion function $f(x)$

$$t(v) = t_0 \cdot f\left(\frac{v}{c}\right) \tag{1}$$

where the argument of the delay function is the v/c ratio, c being a measure of the capacity of the road.

Many different types of volume-delay functions have been proposed and used in practice in the past (for a review article, see BRANSTON[1]). By far the most widely used volume-delay functions are the *BPR functions* (Bureau of Public Roads[2]), which are defined as

$$t^{\mathrm{BPR}}(v) = t_0 \cdot \left(1 + \left(\frac{v}{c}\right)^{\alpha}\right). \tag{2}$$

With higher values of α, the onset of congestion effects becomes more and more sudden. This can be seen in Figures 1, a and b, which shows the BPR-type congestion function

$$f^{\mathrm{BPR}}(x) = 1 + x^{\alpha} \tag{3}$$

for exponents $\alpha = 2, 4, 6, 8, 10$ and 12. This range of alpha values is also indicative of the wide range that is used in practice. The values of α are usually not restricted to integers.

The simplicity of the BPR functions is certainly one reason for their widespread use. It is also very convenient that, for any value of α, we have $f^{\mathrm{BPR}}(1) = 2$, i.e. when traffic volume equals the capacity, the speed is always half the free flow speed.

Unfortunately, these BPR functions also have some inherent drawbacks, especially when used with high values of α:

a) While for any realistic set of travel volumes, we can assume that $v/c \leq 1$ (or at least not much larger than 1), this is usually not the case during the first few iterations of an equilibrium assignment. Values of v/c may well reach values of 3, 5 or even more. To illustrate this, the link time of a link with $\alpha = 12$ and a v/c ratio of 3 is increased by a factor of $1 + 3^{12} = 531443$, which means that every minute of free flow time becomes roughly one year of congested time! These aberrations slow down convergence by giving undue weight to overloaded links with high α-values and can also cause numerical problems, such as overflow conditions and loss of precision.

b) For links that are used far below capacity, the BPR functions, especially when high values of alpha are used, always yield free flow times independent of actual traffic volume. To illustrate this, consider again a link with $\alpha = 12$ and a capacity of 1000. Whether the volume is 0 or 300, the volume delay function yields exactly the same numeric value (assuming single precision calculation). Therefore, the equilibrium model will locally degenerate to an all-or-nothing assignment, where the slightest change (or error) in free flow time may result in a complete shift of volume from one path to another path. In addition, the solution is no longer guaranteed to be unique on the level of link flows, since the volume-delay functions are not strictly increasing functions of the volume any more.

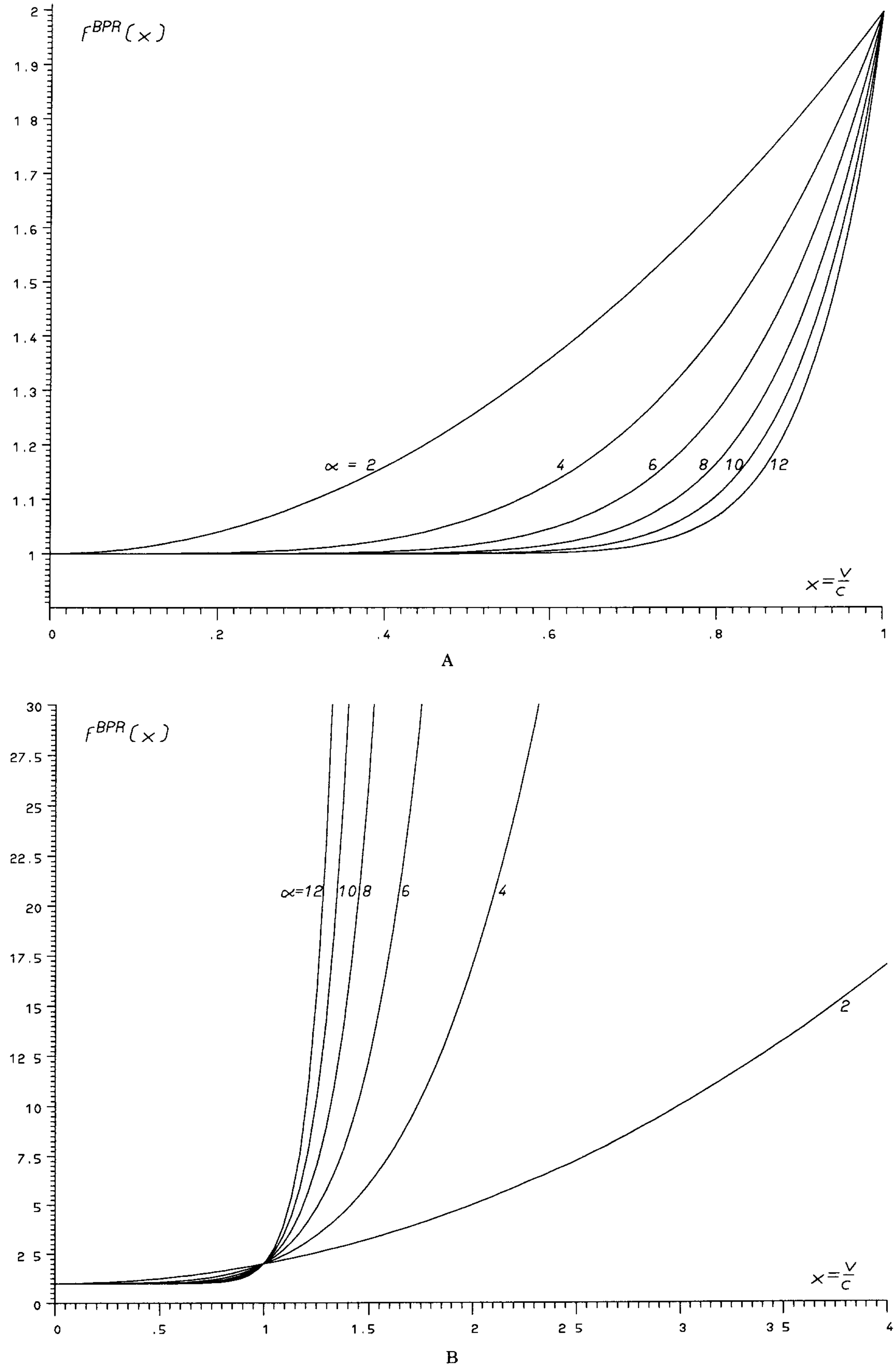


Fig. 1. BPR functions for (A) small and (B) large v/c ratios.

c) Even though the formula for the BPR function is very simple, its evaluation requires the computation of two transcendental functions, i.e. a logarithm and an exponential function to implement the power x^α; this requires a fair amount of computing resources.

REQUIREMENTS FOR A WELL BEHAVED CONGESTION FUNCTION

ARE THERE other types of congestion functions that are not (or are less) subject to the drawbacks of the BPR functions? If yes, what would such a "designer" volume-delay function look like? Let us first set forth some conditions these functions need to satisfy:

1) $f(x)$ *should be strictly increasing.* This is a necessary condition for the assignment to converge to a unique solution.
2) $f(0) = 1$ *and* $f(1) = 2$. These conditions ensure compatibility with the well known BPR type functions. The capacity is thus still defined as the volume at which congested speed is half the free flow speed.
3) $f'(x)$ *should exist and be strictly increasing.* This ensures convexity of the congestion function—not a necessary, but a highly desirable property.
4) $f'(1) = \alpha$. Alpha is, by analogy to the exponent in the BPR functions, the parameter that defines how the congestion effects change when the capacity is reached.
5) $f'(x) < M\alpha$, *where M is a finite positive constant.* The steepness of the congestion curve is thus limited. This in turn also prevents the values of the volume delay function from becoming too high when considering v/c ratios higher than 1, thus avoiding the problems mentioned in a) above.
6) $f'(0) > 0$. This condition guarantees the uniqueness of the link volumes. It also renders the assignment stable in the presence of coding imprecisions or small errors in travel time and distributes volumes on competing uncongested paths in proportion to their capacity.
7) The evaluation of $f(x)$ should not take more *computing time* than does the evaluation of the corresponding BPR function.

Conditions 1 to 4 hold, of course, for the BPR function and are stated to ensure compatibility with them. Conditions 5, 6 and 7 are imposed in order to overcome the BPR functions' drawbacks a), b) and c) mentioned above.

At least one class of such congestion functions exists indeed, as we will show in the next section.

CONICAL CONGESTION FUNCTIONS

CONSIDER an obtuse three-dimensional cone intersected by the two-dimensional X–Y plane. Figure 2 shows the projection of the cone, as well as one possible resulting hyperbolic section. These hyperbolic cone sections have all the desired properties and constitute the base for the *conical congestion functions*, as we shall name them.

The name "hyperbolic congestion functions" would also be appealing, but it has been used in the past for functions of the form $f^{\mathrm{HYP}}(x) = 1/(1 - x)$, see Branston.[1] Furthermore, this name could also lead to confusion with the transcendental hyperbolic functions *sinh* and *cosh*.

Since the mathematical derivation is quite simple but lengthy and only involves basic geometry and elementary algebra, we shall simply state the resulting function and show that it indeed satisfies the conditions 1 to 7 set forth above.

Let the class of *conical congestion functions* be defined as

$$f^{\mathrm{C}}(x) = 2 + \sqrt{\alpha^2(1 - x)^2 + \beta^2} - \alpha(1 - x) - \beta \quad (4)$$

where β is given as

$$\beta = \frac{2\alpha - 1}{2\alpha - 2} \quad (5)$$

and α is any number larger than 1.

In order to prove that the desired properties hold for f^{C}, we need to evaluate the first derivative of f^{C}, which is

$$f^{\mathrm{C}\prime}(x) = \alpha + \frac{\alpha^2(x - 1)}{\sqrt{\alpha^2(1 - x)^2 + \beta^2}}. \quad (6)$$

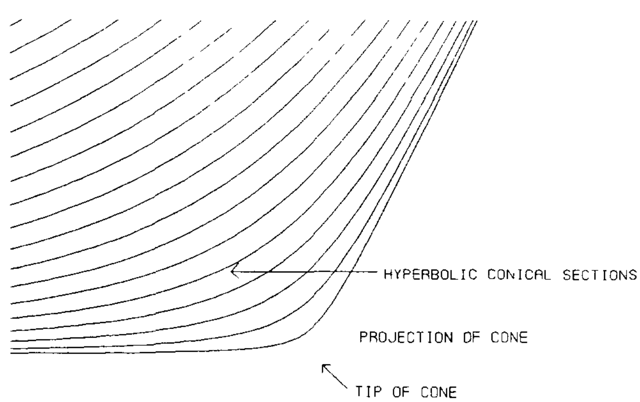
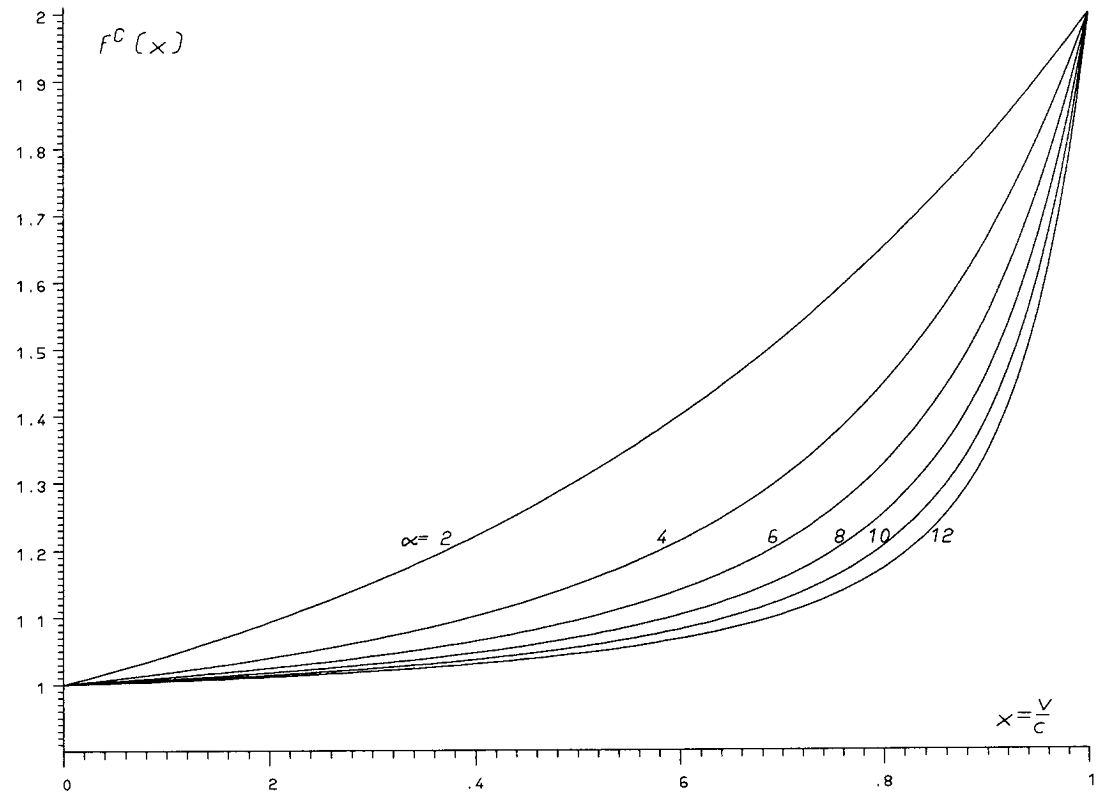
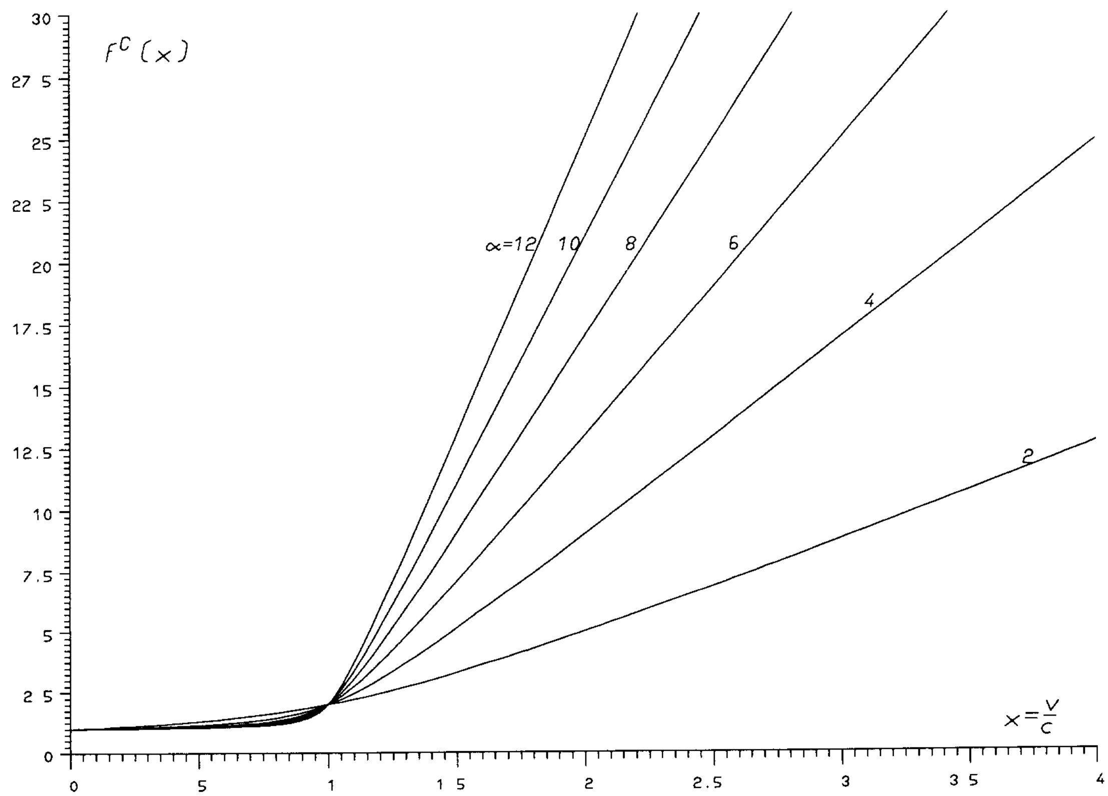


Fig. 2. Hyperbolic conical sections.



A



B

Fig. 3. Conical functions for (A) small and (B) large v/c ratios.

Let us now show that the properties 1 to 7 indeed hold for f^C:

1) By rewriting $f^{C\prime}$, we obtain

$$f^{C\prime}(x) = \alpha\left(1 + \frac{\alpha(x-1)}{\sqrt{\alpha^2(1-x)^2 + \beta^2}}\right). \qquad (7)$$

Using Pythagoras' theorem, it is easy to see that the second term is strictly contained between -1 and 1. Thus $f^{C\prime}(x) > 0$ and it follows that $f^C(x)$ is strictly increasing.

2) For the function value at $x = 0$ we obtain $f^C(0) = 2 + \sqrt{\alpha^2 + \beta^2} - \alpha - \beta$. Using (5), it follows that

$$\alpha^2 + \beta^2 = (\alpha + \beta - 1)^2, \qquad (8)$$

which, once substituted under the square root, shows that indeed we have $f^C(0) = 1$.

3) The existence of $f^{C\prime}(x)$ has already been shown when proving 1). To show that $f^{C\prime}(x)$ is a strictly increasing function, it suffices to show that the second derivative of $f^C(x)$ is strictly positive. The details are left for the interested reader.

4) From (7), it follows immediately that $f^{C\prime}(1) = \alpha$.

5) Using the same reasoning as used to prove 1), we show that $f^{C\prime}(x) < 2\alpha$. This means that for large v/c ratios, the congestion behaves as a quasi-linear function, with a gradient that approaches but never exceeds twice the gradient at capacity.

6) Again using (6) and (8), we obtain $f^{C\prime}(0) = \alpha - \alpha^2/(\alpha + \beta - 1)$ which can be developed using (5) to obtain

$$f^{C\prime}(0) = \frac{\alpha}{2\alpha^2 - 2\alpha + 1} > \frac{1}{2\alpha}.$$

7) As for the computation time, we note that the evaluation of $f^C(x)$ needs: 2 multiplications, 1 square root and 4 additions. This compares very favorably with the 2 transcendentals, 1 multiplication and 1 addition needed to compute the BPR type function. Note that for a given value of α, the values of β and β^2 are constants that can be evaluated just once at the onset.

Figure 3, a and b, shows the functions $f^C(x)$ for values of α = 2, 4, 6, 8, 10 and 12. Note the quasi-linear behavior for $x > 1$ when comparing Figure 3b to the BPR functions in Figure 1b. The non-zero gradient of the functions at $x = 0$ can be seen clearly.

We implemented both the BPR and the conical volume-delay function using a brief program written in C, in order to compare execution times. The program was compiled and run on the three major families of micro-computers. The results in Table I show that the conical functions can be evaluated more efficiently than the BPR functions, in spite of the seemingly more complex formula.

TABLE I

Computational Efficiency

Computer Installation			Execution Time (msec)	
CPU/FPU	Speed (MHz)	Compiler	$f^{BPR}(x)$	$f^C(x)$
NSC, 32016/32081	10	BSD 4.2	1.22	0.98
Intel, 80286/80287	10	MS 5.0	0.65	0.37
Motorola, 68020/68881	17	SVS	0.13	0.09

While, to our best knowledge, the class of conical congestion functions proposed here constitutes a new approach, it is interesting to note that a very similar functional form was proposed in an unpublished report prepared in the context of a transportation study for the City of Winnipeg (Traffic Research Corporation,[5] FLORIAN and NGUYEN[3]). The functions used in that study can also be interpreted as conical sections, but they use more parameters and do not, in general, satisfy the conditions we set forth in the preceding section. For Branston,[1] it was "doubtful whether this functional form could be of more general use" and in his article he proceeded to show that these functions should be approximated by BPR-type functions—a questionable "progress" in light of our findings.

In a recent transportation study for the City of Basel, Switzerland, the proposed conical volume-delay functions have been used successfully in practice. A dramatic improvement in the convergence of the equilibrium assignment was observed when switching from the previously used BPR functions to the corresponding conical functions, with no practically significant changes in the resulting network flows. This study was carried out using the EMME/2 transportation planning software (SPIESS[4]).

CONCLUSIONS

IN TRYING to overcome the known disadvantages of the BPR functions, we have developed a new class of volume-delay functions, the *conical* functions. The interpretation of the parameters used to characterize the specific congestion behavior of a road link, i.e. capacity c and steepness α, is the same for both BPR and conical functions, which makes the transition to conical functions particularly simple. Since the difference between a BPR function and a conical function with the same parameter α is very small within the feasible domain, i.e. $v/c < 1$, the BPR parameters can be transferred directly in most cases.

Further research would be needed to develop statistical methods for directly estimating the parameters of the conical functions, using observed speeds and volumes.

REFERENCES

1. D. BRANSTON, "Link Capacity Functions: A Review", *Trans. Res.* **10,** 223–236 (1976).
2. Bureau of Public Roads, Traffic Assignment Manual, U.S. Dept. of Commerce, Urban Planning Division, Washington D.C., 1964.
3. M. FLORIAN AND S. NGUYEN, "An Application and Validation of Equilibrium Trip Assignment Methods", *Trans. Sci.* **10,** 374–390 (1976).
4. H. SPIESS, Contributions à la théorie et aux outils de planification des réseaux de transport urbain, Ph.D. thesis, Département d'informatique et recherche operationnelle, Université de Montréal, 1984.
5. Traffic Research Corporation, Winnipeg Area Transportation Study, report prepared for the Streets and Transit Division of the Metropolitan Corporation of Greater Winnipeg (unpublished), 1966.

(Received, June 1989; revision received October 1989; accepted October 1989)