



## Operations Research

Publication details, including instructions for authors and subscription information:  
<http://pubsonline.informs.org>

### Technical Note—Minimax Procedure for a Class of Linear Programs under Uncertainty

R. Jagannathan,

To cite this article:

R. Jagannathan, (1977) Technical Note—Minimax Procedure for a Class of Linear Programs under Uncertainty. Operations Research 25(1):173-177. <https://doi.org/10.1287/opre.25.1.173>

Full terms and conditions of use: <https://pubsonline.informs.org/Publications/Librarians-Portal/PubsOnLine-Terms-and-Conditions>

This article may be used only for the purposes of research, teaching, and/or private study. Commercial use or systematic downloading (by robots or other automatic processes) is prohibited without explicit Publisher approval, unless otherwise noted. For more information, contact [permissions@informs.org](mailto:permissions@informs.org).

The Publisher does not warrant or guarantee the article's accuracy, completeness, merchantability, fitness for a particular purpose, or non-infringement. Descriptions of, or references to, products or publications, or inclusion of an advertisement in this article, neither constitutes nor implies a guarantee, endorsement, or support of claims made of that product, publication, or service.

© 1977 INFORMS

Please scroll down for article—it is on subsequent pages



With 12,500 members from nearly 90 countries, INFORMS is the largest international association of operations research (O.R.) and analytics professionals and students. INFORMS provides unique networking and learning opportunities for individual professionals, and organizations of all types and sizes, to better understand and use O.R. and analytics tools and methods to transform strategic visions and achieve better outcomes.

For more information on INFORMS, its publications, membership, or meetings visit <http://www.informs.org>

## Minimax Procedure for a Class of Linear Programs under Uncertainty

R. JAGANNATHAN

*University of Iowa, Iowa City, Iowa*

(Received original December 11, 1975; final, March 29, 1976)

We consider a linear programming problem with random  $a_{ij}$  and  $b_i$  elements that have known (finite) mean and variance, but whose distribution functions are otherwise unspecified. A minimax solution of the stochastic programming model is obtained by solving an equivalent deterministic convex programming problem. We derive these deterministic equivalents under different assumptions regarding the stochastic nature of the random parameters.

---

IN FORMULATING a stochastic linear programming model, we generally assume definite probability distribution for the parameters ( $A$ ,  $b$ ,  $c$ ) of the model. In this note we avoid making the assumption that the precise form of the probability distribution of the parameters is known. What we assume, however, is that the random  $a_{ij}$  and  $b_i$  elements have known (finite) means and variances. The problem is then to obtain a minimax solution that minimizes the maximum of the objective function over all distributions with the given mean and standard deviation.

The situation of a decision maker facing an unknown probability distribution can be viewed as a zero-sum game against nature. Zuckova [3] proves that the general min-max theorem holds in this case if the set  $\Gamma$  of all possible distributions is assumed to be convex and compact (in the sense of Levy's distance). However, neither the above result nor its proof presents any effective method for finding the value of the game or determining explicit solutions.

In Section 1 we obtain some results that are used later in determining minimax solutions under different assumptions regarding the stochastic nature of the random parameters of a linear programming model. Section 2 considers a stochastic linear programming problem with random RHS elements and obtains a minimax solution of the problem as an optimal solution of an equivalent deterministic convex separable programming problem. Section 3 presents similar results for the case of random  $a_{ij}$  elements.

# 1. PRELIMINARY RESULTS

Let  $\Gamma(\mu, \sigma)$  be the class of distribution functions  $F$  such that  $\int t dF(t) = \mu$  and  $\int t^2 dF(t) = \sigma^2 + \mu^2$ , where  $\mu$  and  $\sigma$  are finite constants. Also, let  $\Gamma_s(\mu, \sigma) \subset \Gamma(\mu, \sigma)$  be the subclass of symmetric distributions and let  $\Gamma_+(\mu, \sigma) \subset \Gamma(\mu, \sigma)$  be the subclass of distributions  $F$  of nonnegative random variables (i.e.,  $F(0-) = 0$ ).

**THEOREM 1.**  $\max_{F \in \Gamma(0, \sigma)} \int_x^\infty (t - x) dF(t) = [(\sigma^2 + x^2)^{1/2} - x]/2$ .

*Proof.* By Schwarz's inequality we have  $[\int_x^\infty (t - x) dF(t)]^2 \leq \int_x^\infty dF(t) \int_x^\infty (t - x)^2 dF(t)$ , where equality occurs if the distribution  $F$  attributes a probability mass  $p$  to a point  $y \geq x$  such that  $p = \int_x^\infty dF(t)$ . Thus the problem of maximizing  $\int_x^\infty (t - x) dF(t)$  reduces to one of choosing  $p$  and  $y$  to maximize  $p(y - x)$  subject to the conditions (i)  $y \geq x$ , (ii)  $\int_{-\infty}^x t dF(t) = -py$ , (iii)  $\int_{-\infty}^x t^2 dF(t) = \sigma^2 - py^2$ , and (iv)  $1 - p = \int_{-\infty}^0 dF(t)$ . Again by the Schwarz inequality,  $[\int_{-\infty}^x t dF(t)]^2 \leq [\int_{-\infty}^x t^2 dF(t)][\int_{-\infty}^x dF(t)]$ , where the equality holds if the distribution  $F$  attributes a probability mass  $(1 - p)$  to a point  $w \leq \min(x, 0)$ . Then  $(1 - p)w + py = 0$  and  $(1 - p)w^2 + py^2 = \sigma^2$ . Therefore, maximizing  $\int_x^\infty (t - x) dF(t)$  subject to  $F \in \Gamma(0, \sigma)$  is equivalent to maximizing  $\{\sigma(p(1 - p))^{1/2} - px\}$  subject to  $0 \leq p \leq 1$ . After some simple calculations the required result follows easily.

**COROLLARY 1.1.**  $\max_{F \in \Gamma(\mu, \sigma)} \int_x^\infty (t - x) dF(t) = ([\sigma^2 + (x - \mu)^2]^{1/2} - (x - \mu))/2$ .

**COROLLARY 1.2.**  

$$\max_{F \in \Gamma_s(\mu, \sigma)} \int_x^\infty (t - x) dF(t) = \begin{cases} (\sigma - x)/2, & \text{if } -\sigma/2 \leq x \leq \sigma/2 \\ \sigma^2/8x, & \text{if } x > \sigma/2 \\ -(\sigma^2 + 8x^2)/8x, & \text{if } x < -\sigma/2 \end{cases}$$

*Proof.* It can be shown, as before, that an optimal  $F$  attributes a probability mass  $p$  to points  $y$  and  $-y$  and a probability mass  $(1 - 2p)$  to the point  $\xi = 0$ . Some simple calculations then yield the required result.

**COROLLARY 1.3.**  

$$\max_{F \in \Gamma_+(\mu, \sigma)} \int_x^\infty (t - x) dF(t) = \begin{cases} ([\sigma^2 + (x - \mu)^2]^{1/2} - (x - \mu))/2, & \text{if } x \geq (\sigma^2 + \mu^2)/2\mu \\ \mu - \mu^2x/(\sigma^2 + \mu^2), & \text{if } 0 \leq x < (\sigma^2 + \mu^2)/2\mu \\ \mu - x, & \text{if } x < 0. \end{cases}$$

*Proof.* In this case an optimal  $F$  attributes a probability mass  $p$  to a point  $y \geq \max(x, \mu)$  and a probability mass  $(1 - p)$  to a point  $(\mu - yp)/(1 - p)$ . The required result follows after some simple calculations.

*Remark.* Note that in all the above cases the maximum value of  $\int_x (t - x) dF(t)$  is nondecreasing in  $\sigma$ . Hence the above results hold even if we redefine  $\sigma^2$  as an upper bound on the variance of the distribution  $F$ . Similarly, in dealing with a stochastic programming problem, we need specify only the mean and an upper bound on the variance of the various random parameters of the model.

## 2. STOCHASTIC LINEAR PROGRAMMING, RANDOM RHS ELEMENTS

Consider a stochastic linear programming problem with simple recourse [1, 2].

$$\min c'x + E \sum (\alpha_i z_i^+ + \beta_i z_i^-) \quad (1)$$

$$a_i'x + z_i^+ - z_i^- = b_i, i = 1, \dots, m, x \in S,$$

where  $c$  is an  $n$ -vector,  $b_i$  is an  $m$ -vector,  $a_i$  is an  $n$ -vector,  $x$  is the unknown  $n$ -vector,  $S$  is a convex polyhedron, and  $b_i$  is a random variable whose distribution function is  $F_i$ .

If  $(\alpha_i + \beta_i) \geq 0, i = 1, \dots, m$ , Problem 1 is equivalent to a deterministic convex separable program:

$$\begin{aligned} \min c'x + \sum \beta_i(y_i - \bar{b}_i) + \sum (\alpha_i + \beta_i) \int_{y_i}^{\infty} t - y_i dF_i(t) \\ a_i'x - y_i = 0, i = 1, \dots, m, x \in S. \end{aligned}$$

Now we can state the deterministic equivalents whose optimal solutions are the minimax solutions for Problem 1.

(a)  $F_i \in \Gamma(\bar{b}_i, \sigma_i)$ .

An application of Corollary 1.1 yields the convex separable program:

$$\begin{aligned} \min c'x + \sum \{\beta_i y_i + (\alpha_i + \beta_i)[(\sigma_i^2 + y_i^2)^{1/2} - y_i]/2\} \\ a_i'x - y_i = \bar{b}_i, i = 1, \dots, m, x \in S. \end{aligned}$$

(b)  $F_i \in \Gamma_s(\bar{b}_i, \sigma_i)$ .

Applying Corollary 1.2 we have the convex separable program:

$$\begin{aligned} \min c'x + \sum \beta_i(y_{i1}^- + y_{i2}^- - y_{i1}^+) \\ + \sum (\alpha_i + \beta_i)[\sigma_i^2/8y_{i1}^+ + y_{i1}^+ - y_{i1}^-/2 \\ - \sigma_i y_{i2}^-/(2\sigma_i + 4y_{i2}^-)] \\ \left. \begin{aligned} a_i'x + y_{i1}^+ - y_{i1}^- - y_{i2}^- &= \bar{\xi}_i \\ y_{i1}^+ &\geq \sigma_i/2 \\ 0 &\leq y_{i1}^- \leq \sigma_i \\ y_{i2}^- &\geq 0 \end{aligned} \right\} i = 1, \dots, m, x \in S. \end{aligned}$$

(c)  $F_i \in \Gamma_+(\bar{b}_i, \sigma_i)$ .

The deterministic equivalent in this case is

$$\begin{aligned} \min c'x + \sum \beta_i (a_i'x - \bar{\xi}_i) \\ + \sum (\alpha_i + \beta_i) [\bar{\xi}_i/2 + y_{i1}^+ - \bar{\xi}_i^2 y_{i1}^- / (\sigma_i^2 + \bar{\xi}_i^2) \\ + (\sigma_i^2 + [(\sigma_i^2 - \bar{\xi}_i^2)/2\bar{\xi}_i + y_{i2}^-]^2)^{1/2} - (\sigma_i^2/2\bar{\xi}_i + y_{i2}^-)] \\ \left. \begin{aligned} a_i'x + y_{i1}^+ - y_{i1}^- - y_{i2}^- &= 0 \\ y_{i1}^+ &\geq 0 \\ 0 \leq y_{i1}^- \leq (\sigma_i^2 + \bar{\xi}_i^2)/2\bar{\xi}_i \\ y_{i2}^- &\geq 0 \end{aligned} \right\} i = 1, \dots, m, x \in S. \end{aligned}$$

### 3. RANDOM $a_{ij}$ ELEMENTS

In this section we again consider the simple recourse model (1), but its  $a_{ij}$  elements are assumed to be random variables. Then  $a_i'x$  is a random variable whose distribution function is denoted as  $F_i(x, t)$ ; i.e.,  $F(x, t) = \Pr(a_i'x \leq t)$ .

Problem (1) can be recast as an equivalent deterministic programming problem:

$$\min_{x,u} c'x + \sum \alpha_i (b_i(b_i - \bar{a}_i'x) + \sum (\alpha_i + \beta_i) \int_{b_i}^{\infty} (t - b_i) dF_i(x, t).$$

Note that the random variable  $a_i'x$  has mean  $\bar{a}_i'x$  and variance  $x'V_i x$ , where  $V_i$  is the covariance matrix of the random vector  $a_i$ .

Again, if  $F_i(x, t) \in \Gamma(\bar{a}_i'x, (x'V_i x)^{1/2})$ , then a minimax solution of Problem (1) can be obtained by solving the convex program:

$$\begin{aligned} \min c'x + \sum \{ \alpha_i y_i + (\alpha_i + \beta_i) [(x'V_i x + y_i^2)^{1/2} - y_i]/2 \} \\ \bar{a}_i'x + y_i = b_i, i = 1, \dots, m, x \in S. \end{aligned}$$

Also, if  $F_i(x, t) \in \Gamma_*(\bar{a}_i'x, (x'V_i x)^{1/2})$ , we can obtain a convex program similar to Problem (2) as a deterministic equivalent.

It is interesting to compare the above problem to the deterministic equivalent of Problem 1, where we assume that  $a_i'x$  is normally distributed with mean  $\bar{a}_i'x$  and variance  $x'v_i x$ :

$$\begin{aligned} \min c'x + \sum \{ \alpha_i y_i + (\alpha_i + \beta_i) \int_{-\infty}^{y_i/\sigma_i} (y_i - \sigma_i u) \phi(u) du \} \\ \left. \begin{aligned} \bar{a}_i'x + y_i &= b_i \\ (x'v_i x)^{1/2} &= \sigma_i \end{aligned} \right\} i = 1, \dots, m, x \in S \end{aligned}$$

where  $\phi(\cdot)$  is a unit normal density function.

### REFERENCES

1. D. W. WALKUP AND R. J. B. WETS, "Stochastic Programs with Recourse: Special Forms," in *Proceeding Princeton Symposium on Mathematics Pro-*

- gramming*, pp. 139–161, H. KURN (ed.), Princeton University Press, Princeton, N. J., 1970.
2. A. C. WILLIAMS, "Approximation Formulas for Stochastic Linear Programming," *SIAM J. Appl. Math.* **14**, 668–677 (1966).
  3. J. ZACKOVA, "On Minimax Solutions of Stochastic Linear Programming Problems," *Casopis pro Pěstování. Matematiky* **91**, 423–430 (1966).

Copyright 1977, by INFORMS, all rights reserved. Copyright of Operations Research is the property of INFORMS: Institute for Operations Research and its content may not be copied or emailed to multiple sites or posted to a listserv without the copyright holder's express written permission. However, users may print, download, or email articles for individual use.