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Technical Notes

Finding Some Essential Characteristics of the Feasible Solutions for a Scheduling Problem

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This note is concerned with some essential characteristics of the feasible solutions for a job-shop scheduling problem in which the jobs are constrained by fixed starting times and due dates. These characteristics are related to the scheduling order of operations on each machine and to the fixed starting and completion times for each operation. We present the basic principles enabling one to define these characteristics and propose a procedure for finding them.

THE SCHEDULING of n jobs on m machines in a manufacturing workshop is a problem that has given rise to a great deal of research in the last twenty years.[1,4,5] We can identify two main trends in this research. The first consists of developing methods that enable one to achieve schedules optimizing a certain criterion (minimizing the average length of time taken to perform the jobs, minimizing the time over which the group of machines is occupied, etc.). As a result, certain specific methods have been found that enable the solution of small-scale problems.[7,8] General methods such as linear programming and branch-and-bound[2,9] have been proposed for problems of any dimension; however, they are too unwieldy to use in large-scale problems. In the light of these difficulties the second main trend has been to develop heuristic procedures for establishing schedules[3,6] and to evaluate these procedures according to various criteria.

In this note we try to find the essential characteristics of feasible solutions (orders to be filled between operations, slack times on the operations) subject to certain constraints. This approach seems to be realistic in that the job-shop scheduling problem, as it is often found in production units, is generally at a level where the main criteria have already been taken into account. Hence, the aim of scheduling the operations in the workshop is, essentially, to enable a set of jobs to be performed subject to certain constraints and in the presence of uncertainty. Under these conditions it is important to know how to isolate the essential characteristics of the feasible solutions and, as a result, to have an idea of the amount of freedom available for establishing the schedule. This was the idea that led to the development of the method presented here.

1. FORMULATION OF THE PROBLEM

This paper deals with the 'simple job-shop process,'[4] in which the shop is made up of a certain number of machines that are represented by the integers 1, 2, $\cdots$, m. The jobs are identified by the integers 1, 2, $\cdots$, n.

A job is made up of a set of g_i operations to be carried out in a fixed order, and an operation is able to start only if the one before it is completed. In addition, for each operation it is necessary to use a single, well-defined machine, and a machine can carry out only one operation at a time. The jth operation of a job i, which is called (i, j), is characterized by the following deterministic values: p_{ij}=processing time and m_{ij}=number of machine employed.

Finding a schedule means determining the order in which the operations are carried out to satisfy two sorts of constraints: sequential constraints arising from the jobs [operation (i,j) can start only if operation $(i, j-1)$ is completed] and disjunctive constraints arising from the machines [if $m_{ij}=m_{kl}$, the time intervals during which (i,j) and (k, l) are carried out should be separate from one another].

Constraints on the time allocated to execute the different jobs can be defined by: r_i=earliest starting time of the job i and d_i=due date of the job i.

The aim of this paper is to bring out certain essential characteristics of the set of schedules satisfying the start time and due date constraints when this set is not empty. These characteristics are related to the order in which certain operations are carried out on the same machine, as well as to the definition of the time intervals within which these operations can be performed.

It should be noted that by setting $r_i=0$ and $d_i=F_{\max}$ for all i, we can arrive at a more classical problem (the $n/m/G/F_{\max}$ problem), which consists of developing schedules that minimize $F_{\max}$ or, in the light of the approach proposed here, looking for the smallest value of $F_{\max}$ such that the set of feasible solutions is not empty.

2. DEVELOPMENT OF THE SEQUENCING RELATIONS BETWEEN OPERATIONS USING THE SAME MACHINE

Suppose that it is possible to associate with each operation (i,j) an earliest starting time c_{ij} and a latest completion time f_{ij}. For two operations (i,j), (k,l) such that $m_{ij}=m_{kl}$, the time interval allocated to the ordered pair of operations $[(i,j),\ (k,l)]$ can be defined by

$\Delta_{ij}^{kl}=f_{kl}-c_{ij}$. The sequencing relation between the two operations of the pair $[(i,j),\ (k,l)]$ can be represented by a binary variable X_{ij}^{kl} such that:

$$X_{ij}^{kl}=\begin{cases}1 & \text{if} \quad (i,j) \text{ precedes } (k,l),\\ 0 & \text{if} \quad (k,l) \text{ precedes } (i,j).\end{cases}$$

By comparing the intervals allocated and the sum of the processing times of the operations, we can define the following two relations:

$$\begin{aligned}&\Delta_{kl}^{ij}<p_{ij}+p_{kl}\Rightarrow X_{ij}^{kl}=1;\\ &\Delta_{ij}^{kl}<p_{ij}+p_{kl}\Rightarrow X_{ij}^{kl}=0.\end{aligned} \tag{1}$$

When one of the two relations is satisfied, it is possible to order the two corresponding operations.

We will call the sequencing relation that results 'essential,' for it is not possible to find a schedule that satifies the given constraints and does not correspond to this relation.

Consider now an operation (i,j) and a set of operations $O_{kl}=\{(k_1,l_1),\ (k_2,l_2),\ \cdots,\ (k_q,l_q)\}$ such that $m_{ij}=m_{k_v l_v}$, $v=1,\cdots,q$. It is possible to associate a fictitious operation (k,l) to the set of operations O_{kl} such that $p_{kl}=\sum_{v=1}^{v=q} p_{k_v l_v}$, $c_{kl}=\min_v c_{k_v l_v}$, and $f_{kl}=\max_v f_{k_v l_v}$.

Taking (1) into account, we can establish the following relations:

$$\begin{aligned}&\Delta_{kl}^{ij}<p_{ij}+p_{kl}\Rightarrow\Pi_v\ X_{k_v l_v}^{ij}=0 \quad \text{or} \quad \textstyle\sum_v X_{ij}^{k_v l_v}=1 \quad \text{and}\\ &\Delta_{ij}^{kl}<p_{ij}+p_{kl}\Rightarrow\Pi_v\ X_{ij}^{k_v l_v}=0 \quad \text{or} \quad \textstyle\sum_v X_{k_v l_v}^{ij}=1,\end{aligned} \tag{2}$$

where Π represents the Boolean operator AND and $\sum$ the Boolean operator OR.

These relations enable one to conclude that there necessarily exists a fixed sequencing relation between operation (i,j) and one of the operations of the set O_{kl}. They will be called 'conditional.'

By generalizing from the above, we could compare two sets of operations O_{ij} and O_{kl} belonging to the same machine, but the relations that result do not bear directly on our procedure.

3. SEARCH FOR ESSENTIAL CHARACTERISTICS IN A SIMPLE JOB-SHOP PROCESS

The problem defined in Section 1 is characterized by the existence of a set of operations partially ordered by the sequence constraints. For given values of r_i and d_i one can define earliest starting times and latest completion times for each operation by taking into account the initial sequencing relations:

$$c_{ij}^0 = r_i + \sum_{l=1}^{j-1} p_{il}, \qquad f_{ij}^0 = d_i - \sum_{l=j+1}^{q_i} p_{il}. \tag{3}$$

These times define necessary conditions for feasible solutions of the problem posed. The procedure described below retains this characteristic for the times c_{ij} and f_{ij}.

The search for essential sequencing relations between operations using the same machine can then be carried out with the following iterative procedure:

Consider the iteration z. Each operation is characterized by the times c_{ij}^z, f_{ij}^z.

Since certain pairs of operations using the same machine have been ordered during the previous steps, certain variables X_{ij}^{kl} have been fixed and c_{ij} and f_{ij} have consequently been updated. In order to develop new sequencing relations, it is necessary to apply the relations (1) to each pair $[(i,j),\ (k,l)]$ that is not yet ordered, such that $m_{ij}=m_{kl}$. When this phase enables the construction of new sequencing relations, c_{ij} and f_{ij} are updated, thus producing the data for the following iteration. When no further new sequencing relations can be obtained in this way, it is possible to compare each operation with each pair of operations with which it has not yet been sequenced and to try to apply relations (2) (for $q=2$). These relations do not, strictly speaking, enable one to construct new sequencing relations, but they can bring about a modification in the times c_{ij}, f_{ij} and this in itself can result in the development of new sequencing relations by application of the relations (1). If no new sequencing relation is produced, it is still possible to try to modify the times c_{ij} and f_{ij} by attempting to apply (2) for $q=3, \cdots, (h_m-1)$, where h_m is the number of operations on machine m. The procedure is terminated either when it is no longer possible to modify the times c_{ij} and f_{ij} by the application of (2) or when one encounters a violation of feasibility in one of the two following forms: either $f_{ij}-c_{ij}<p_{ij}$ or both $\Delta_{kl}^{ij}<p_{ij}+p_{kl}$ and $\Delta_{ij}^{kl}<p_{ij}+p_{kl}$, where $[(i,j),\ (k,l)]$ make up a pair of operations that are not yet sequenced with respect to each other. When such a violation arises, there does not exist a schedule that is compatible with the problem constraints.

When the procedure so defined is terminated without feasibility violation, a set of sequencing relations has been defined between operations that take place on the same machine. These sequencing relations are essential for the constraints given by the times r_i and d_i; i.e., if any feasible schedule exists, it must satisfy these relations. However, it is not at all certain that such a feasible schedule exists. Indeed, there may be constraints between sequencing relations that impose new essential sequencing relations or generate a feasibility violation (see Section 4).

Our procedure is represented by the flow chart in Fig. 1. The updating of c_{ij} and f_{ij} can be facilitated by attributing a row s_{ij} to each operation: $s_{ij}=1$ if the operation (i,j) is the first one within the job and on the machine; $s_{ij}=\max_{kl} s_{kl} +1$ with kl taken over preceding operations within the job and on the machine.

The only sequencing relations that exist at iteration 0 are due to constraints on the order of operations carried out on the same job. Thus $s^0_{ij}=j$. As the procedure evolves by (1), new essential sequencing relations appear between operations on the same machine. This has the effect of modifying (increasing) the row of certain operations. This row concept enables us to update the c_{ij} and f_{ij} in a single pass since the updating of a time cannot affect one already updated. We carry out the updating of c_{ij} in the sequence of increasing rows, using the relation $c_{ij}=\max [c(m_{ij}),\ c_{i,j-1}+p_{i,j-1}]$, where $c(m_{ij})$, the earliest starting time of (i, j) when only the constraint arising from the machine m_{ij} is considered, is calculated by programming the operations preceding (i, j) on m_{ij} up to the earliest starting time.

We update f_{ij} in a decreasing rows sequence using the relation $f_{ij}=\min [f(m_{ij}), f_{i,j+1}-p_{i,j+1}]$, where $f(m_{ij})$, the latest completion time of (i,j) when only the constraint arising from the machine m_{ij} is considered, is calculated by programming the operations following (i, j) on m_{ij} up to the latest completion time.

In the case where conditional sequencing relations are found, only the relations that minimize the variation of the times c_{ij} and f_{ij} are taken into account when these quantities are updated. In the course of the procedure the evolution of c_{ij} is monotone and nondecreasing and that of f_{ij} is monotone and nonincreasing.

4. EXAMPLES

We present two extreme examples to illustrate the advantages and the limitations of the method. In the first the choice of constraints is such

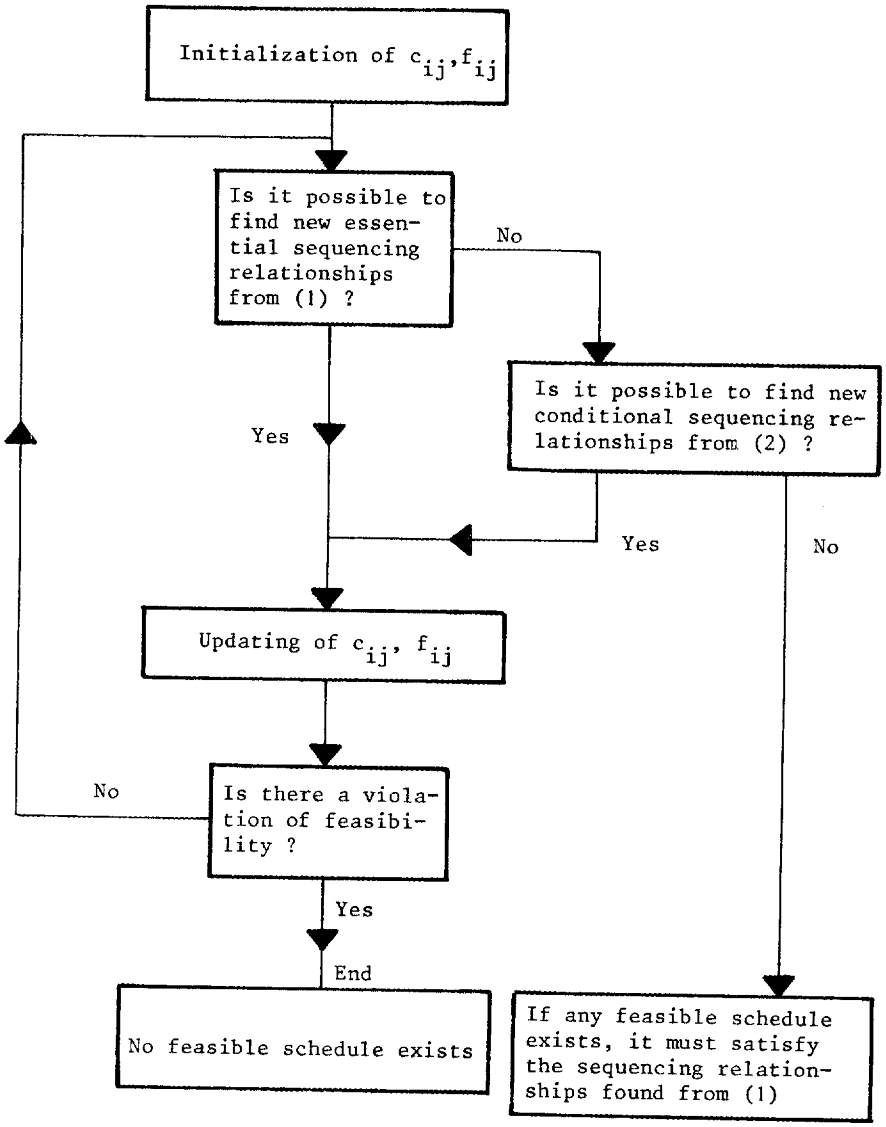

Figure 1

that the procedure directly provides the set of feasible solutions in relation to these constraints. In the second the procedure does not enable the set of feasible solutions to be defined directly, and in this case we present a solution for the whole problem.

Example I

Consider the following 5/3 problem taken from reference 4:

$$[m_{ij}] = \begin{bmatrix} 2 & 3 & 1 & 0 \\ 2 & 1 & 2 & 3 \\ 3 & 1 & 2 & 0 \\ 2 & 3 & 1 & 2 \\ 3 & 2 & 0 & 0 \end{bmatrix}, \qquad [p_{ij}] = \begin{bmatrix} 3 & 6 & 3 & 0 \\ 2 & 5 & 2 & 7 \\ 5 & 7 & 3 & 0 \\ 4 & 6 & 7 & 4 \\ 2 & 6 & 0 & 0 \end{bmatrix},$$

$m_{ij} = p_{ij} = 0$ means that the operation (i, j) does not exist. Suppose that $r_i = 0$ and $d_i = 26$ for all i.

The different steps of the method are represented in Table I, where the

TABLE I

	(i, j)	p_{ij}	s^0_{ij}	c^0_{ij}	f^0_{ij}	s^1_{ij}	c^1_{ij}	f^1_{ij}	s^2_{ij}	c^2_{ij}	f^2_{ij}	s^3_{ij}	c^3_{ij}	f^3_{ij}	s^4_{ij}	c^4_{ij}	f^4_{ij}
$m_{ij} = 1$	(1, 3)	3	3	9	26	3	9	26	4	14	26	5	21	26	5	21	26
	(2, 2)	5	2	2	17	2	2	15	2	2	8	2	2	8	2	2	8
	(3, 2)	7	2	5	23	2	5	15	3	7	15	3	7	15	3	7	15
	(4, 3)	7	3	10	22	3	14	22	4	14	22	4	14	22	4	14	22
	Essential sequencing relations on machine 1		(2, 2) < (4, 3)			(2, 2) < (1, 3)			(4, 3) < (1, 3)								
			(3, 2) < (4, 3)			(3, 2) < (1, 3)											
						(2, 2) < (3, 2)											
$m_{ij} = 2$	(1, 1)	3	1	0	17	1	0	17	1	0	13	2	2	13	3	6	13
	(2, 1)	2	1	0	12	1	0	10	1	0	3	1	0	3	1	0	3
	(2, 3)	2	3	7	19	3	7	19	3	7	19	3	7	19	3	7	19
	(3, 3)	3	3	12	26	3	12	26	4	14	22	4	15	22	5	17	22
	(4, 1)	4	1	0	9	1	0	9	1	0	9	2	2	7	2	2	7
	(4, 4)	4	4	17	26	4	21	26	5	21	26	5	21	26	6	21	26
	(5, 2)	6	2	2	26	2	4	22	2	6	22	3	9	19	4	9	19
	Essential sequencing relations on machine 2		(1, 1) < (3, 3)			(2, 1) < (2, 3)			(2, 1) < (1, 1)			(4, 1) < (1, 1)					
			(1, 1) < (4, 4)			(2, 1) < (5, 2)			(2, 1) < (4, 1)			(2, 3) < (3, 3)					
			(2, 1) < (3, 3)			(3, 3) < (4, 4)			(5, 2) < (3, 3)								
			(2, 1) < (4, 4)						(1, 1) < (5, 2)								
			(4, 1) < (2, 3)														
			(2, 3) < (4, 4)														
			(4, 1) < (3, 3)														
			(4, 1) < (4, 4)														
			(4, 1) < (5, 2)														
			(5, 2) < (4, 4)														
$m_{ij} = 3$	(1, 2)	6	2	3	23	2	3	23	2	5	19	4	12	19	4	13	19
	(2, 4)	7	4	9	26	4	11	26	4	19	26	5	19	26	5	19	26
	(3, 1)	5	1	0	16	1	0	8	1	0	7	1	0	7	1	0	7
	(4, 2)	6	2	4	15	2	4	15	2	5	15	3	6	13	3	7	13
	(5, 1)	2	1	0	20	1	0	16	1	0	16	1	0	13	1	0	7
	Essential sequencing relations on machine 3		(3, 1) < (2, 4)			(1, 2) < (2, 4)			(4, 2) < (1, 2)			(5, 1) < (1, 2)					
			(4, 2) < (2, 4)			(3, 1) < (1, 2)						(5, 1) < (4, 2)					
						(5, 1) < (2, 4)											
						(3, 1) < (4, 2)											

operations are grouped by machine. The essential sequencing relations are written as $(i,j)<(k,l)\Leftrightarrow X_{ij}^{kl}=1$.

For $d_i=26$ no feasibility violation was detected, and at the end of the procedure only three sequencing relations remain undefined. They are those related to the pairs:

$$\left.\begin{array}{l}[(1,1),\ (2,3)]\\ [(2,3),\ (5,2)]\end{array}\right\}\ \text{on machine 2,}$$

$$[(3,1),\ (5,1)]\quad \text{on machine 3.}$$

These free sequencing relations define the 6 feasible schedules. This example has two conditions that are particularly favorable to the application of this method:

(a) the procedure makes it possible to develop all the essential sequencing relations without having to use (2);

(b) there is only a very small number of nondefined sequencing relations, and the essential sequencing relations obtained are in this case necessary and sufficient conditions.

Example II

Consider the following 4/4 problem:

$$[m_{ij}]=\begin{bmatrix}1&2&4\\1&4&1\\3&4&2\\3&2&1\end{bmatrix},\qquad [p_{ij}]=\begin{bmatrix}1&1&1\\1&1&1\\1&1&1\\1&1&1\end{bmatrix}.$$

The application of the procedure described in Section 3 to such a problem only enables one to define the following sequencing relations: $(1,1)<(2,3)$ $(1,1)<(4,3)$ $(2,1)<(2,3)$ $(2,1)<(4,3)$ $(4,2)<(3,3)$ $(1,2)<(3,3)$ $(2,2)<(1,3)$ $(3,2)<(1,3)$.

However, it is easy to see (on a Gantt chart, for example) that the undefined sequences cannot be chosen arbitrarily. Then, to find all the characteristics of the set of feasible solutions, the procedure should be elaborated.

Let (k_1,l_1), (i,j) be two operations such that $m_{k_1l_1}=m_{ij}=m_1$, and let $(i,j+1)$, (k_2,l_2) be two operations such that $m_{i,j+1}=m_{k_2l_2}=m_2\neq m_1$. Suppose that $X_{k_1l_1}^{ij}$ and $X_{ij+1}^{k_2l_2}$ are not defined by the procedure. One can then write

$$[f_{k_2l_2}-c_{k_1l_1}<p_{k_1l_1}+p_{ij}+p_{ij+1}+p_{k_2l_2}]\Rightarrow X_{k_1l_1}^{ij}\cdot X_{ij+1}^{k_2l_2}=0$$

or

$$X_{ij}^{k_1l_1}+X_{k_2l_2}^{ij+1}=1,$$

where + and · represent the Boolean operators OR and AND.

If this relation is applied to the present example, we obtain $(X_{12}^{21}+X_{41}^{12})\cdot(X_{21}^{11}+X_{32}^{22})\cdot(X_{42}^{12}+X_{23}^{43})\cdot(X_{22}^{32}+X_{43}^{23})\cdot(X_{41}^{31}+X_{12}^{42})\cdot(X_{31}^{41}+X_{22}^{32})=1$ whence $X_{21}^{11}\cdot X_{41}^{31}\cdot X_{22}^{32}\cdot X_{42}^{12}=1$, i.e., $X_{21}^{11}=1\Leftrightarrow(2,1)<(1,1)$ $X_{41}^{31}=1\Leftrightarrow(4,1)<(3,1)$ $X_{22}^{32}=1\Leftrightarrow(2,2)<(3,2)$ $X_{42}^{12}=1\Leftrightarrow(4,2)<(1,2)$.
These four sequencing relations are therefore also essential.

It is easy to verify that the sequence of the pair $[(2,3),(4,3)]$ is in fact free, and thus we can define two feasible solutions.

Although the procedure defined here does give necessary conditions, it should in certain cases be elaborated if one wants to find all the characteristics of the set of feasible solutions. It might then be useful to use a Boolean representation to characterize the couplings between the sesequences of jobs on the machines.

5. CONCLUSION

The basic procedure presented here enables us to point out certain essential characteristics of schedules that must satisfy constraints on the starting times and due dates of the jobs. These characteristics include sequencing relations between operations on the same machine and earliest starting times and latest finishing times for each operation. The conditions found in this way are necessary but not sufficient. Indeed, if they are verified, the scheduling is not necessarily feasible. However, since the search for these characteristics was systematic and relatively straightforward, it can serve as a basis for the solution of the scheduling problem.

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Exact Solutions of Inexact Linear Programs

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We address the problem of solving a linear program whose objective function coefficients are known only to lie in a given convex set. We seek a solution that is optimal against the worst possible realization of the objective function, i.e., a max-min solution. We present optimality criteria that characterize the desired solution and strengthen earlier results due to Soyster. The conditions are computationally implementable.

IN THE modeling of a given problem by means of a linear program

$$\min\{cx \mid Ax \leqq b,\ x \geqq 0\}, \tag{1}$$

the coefficients b_i, a_{ij}, and c_j are assumed to be known exactly. In practice, however, this is seldom the case. Sensitivity analysis offers local information near the assumed value of the coefficients. Stochastic and chance-constrained programming approaches do not require exact knowledge of the coefficients but require probability distributions of the random variables whose realizations are the coefficients.

There are numerous situations, however, in which the coefficients are known only to lie in some set described by exact functional relations.