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Technical Note—Construction of Difficult Linearly Constrained Concave Minimization Problems

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formulations and, in the longer run, to developing automatic model reformulation techniques.

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Construction of Difficult Linearly Constrained Concave Minimization Problems

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Given a polytope and an arbitrary subset of its vertices, we show how to construct a differentiable concave function that assumes any arbitrary value (within a specified ϵ-tolerance) at each vertex of the subset, with each vertex in the subset a strong local constrained minimum. We also show how this construction method can be used to generate test problems for linearly constrained concave minimization algorithms.

MANY AUTHORS (see, for example, Tui [1964], Zwart [1974], Falk and Hoffman [1976], and Rosen [1983]) have considered the global minimization of a concave function subject to linear inequality

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constraints. The problem can formally be defined as follows:

$$(P)\text{minimize}_{x\in\Omega} f(x),$$

where f is a concave function, $\Omega = \{x: Ax \le b\}$, A is an $m \times n$ matrix, $x \in R^n$, and $b \in R^m$. We assume Ω is nonempty and bounded.

As is well-known, any global or local minimum solution of problem (P) occurs at a vertex of Ω. Methods for solving problem (P) are based on this property. A simple, but computationally impractical algorithm is to enumerate all the vertices of Ω, evaluate the function f for each vertex, and select a vertex with minimum function value. Most algorithms attempt to enumerate a small portion of the vertices before finding a global minimum solution. For a survey of different techniques see Heising [1981].

The major difficulty faced in solving problem (P) is that it can have many local solutions. In the next section we construct smooth concave functions with an arbitrary number of strong local optimal solutions. More specifically, given a bounded polyhedron Ω, and a subset of its vertices, V, we construct a differentiable concave function f, which assumes any arbitrary value (within a specified ε-tolerance) at each vertex in V, and such that every vertex in V is a strong local constrained minimum of f. In particular, this construction shows that in the worst case (even when f is differentiable) problem (P) can have as many strong local solutions as there are vertices in Ω. For such an extreme case an algorithm could require enumeration of all the vertices of Ω.

In Section 2, we show how to use the construction method developed in Section 1 to generate practical test problems for concave minimization algorithms.

1. THE CONSTRUCTION METHOD

THEOREM 1. *Let $\Omega = \{x: Ax \le b\}$ be nonempty and bounded. Let $V = \{v_1, \cdots, v_k\}$ be a subset of the vertices of Ω, and let β_i for $i = 1, \cdots, k$ an arbitrary set of real numbers. Given $\varepsilon > 0$, there exists a function f_ε: $R^n \to R$ such that*:

(i) f_ε is differentiable and concave.

(ii) $\beta_i - \varepsilon \le f_\varepsilon(v_i) \le \beta_i$ for all $i = 1, \cdots, k$,

(iii) *If ε is sufficiently small, then each vertex $v_i \in V$ is a strong local minimum of f_ε subject to $x \in \Omega$.*

Proof. Case 1: $\beta_i < 0$ for all $i = 1, \cdots, k$.

(i): For each $i = 1, \cdots, k$, there exists $c_i \in R^n$ such that v_i is the unique optimal solution of problem (P_i) defined as:

$$(P_i)\max_{x\in\Omega} c_i^T x.$$

For each $i = 1, \cdots, k$, let $\bar{v}_i$, a vertex in Ω, be the second best solution to problem (P_i). That is, for every vertex $v \in \Omega$, $v \neq v_i$, we have

$$c_i^T v_i > c_i^T \bar{v}_i \geq c_i^T v. \quad (1)$$

Pick $\mu_i > 0$ such that

$$\exp(\mu_i c_i^T(\bar{v}_i - v_i)) \leq \varepsilon/(-\beta_i k), \quad (2)$$

by selecting $\mu_i \geq \mu_i(\varepsilon)$, where $\mu_i(\varepsilon)$ is given by

$$\mu_i(\varepsilon) = |\log(\varepsilon/(-\beta_i k))/c_i^T(\bar{v}_i - v_i)|. \quad (3)$$

From (1), and since $\beta_i < 0$, for every vertex $v \in \Omega$, $v \neq v_i$ we have

$$\beta_i \exp(\mu_i c_i^T(v - v_i)) \geq \beta_i \exp(\mu_i c_i^T(\bar{v}_i - v_i)). \quad (4)$$

For $i = 1, \cdots, k$, define

$$f_i(x) = \beta_i \exp(\mu_i c_i^T(x - v_i)),$$

and note that $f_i(v_i) = \beta_i$. We now define f_ε by

$$f_\varepsilon(x) = \textstyle\sum_{i=1}^k f_i(x).$$

To prove that $f_\varepsilon(x)$ is concave, note that if $c, z \in R^n$, then $\exp(c^T z)$ is convex in z, since its Hessian, given by $(c^T c)\exp(c^T z)$ is positive semidefinite. Since $\beta_i < 0$, each $f_i(x)$ is concave and hence $f_\varepsilon(x)$ is concave. Clearly $f_\varepsilon(x)$ is differentiable.

(ii): Let $l \in \{1, \cdots, k\}$, then

$$f_\varepsilon(v_l) = \textstyle\sum_{i=1}^k f_i(v_l) = \beta_l + \sum_{i=1, i\neq l}^k f_i(v_l).$$

Applying (4) and (2) to the above, we obtain

$$f_\varepsilon(v_l) \geq \beta_l + \textstyle\sum_{i=1, i\neq l}^k \beta_i \exp(\mu_i c_i^T(\bar{v}_i - v_i))$$

$$\geq \beta_l + \textstyle\sum_{i=1, i\neq l}^k - \varepsilon/k = \beta_l - \varepsilon((k-1)/k) > \beta_l - \varepsilon.$$

Also note that $f_\varepsilon(v_l) \leq \beta_l$.

(iii): Note that for all $i = 1, \cdots, k$, v_i is the global minimum of $f_i(x)$ subject to $x \in \Omega$. If ε is sufficiently small (forcing the μ_i's to be sufficiently large), then for a small neighborhood of v_i, $f_\varepsilon(x)$ behaves like $f_i(x)$ and hence v_i will be a strong local minimum of $f_\varepsilon(x)$ subject to $x \in \Omega$. More precisely:

Choose $\varepsilon < \min\{-\beta_i\}_{i=1}^k$, and let the μ_i's in f_ε satisfy

$$\mu_i \geq 2\mu_i(\varepsilon), \qquad i = 1, \cdots, k, \quad (5)$$

where $\mu_i(\varepsilon)$ is given by (3). For $l \in \{1, \cdots, k\}$, we show that v_l is a strong local constrained minimum of f_ε. Without loss of generality, we assume that v_l is nondegenerate. Let $\{v_l^j\}_{j=1}^n$ be the vertices adjacent to v_l.

For $j = 1, \cdots, n$, define

$$\bar{v}_l^j = \tfrac{1}{2}(v_l + v_l^j), \tag{6}$$

i.e., the midpoint of the edge connecting v_l and v_l^j. Let S_l be the simplex generated by the $n + 1$ points $\{\bar{v}_l^j\}_{j=0}^n$, with $\bar{v}_l^0 \equiv v_l$.

We show that if $x \in S_l$, $x \neq v_l$, then

$$f_\varepsilon(x) > f_\varepsilon(v_l). \tag{7}$$

Note that (7) would then imply that v_l is a strong local minimum of f_ε.

Each $x \in S_l$ distinct from v_l is given by

$$x = \textstyle\sum_{j=0}^n \alpha_j \bar{v}_l^j,$$

where

$$\textstyle\sum_{j=0}^n \alpha_j = 1, \qquad \alpha_j \geq 0, j = 1, \cdots, n, \qquad \alpha_j > 0, \quad \text{for some} \quad j \in \{1, \cdots, n\}. \tag{8}$$

Since f_ε is concave, we have

$$f_\varepsilon(x) \geq \textstyle\sum_{j=0}^n \alpha_j f_\varepsilon(\bar{v}_l^j). \tag{9}$$

We claim that the following is true:

$$f_\varepsilon(\bar{v}_l^j) > f_\varepsilon(v_l), \quad \text{for all} \quad j = 1, \cdots, n. \tag{10}$$

Before proving (10), note that by using it in (9), and from (8), we obtain the desired inequality in (7), namely

$$f_\varepsilon(x) > \textstyle\sum_{j=0}^n \alpha_j f_\varepsilon(v_l) = f_\varepsilon(v_l).$$

We now prove (10), thereby completing the proof of (iii).

Let $i \in \{1, \cdots, k\}$. From (6) we have

$$c_i^T(\bar{v}_l^j - v_i) = \tfrac{1}{2}c_i^T(v_l^j - v_i) + \tfrac{1}{2}c_i^T(v_l - v_i), \qquad j = 1, \cdots, n.$$

Either $v_l^j \neq v_i$ or $v_l \neq v_i$. Using this fact and relation (1), we obtain

$$c_i^T(\bar{v}_l^j - v_i) \leq \tfrac{1}{2}c_i^T(\bar{v}_i - v_i), \qquad j = 1, \cdots, n.$$

Thus, we have

$$\begin{aligned} f_\varepsilon(\bar{v}_l^j) &= \textstyle\sum_{i=1}^k \beta_i \exp(c_i^T(\bar{v}_l^j - v_i)) \\ &\geq \textstyle\sum_{i=1}^k \beta_i \exp(\tfrac{1}{2}\mu_i c_i^T(\bar{v}_i - v_i)), \qquad j = 1, \cdots, n. \end{aligned} \tag{11}$$

From (5) we have

$$\exp(\tfrac{1}{2}\mu_i c_i^T(\bar{v}_i - v_i)) \leq \varepsilon/(-\beta_i k), \qquad i = 1, \cdots, k. \tag{12}$$

Using (12) in (11), we obtain $f_\varepsilon(\bar{v}_l^j) \geq \sum_{i=1}^k - \varepsilon/k = -\varepsilon$, $\quad j = 1, \cdots, n$. The proof of (10) now follows by recalling that $\varepsilon < -\beta_l$, and that $f_\varepsilon(v_l) \leq \beta_l$ (from part (ii) of the theorem).

Case 2: The β_i's are not necessarily all negative.

Let $\beta_i' = \beta_i - \beta$, where $\beta = \max\{\beta_i\}_{i=1}^k + 1$. Since $\beta_i' < 0$, we may apply Case 1 to obtain f_ε' satisfying (i), (ii) and (iii). Define $f_\varepsilon(x) = f_\varepsilon'(x) + \beta$ and note that f_ε satisfies (i), (ii), and (iii).

2. CONSTRUCTION OF TEST PROBLEMS

In this section we use the results of the previous sections to generate test problems for linearly constrained concave minimization algorithms.

Enumeration of a large subset of the vertices of a polytope is both undesirable and impractical. Moreover, the corresponding objective function $f_\varepsilon(x)$ would have as many terms as there are vertices in the subset. Thus, we want k, the size of the subset, to be "small." We propose the following scheme:

Step 1. Generate a random polytope $\Omega \in R^n$, and choose a value for k (say n, $2n$, or some other multiple of n).

Step 2. Randomly generate c_i vectors for $i = 1, \cdots, k$. Maximize $c_i^T x$ subject to $x \in \Omega$ to obtain a vertex v_i (perturb c_i, if necessary, to obtain a unique v_i).

Step 3. Determine $\bar{v}_i$ (the second best solution to problem P_i) for all $i = 1, \cdots, k$. Randomly generate the β_i's from a specified interval $[a, b]$ with $b < 0$. Select $\varepsilon < -b$ and compute the $\mu_i(\varepsilon)$'s according to (3). For all $i = 1, \cdots, k$, select $\mu_i \geq 2\mu_i(\varepsilon)$, and, as before, let $f_\varepsilon(x) = \sum_{i=1}^k \beta_i \exp(\mu_i c_i^T(x - v_i))$.

Since $\bar{v}_i$ is always a vertex adjacent to v_i, in order to obtain it, we would need to generate all the vertices adjacent to v_i, which requires additional computation. To avoid calculation of the $\bar{v}_i$'s, we could simply choose a large value for the μ_i's. We remark that regardless of the option in Step 3, it is possible that the resulting test problem is ill-conditioned.

The above scheme is substantially simplified if we begin with a feasible region of the form $\Omega = \{x : Ax \leq b,\ 0 \leq x_i \leq 1,\ i = 1, \cdots, n\}$. Assume Ω contains integral vertices (i.e., each component is either 0 or 1), and let $V = \{v_1, \cdots, v_k\}$ be a subset of such vertices. In this case, we can immediately obtain μ_i's (hence f_ε) for which each $v_i \in V$ is a strong local minimum of f_ε over the unit hypercube. Since Ω is contained in the unit hypercube, each v_i is also a strong local minimum of f_ε over Ω.

Given $v_i = (e_1, \cdots, e_n)^T \in V$, define

$$c_i = (u_1, \cdots, u_n)^T, \qquad \text{where} \quad u_j = \begin{cases} 1, & \text{if } e_j = 1. \\ -1, & \text{if } e_j = 0. \end{cases}$$

It is easy to show that

$$\max_{x \in \Omega} c_i^T x = c_i^T v_i = \sum_{j=1}^n e_j. \tag{13}$$

Each vertex adjacent to v_i (with respect to the unit hypercube) differs from it in only one component. Since $\bar{v}_i$ is a vertex adjacent to v_i, from (13), we have $c_i^T(\bar{v}_i - v_i) = -1$. Substituting in (5), we obtain $\mu_i \geq 2\,|\log(\varepsilon/(-\beta_i k))\,|$. Suppose we choose the β_i's from the interval $[-99, -2]$, $\varepsilon = 1$, and $k \leq 30$, then it suffices to select $\mu_i \geq 16$ for $i = 1, \cdots, k$ in f_ε.

3. CONCLUSION

We have shown that given a polytope and an arbitrary subset of its vertices, we can construct a differentiable concave function that assumes any arbitrary value (within a specified ε-tolerance) at each vertex in the subset, with each vertex in the subset a strong local constrained minimum. In particular, this construction shows that in the worst case, even when its objective function is differentiable, the concave optimization problem (P) can have as many strong local solutions as there are vertices of the polytope. For such an extreme case, an algorithm could require enumeration of all the vertices of the polytope. Although our main objective was to demonstrate the "difficulty" of concave programming, we also showed how the construction method can be used to generate test problems for linearly constrained concave minimization algorithms.

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