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Technical Note—Minimax Procedure for a Class of Linear Programs under Uncertainty

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Minimax Procedure for a Class of Linear Programs under Uncertainty

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We consider a linear programming problem with random a_{ij} and b_i elements that have known (finite) mean and variance, but whose distribution functions are otherwise unspecified. A minimax solution of the stochastic programming model is obtained by solving an equivalent deterministic convex programming problem. We derive these deterministic equivalents under different assumptions regarding the stochastic nature of the random parameters.

IN FORMULATING a stochastic linear programming model, we generally assume definite probability distribution for the parameters (A, b, c) of the model. In this note we avoid making the assumption that the precise form of the probability distribution of the parameters is known. What we assume, however, is that the random a_{ij} and b_i elements have known (finite) means and variances. The problem is then to obtain a minimax solution that minimizes the maximum of the objective function over all distributions with the given mean and standard deviation.

The situation of a decision maker facing an unknown probability distribution can be viewed as a zero-sum game against nature. Zackova [3] proves that the general min-max theorem holds in this case if the set Γ of all possible distributions is assumed to be convex and compact (in the sense of Levy's distance). However, neither the above result nor its proof presents any effective method for finding the value of the game or determining explicit solutions.

In Section 1 we obtain some results that are used later in determining minimax solutions under different assumptions regarding the stochastic nature of the random parameters of a linear programming model. Section 2 considers a stochastic linear programming problem with random RHS elements and obtains a minimax solution of the problem as an optimal solution of an equivalent deterministic convex separable programming problem. Section 3 presents similar results for the case of random a_{ij} elements.

1. PRELIMINARY RESULTS

Let $\Gamma(\mu, \sigma)$ be the class of distribution functions F such that $\int t\, dF(t) = \mu$ and $\int t^2\, dF(t) = \sigma^2 + \mu^2$, where μ and σ are finite constants. Also, let $\Gamma_s(\mu, \sigma) \subset \Gamma(\mu, \sigma)$ be the subclass of symmetric distributions and let $\Gamma_+(\mu, \sigma) \subset \Gamma(\mu, \sigma)$ be the subclass of distributions F of nonnegative random variables (i.e., $F(0-) = 0$).

THEOREM 1. $\max_{F\in\Gamma(0,\sigma)} \int_x^\infty (t - x)\, dF(t) = [(\sigma^2 + x^2)^{1/2} - x]/2$.

Proof. By Schwarz's inequality we have $[\int_x^\infty (t - x)\, dF(t)]^2 \leqq \int_x^\infty dF(t) \int_x^\infty (t - x)^2\, dF(t)$, where equality occurs if the distribution F attributes a probability mass p to a point $y \geqq x$ such that $p = \int_x^\infty dF(t)$. Thus the problem of maximizing $\int_x^\infty (t - x)\, dF(t)$ reduces to one of choosing p and y to maximize $p(y - x)$ subject to the conditions (i) $y \geqq x$, (ii) $\int_{-\infty}^x t\, dF(t) = -py$, (iii) $\int_{-\infty}^x t^2\, dF(t) = \sigma^2 - py^2$, and (iv) $1 - p = \int_{-\infty}^{x-0} dF(t)$. Again by the Schwarz inequality, $[\int_{-\infty}^x t\, dF(t)]^2 \leqq [\int_{-\infty}^x t^2\, dF(t)][\int_{-\infty}^x dF(t)]$, where the equality holds if the distribution F attributes a probability mass $(1 - p)$ to a point $w \leqq \min(x, 0)$. Then $(1 - p)w + py = 0$ and $(1 - p)w^2 + py^2 = \sigma^2$. Therefore, maximizing $\int_x^\infty (t - x)\, dF(t)$ subject to $F \in \Gamma(0, \sigma)$ is equivalent to maximizing $\{\sigma(p(1 - p))^{1/2} - px\}$ subject to $0 \leqq p \leqq 1$. After some simple calculations the required result follows easily.

COROLLARY 1.1. $\max_{F\in\Gamma(\mu,\sigma)} \int_x^\infty (t - x)\, dF(t) = ([\sigma^2 + (x - \mu)^2]^{1/2} - (x - \mu))/2$.

COROLLARY 1.2.

$$\max_{F\in\Gamma_s(\mu,\sigma)} \int_x^\infty (t - x)\, dF(t) = \begin{cases} (\sigma - x)/2, & \text{if } -\sigma/2 \leqq x \leqq \sigma/2 \\ \sigma^2/8x, & \text{if } x > \sigma/2 \\ -(\sigma^2 + 8x^2)/8x, & \text{if } x < -\sigma/2 \end{cases}$$

Proof. It can be shown, as before, that an optimal F attributes a probability mass p to points y and $-y$ and a probability mass $(1 - 2p)$ to the point $\xi = 0$. Some simple calculations then yield the required result.

COROLLARY 1.3.

$$\max_{F\in\Gamma_+(\mu,\sigma)} \int_x^\infty (t - x)\, dF(t) = \begin{cases} ([\sigma^2 + (x - \mu)^2]^{1/2} - (x - \mu))/2, & \text{if } x \geqq (\sigma^2 + \mu^2)/2\mu \\ \mu - \mu^2 x/(\sigma^2 + \mu^2), & \text{if } 0 \leqq x < (\sigma^2 + \mu^2)/2\mu \\ \mu - x, & \text{if } x < 0. \end{cases}$$

Proof. In this case an optimal F attributes a probability mass p to a point $y \geqq \max(x, \mu)$ and a probability mass $(1 - p)$ to a point $(\mu - yp)/(1 - p)$. The required result follows after some simple calculations.

Remark. Note that in all the above cases the maximum value of $\int_x (t - x)\, dF(t)$ is nondecreasing in σ. Hence the above results hold even if we redefine σ^2 as an upper bound on the variance of the distribution F. Similarly, in dealing with a stochastic programming problem, we need specify only the mean and an upper bound on the variance of the various random parameters of the model.

2. STOCHASTIC LINEAR PROGRAMMING, RANDOM RHS ELEMENTS

Consider a stochastic linear programming problem with simple recourse [1, 2].

$$\min c'x + E\sum (\alpha_i z_i^+ + \beta_i z_i^-) \tag{1}$$

$$a_i'x + z_i^+ - z_i^- = b_i,\ i = 1, \cdots, m,\ x \in S,$$

where c is an n-vector, b_i is an m-vector, a_i is an n-vector, x is the unknown n-vector, S is a convex polyhedron, and b_i is a random variable whose distribution function is F_i.

If $(\alpha_i + \beta_i) \geqq 0, i = 1, \cdots, m$, Problem 1 is equivalent to a deterministic convex separable program:

$$\min c'x + \sum \beta_i(y_i - \bar{b}_i) + \sum (\alpha_i + \beta_i) \int_{y_i}^{\infty} t - y_i)\, dF_i(t)$$

$$a_i'x - y_i = 0,\ i = 1, \cdots, m,\ x \in S.$$

Now we can state the deterministic equivalents whose optimal solutions are the minimax solutions for Problem 1.

(a) $F_i \in \Gamma(\bar{b}_i, \sigma_i)$.

An application of Corollary 1.1 yields the convex separable program:

$$\min\ c'x + \sum \{\beta_i y_i + (\alpha_i + \beta_i)[(\sigma^2 + y_i^2)^{\frac{1}{2}} - y_i]/2\}$$

$$a_i'x - y_i = \bar{b}_i,\ i = 1, \cdots, m,\ x \in S.$$

(b) $F_i \in \Gamma_s(\bar{b}_i, \sigma_i)$.

Applying Corollary 1.2 we have the convex separable program:

$$\begin{aligned}\min c'x + \sum \beta_i(y_{i1}^- + y_{i2}^- - y_{i1}^+) & \\ + \sum (\alpha_i + \beta_i)[\sigma_i^2/8y_{i1}^+ + y_{i1}^+ - y_{i1}^-/2 & \\ - \sigma_i y_{i2}^-/(2\sigma_i + 4y_{i2}^-)] &\end{aligned} \tag{2}$$

$$\left.\begin{aligned} & a_i'x + y_{i1}^+ - y_{i1}^- - y_{i2}^- = \xi_i \\ & y_{i1}^+ \geqq \sigma_i/2 \\ & 0 \leqq y_{i1}^- \leqq \sigma_i \\ & y_{i2}^- \geqq 0 \end{aligned}\right\} i = 1, \cdots, m, x \in S.$$

(c) $F_i \in \Gamma_+(\bar{b}_i, \sigma_i)$.

The deterministic equivalent in this case is

$$\min c'x + \sum \beta_i(a_i'x - \xi_i) + \sum (\alpha_i + \beta_i)[\xi_i/2 + y_{i1}^+ - \xi_i^2 y_{i1}^-/(\sigma_i^2 + \xi_i^2) + (\sigma_i^2 + [(\sigma_i^2 - \xi_i^2)/2\xi_i + y_{i2}^-]^2)^{\frac{1}{2}} - (\sigma_i^2/2\xi_i + y_{i2}^-)]$$

$$\left.\begin{aligned} & a_i'x + y_{i1}^+ - y_{i1}^- - y_{i2}^- = 0 \\ & y_{i1}^+ \geqq 0 \\ & 0 \leqq y_{i1}^- \leqq (\sigma_i^2 + \xi_i^2)/2\xi_i \\ & y_{i2}^- \geqq 0 \end{aligned}\right\} i = 1, \cdots, m, x \in S.$$

3. RANDOM a_{ij} ELEMENTS

In this section we again consider the simple recourse model (1), but its a_{ij} elements are assumed to be random variables. Then $a_i'x$ is a random variable whose distribution function is denoted as $F_i(x, t)$; i.e., $F(x, t) = Pr(a_i'x \leqq t)$.

Problem (1) can be recast as an equivalent deterministic programming problem:

$$\min_{x \in S} c'x + \sum \alpha_i(b_i(b_i - \bar{a}_i'x) + \sum (\alpha_i + \beta_i) \int_{b_i}^{\infty} (t - b_i)\, dF_i(x, t).$$

Note that the random variable $a_i'x$ has mean $\bar{a}_i'x$ and variance $x'V_ix$. where V_i is the covariance matrix of the random vector a_i.

Again, if $F_i(x, t) \in \Gamma(\bar{a}'x, (x'V_ix)^{\frac{1}{2}})$, then a minimax solution of Problem (1) can be obtained by solving the convex program:

$$\min c'x + \sum \{\alpha_i y_i + (\alpha_i + \beta_i)[(x'V_ix + y_i^2)^{\frac{1}{2}} - y_i]/2\}$$
$$\bar{a}_i'x + y_i = b_i, i = 1, \cdots, m, x \in S.$$

Also, if $F_i(x, t) \in \Gamma_s(\bar{a}x, (x'V_ix)^{\frac{1}{2}})$, we can obtain a convex program similar to Problem (2) as a deterministic equivalent.

It is interesting to compare the above problem to the deterministic equivalent of Problem 1, where we assume that a'_ix is normally distributed with mean $\bar{a}'_ix$ and variance $x'v_ix$:

$$\min c'x + \sum \{\alpha_i y_i + (\alpha_i + \beta_i) \int_{-\alpha}^{y_i/\sigma_i} (y_i - \sigma_i u)\, \phi(u)\, du\}$$

$$\left.\begin{aligned} \bar{a}_i'x + y_i &= b_i \\ (x'v_ix)^{\frac{1}{2}} &= \sigma_i \end{aligned}\right\} i = 1, \ldots, m, x \in S$$

where $\phi(\cdot)$ is a unit normal density function.

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