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Technical Note—Duality Theory for Generalized Linear Programs with Computational Methods

David J. Thuente,

162 patterns and 93 rolls underrun because of rounding problems. The modified Gilmore-Gomory algorithm took 8.074 seconds and gave solutions with 153 patterns and 56 rolls underrun. The modified algorithm took 45% longer to solve the problem, but it reduced the number of patterns by 5.5% and the number of rolls underrun by 49%. The increased running time was only 2.5 seconds and would be economically insignificant relative to the reduction in pattern changes and rounding difficulties. Before these solutions could be used, additional work would have to be done on them to reduce the underruns. This would require that some sizes be overrun to increase the usage of patterns containing the sizes underrun, or new patterns be generated containing only the sizes underrun. Regardless of what is done, the modified algorithm provides a substantially better starting point.

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Duality Theory for Generalized Linear Programs with Computational Methods

DAVID J. THUENTE
Indiana University-Purdue University, Fort Wayne, Indiana

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This paper presents a duality theory for generalized linear programs which parallels the usual duality results for linear programming The duals are a form of inexact linear programs and can be solved by the simplex method Computational methods with examples and applications are given.

IN THE USUAL application of a linear program, min $\{cx|Ax \leq b, x \geq 0\}$ it is assumed that the coefficients a_{ij}, b_i, and c_j are known exactly. It has been observed by Falk [2], Gass [3, p. 147], and others that this is frequently not an accurate assumption. Chance-constrained and stochastic programming and sensitivity analysis are among the ap-

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proaches to handle this uncertainty. However, in many cases it is possible to determine that the technological coefficients of a given output in a chemical reaction or plant operation may/must satisfy certain relations and hence the activity vectors belong to a given set.

The problems we will consider have the general form

$$\begin{aligned}&\min cx\\&\text{s.t. } a_1x_1 + a_2x_2 + \cdots + a_nx_n \leq b,\ x_j \geq 0\end{aligned} \tag{1}$$

where each column vector $a_j \in P_j$, $j = 1, \cdots, n$ and P_j is a convex polyhedron in R^m.

This problem can be examined from two quite different strategies. The conservative strategy seeks the optimal solution which is feasible for *all* possible $a_j \in P_j$. This approach is called inexact linear programming which has been solved for arbitrary convex sets by Soyster [4-6] and Falk [3]. The optimistic strategy seeks the optimal solution where the solution is feasible for some $a_j \in P_j$. We can view this as the best possible value of the problem, and in this case the activity vectors a_j along with the vector x are the decision variables. The optimistic approach is a generalized linear program (GLP) as given by Dantzig [1, p. 434]. This paper presents duality and computational results for the optimistic strategy.

1. THE DUAL PROBLEMS

Let $P_j, j = 1, \cdots, n$ be bounded polyhedral convex sets in R^m and let a_j denote columns of the matrix A with a_j from P_j.

The GLP as given by Dantzig [1] can be written as

$$\begin{aligned}&\min cx\\&\text{s.t. } Ax = b,\ x \geq 0 \text{ where } Ax = b \text{ for } \textit{some } a_j \in P_j.\end{aligned} \tag{2}$$

This will be called the primal problem. An optimal solution to (2) is a pair (x^*, A^*) such that $A^*x^* = b$, $x^* \geq 0$ and $cx^* \leq cx$ for all other pairs (x, A) such that $Ax = b$ and $x \geq 0$.

The "dual" problem of (2) is defined to have the form of an inexact linear program

$$\begin{aligned}&\max wb\\&\text{s.t. } wA \leq c \text{ for } \textit{all } a_j \in P_j.\end{aligned} \tag{3}$$

We will show that (2) and (3) are primal-dual problems in the usual sense of linear programming. The (GLP) can also be written with inequality constraints and a corresponding duality theorem proved. Then it can also be shown that complementary slackness holds for generalized and inexact linear programs.

THE DUALITY THEOREM. *If either the primal* (2) *or the dual* (3) *problem has a finite optimal solution, then the other problem has a finite optimal solution and their values are the same. If either problem is unbounded, then the other problem has no feasible solution.*

Proof. Since each P_j is a bounded polyhedral convex set, any point in P_j can be written as a convex combination of its extreme points $\{a_j^1, a_j^2, \cdots, a_j^{r_j}\}$. For any column vector $a_j \in P_j$, a_jx_j can be written as

$$a_jx_j = \sum_{i=1}^{r_j} \alpha_i a_j^i x_j = \sum_{i=1}^{r_j} a_j^i x_j^i$$

where $\alpha_i \geq 0, \sum_{i=1}^{r_j} \alpha_i = 1$ and $x_j^i = \alpha_i x_j$. Then $x_j = \sum_{i=1}^{r_j} x_j^i$ and program (2) can be written as

$$\begin{aligned} &\min c_1 \textstyle\sum_{i=1}^{r_1} x_1^i + c_2 \sum_{i=1}^{r_2} x_2^1 + \cdots + c_n \sum_{i=1}^{r_n} x_n^i \\ &\text{s.t.} \textstyle\sum_{i=1}^{r_1} a_1^i x_1^i + \sum_{i=1}^{r_2} a_2^i x_2^i + \cdots + \sum_{i=1}^{r_n} a_n^i x_n^i = b \end{aligned} \tag{4}$$

all $x_j^i \geq 0,\ i = 1, \cdots, r_j,\ j = 1, \cdots, n.$

The linear programming dual of (4) is

$$\begin{aligned} &\max wb \\ &\text{s.t. } wa_j^i \leq c_j,\ i = 1, \cdots, r_j, \qquad j = 1, \cdots, n. \end{aligned} \tag{5}$$

Since $\{a_j^1, a_j^2, \cdots, a_j^{r_j}\}$ generates the convex set P_j, the constraints in (5) can be written as $wA \leq c$ for all $a_j \in P_j$. But then (5) is the same program as (3). Thus we have shown if either (2) or (3) has a finite optimal solution, then the other has a finite optimal solution with the same optimal value. If (2) is unbounded, then (4) is also unbounded, and hence (5) and therefore (3) are unfeasible. The same reasoning shows that (3) unbounded implies that (2) is unfeasible and this concludes the proof.

A tremendous simplification accrues when the activity sets P_j are parallelepipeds in R^m.

COROLLARY. *Let* $P_j = \{a_j \mid \underline{a}_j \leq a_j \leq \bar{a}_j\}$ *and let* $\underline{A}$ *and* $\bar{A}$ *be the matrices with columns* $\underline{a}_j$ *and* $\bar{a}_j$, *respectively. Then the dual of* (2) *is given by*

$$\begin{aligned} &\max w'b - w''b \\ &\text{s.t. } w'\bar{A} - w''\underline{A} \leq c, \qquad w', w'' \geq 0. \end{aligned} \tag{6}$$

Proof. Since P_j is a parallelepiped in R^m, each technological coefficient a_{ij} is bound by $\underline{a}_{ij} \leq a_{ij} \leq \bar{a}_{ij}$. Let a^i be the ith row of the matrix A and let P^i be the corresponding parallelepiped in R^n.

By the previous theorem, the dual of (2) is (3) and for this special case (3) can be written as

$$\max wb \qquad \text{s.t. } wA \le c \text{ where } wA \le c \text{ for } \textit{all } a^i \in P^i. \tag{7}$$

Setting $w = w' - w''$, (7) has the form

$$\max w'b - w''b$$

$$\text{s.t. } w'A - w''A \le c \text{ for all } a^i \in P^i, \qquad w', w'' \ge 0$$

or

$$\max (w', w'') \begin{bmatrix} b \\ -b \end{bmatrix} \qquad \text{s.t. } (w', w'') \begin{bmatrix} A \\ -A \end{bmatrix} \le c \text{ for all } a^i \in P^i, \qquad w', w'' \ge 0. \tag{8}$$

By taking the support functionals for P^i as in [4] and utilizing the transposed form of an inexact linear programming theorem on page 1156 of [4], (8) can be written as

$$\max (w', w'') \begin{bmatrix} b \\ -b \end{bmatrix} \text{ s.t. } (w', w'') \begin{bmatrix} \bar{A} \\ -\underline{A} \end{bmatrix} \le c, \qquad w', w'' \ge 0$$

which is the same as (6), and this concludes the proof.

Program (6) gives an effective method of computing the optimal value of program (2), and the dual variables to (6) give the optimal solution to (2). However, it does not indicate how to compute A^*. This will be done in the next section. The computational efficiency when the P_j are parallelepipeds make it an important case of the duality theory. If we use program (4) or (5) to solve the generalized linear program for these P_j, we must consider all 2^m extreme points for each P_j with the corresponding number of variables or constraints. However, in (6) we need to consider only the largest and smallest extreme points (corners) of each P_j.

2. COMPUTATIONAL STRATEGY, EXAMPLES AND APPLICATIONS

The solution to the generalized linear program (2) is a pair (x^*, A^*). We have shown that (2) is equivalent to (5) and if x_j^i are the primal variables in (4), then $x_j = \sum_{i=1}^{r_j} x_j^i$. Further

$$a_j^* = \begin{cases} \sum_{i=1}^{r_j} (x_j^i/x_j) a_j^i, & x_j \ne 0 \\ \text{any } a_j \text{ in } P_j, & x_j = 0. \end{cases}$$

It can also be shown by contradiction that if the x_j variable is nonzero then a_j^* will be a boundary point of P_j. Hence, for a fixed j, the only x_j^i that can be nonzero are those x_j^i corresponding to a_j^i on a particular face of P_j since otherwise a_j^* would be an interior point of P_j. Consequently,

P_j need not be convex but only each face must be convex. If the a_j^i are determined by a control parameter, say temperature, then linear interpolation between various settings need be valid only on each face. This generalizes the conditions given by Dantzig [1, Ch. 22-2]. However, it is usually difficult to determine convexity of the faces. The first example illustrates the above theory for polyhedral P_j.

Example 1.

$$\min 3x_1 + 2x_2 - 6x_3$$

$$\text{s.t. } a_1x_1 + a_2x_2 + a_3x_3 = \begin{bmatrix} 12 \\ 8 \\ 9 \end{bmatrix} \quad x_1, x_2, x_3 \geq 0 \text{ for some } a_j \in P_j$$

where the extreme points of the constraint sets P_j are given by

P_1: {(2, 2, 3), (1.5, 2.3, 2.4), (3, 2, 2), (2, 1.3, 2), (1.9, 2.7, 2.1), (2.1, 1.8, 1.0)}

P_2: {(1, 1, 1), (4, 3, 2)}, P_3: {(3, 4, 2), (3, 5, 3), (4, 4, 4), (7, 2, 2)}.

Using program (4) with the previous remarks, we get the solution $x_1^1 = 1.00$, $x_3^3 = 1.10$, $x_3^4 = 0.80$ with all other $x_j^i = 0$. Then $x_1^* = 1.0$, $x_2^* = 0$, $x_3^* = 1.9$ and $a_1^* = [2, 2, 3]$, $a_2^* = [2.5, 2, 1.5]$, $a_3^* = [5.74, 2.84, 2.84]$. This example illustrates the previous observation that the only nonzero x_j^i must correspond to extreme points of at most one face of P_j.

The next example considers the special case when P_j are parallelepipeds and uses the corollary to the duality theorem. To solve this special case we will use program (6) or its dual; min cx subject to $\bar{A}x \geq b$, $\underline{A}x \leq b$, $x \geq 0$. Suppose w'^*, w''^*, x^* are the optimal primal and dual solutions of (6). These give the optimal value to the problem and can be used for the optimal setting of A^* if this setting is needed.

Let $\bar{a}^i(\underline{a}^i)$ be the ith row of $\bar{A}(\underline{A})$, $\underline{b}_i = \underline{a}^i \cdot x^*$, and $\bar{b}_i = \bar{a}^i \cdot x^*$. If $\bar{b}_i \neq \underline{b}_i$, let $\lambda = (\bar{b}_i - b_i)/(\bar{b}_i - \underline{b}_i)$ and $1 - \lambda = (b_i - \underline{b}_i)/(\bar{b}_i - \underline{b}_i)$. Then, by examining program (6), it can be seen that the optimal A^* is given by: for $i = 1, \cdots, m$ and $j = 1, \cdots, n$

$$(a_{ij}^*) = \begin{cases} \bar{a}_{ij}, & \text{if } w_i'^* > 0 \\ \underline{a}_{ij}, & \text{if } w_i'^* > 0 \\ \lambda \underline{a}_{ij} + (1 - \lambda)\bar{a}_{ij}, & \text{if } w_i'^* = w_i''^* = 0 \text{ and } \underline{b}_i \neq \bar{b}_i \\ \bar{a}_{ij}, & \text{if } \underline{b}_i = \bar{b}_i. \end{cases} \quad (9)$$

It appears that $w_i'^*$ and $w_i''^*$ may both be positive but this will never happen using the simplex method. The next example will illustrate these ideas.

Example 2. This example is a modified version of a metal blending problem given in [1, pp. 42–50]. The problem is to minimize the cost of producing an alloy of 15% lead, 30% zinc, 20% tin and 35% other materials by blending five alloys. The exact metal content of each alloy can be controlled within given limits, and because of extraneous materials, some alloys consume (cf., minus signs in columns a_j) lead, zinc, or tin in the blending process. Let x_i be the amount of alloy i used to make one pound of the desired alloy. The problem can be formulated

$$\text{minimize } 4.1x_1 + 4.3x_2 + 2.2x_3 + 3.1x_4 + 1.5x_5$$

$$\text{s.t. } a_1x_1 + a_2x_2 + a_3x_3 + a_4x_4 + a_5x_5 = \begin{bmatrix} 1 \\ .15 \\ .3 \\ .2 \end{bmatrix} \quad \text{for some } a_j \in P_j \qquad (10)$$

(material balance equations) $x_1, x_2, x_3, x_4, x_5 \geq 0$, where $[1, 0.1, 0.3, -0.1] \leq a_1 \leq [1, 0.2, 0.35, 0.0]$, $[0.9, 0.1, 0.2, 0.1] \leq a_2 \leq [1, 0.2, 0.3, 0.3]$, $[1, 0.4, -0.3, 0.1] \leq a_3 \leq [1, 0.5, -0.2, 0.3]$, $[1, 0.3, -0.2, -0.2] \leq a_4 \leq [1, 0.4, 0.3, -0.1]$, $[1, 0.5, 0.0, -0.3] \leq a_5 \leq [1, 0.6, 0.2, -0.2]$.

The columns of $\bar{A}(\underline{A})$ are the upper (lower) bounds on the a_j. Using (6) to solve (10) we get $w_3'^* = 11.92$, $w_4'^* = 2.65$, $w_2''^* = 0.70$ and all other variables are 0. The dual variables yield $x_1^* = 0.15$, $x_2^* = 0.75$, $x_5^* = 0.12$, $x_3^* = x_4^* = 0$. Since $w_3'^*$ and $w_4'^*$ are positive, the third and fourth rows of A^* are from $\bar{A}$ while $w_2''^*$ positive implies that the second row of $\underline{A}$ is used. Since $w_1'^* = w_1''^* = 0$, row one of A^* is computed using $\bar{b}_1$ and $\underline{b}_1$. Thus

$$A^* = \begin{bmatrix} 1 & .98 & 1 & 1 & 1 \\ .1 & .1 & .4 & .3 & .5 \\ .35 & .3 & -.2 & .3 & .2 \\ .0 & .3 & .3 & -.1 & -.2 \end{bmatrix}.$$

Also, $A^*x^* = b$ and the optimal value of the problem is 4.00.

The technological coefficients in this example are perhaps more characteristic of a chemical blending problem. The example illustrates problems that involve material balance equations where there is some choice or uncertainty about the technological coefficients. Frequently, industrial processes have a convex set of possible coefficients depending on the catalyst, temperature, pollution controls, etc., and the setting must be chosen. In other cases there is uncertainty about the coefficients and the generalized linear program represents the best possible value of the problem. For additional applications see [1, Ch. 22].

3. RELATED WORK

The approach used here is related to [2], [4] and [5] but we have pursued the optimistic strategy and thus have a partial solution to the comment by Falk [2, p. 787]. Of course, Dantzig [1] presents a solution for the generalized linear program which would solve both of our examples. Dantzig's algorithm requires a starting feasible solution (x, A) and does not give a Phase I procedure to obtain it. Any Phase I procedure must involve the sets P_j and hence is unlikely to be more efficient than our entire method. Even after a feasible solution (x, A) has been obtained for Example 2, the master program in Dantzig's algorithm will be enlarged to at least 11 variables and several smaller programs, whereas our method solved the problem with only 8 variables. Most importantly, coded versions of Dantzig's algorithm are not generally available while we have reduced the generalized linear programming problem to an ordinary linear program. Our approach does require, however, the generation of all extreme points of each polyhedron P_j or the bounds of the parallelepipeds.

The results presented in this paper can be extended to include uncertainty in the cost and demand vectors using an approach similar to [1].

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