Operations Research

Publication details, including instructions for authors and subscription information:
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Technical Note—Multidimensional Sequencing Rule

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REFERENCE

1. E. Brodheim, C. Derman, and G. Prastacos, "On the Evaluation of a Class of Inventory Policies for Perishable Products Such as Blood," to appear in *Management Sci.*
2. A. F. Veinott, Jr., "Optimal Policy in a Dynamic, Single Product, Nonstationary Inventory Model with Several Demand Classes," *Opns. Res.* **13,** 761–778 (1965).

Multidimensional Sequencing Rule

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(Received original May 8, 1974; final, August 5, 1974)

This note derives a simple heuristic sequencing rule for determining a minimum discounted cost development sequence of multiple-purpose projects that simultaneously satisfy a number of increasing demands over a finite planning horizon. An evaluation of the heuristic's performance on the solution of a number of real-world water resources problems indicates that it can be an extremely effective planning aid.

WE CONSIDER a sequencing problem that arises in the capital expenditure planning of capacity expansion[5] and the exploitation of natural resources.[11] Suppose that we have identified a finite set Ψ of N potential independent multiple-purpose development projects and have estimated both the capital costs and the capacities (outputs) of each of the projects. Each multiple-purpose project $i\epsilon\Psi$ can be described by an $(M+1)$-tuple $(C_i, Q_{i1}, Q_{i2}, \cdots, Q_{iM})$ in which $C_i(>0)$ is the capital cost of the ith project at its completion date and $Q_{ij}(\geqq 0)$ is the capacity (output) of the ith project to satisfy the jth demand, $1\leqq j\leqq M$. Let $D_j(t)$ denote the projected increase in the jth demand from time 0 to time $t\epsilon[0, T]$ and let $r(>0)$ denote the continuous interest rate. For ease of exposition, we assume that for each $1\leqq j\leqq M$ the demand projection $D_j(t)$ is continuous and strictly increasing on $[0, T]$ with $D_j(0)=0$ and $D_j(T)=\sum_{i\epsilon\Psi} Q_{ij}$, and that all the projected demands must be satisfied entirely by output from projects in Ψ.

The *multidimensional project sequencing problem* is to determine a development sequence of the N multi-purpose projects that provides the capacity to satisfy each of the M demands over the entire planning horizon at the minimum discounted cost. That is, we wish to find a sequence φ so as to

$$\min_{\varphi \epsilon S} \sum_{i=1}^{i=N} C_{\varphi(i)} \exp\{-rt_{\varphi(i)}\}, \tag{1}$$

where S denotes the set all $N!$ permutations of the projects in Ψ and $t_{\varphi(i)}$ denotes completion time of the ith project in the sequence given by:

$$t_{\varphi(i)} = t[\psi(i-1)] = \min_j \{D_j^{-1}(\textstyle\sum_{k\epsilon\psi(i-1)} Q_{kj})\} \quad (i=1, 2, \cdots, N) \tag{2}$$

In (2) $\psi(i-1) \subset \Psi$ is the unordered subset whose $(i-1)$ elements are the projects $[\varphi(1), \varphi(2), \cdots, \varphi(i-1)]$, and D_j^{-1} is the (time) inverse of the jth demand projection. Simply stated, (2) expresses the fact that each project should be completed as late as possible without violating any of the M demand requirements. Notice that the completion time of the ith project in the sequence is dependent upon the identity of the $(i-1)$ projects preceding it, but not upon their sequence.

Dynamic programming has recently been proposed as a solution procedure for one-dimensional $(M=1)$ special cases of (1)[1,2,4] as well as for the general multidimensional project sequencing problem considered here.[5,7] However, as a result of the combinatorial nature of the problem, optimum-seeking algorithms such as dynamic programming are seriously inhibited in the solution of even moderate $(N \geqq 15)$ size problems. In Section 1 we develop a simple heuristic sequencing rule that surmounts this computational difficulty and that can be applied to sequencing problems involving practically any number of multiple-purpose projects. The heuristic's performance on a number of real-world problems is summarized in Section 2.

1. A SEQUENCING RULE BASED UPON NECESSARY CONDITIONS FOR ADJACENT PROJECTS

Let Φ^* denote the set of optimal solutions to the multidimensional project sequencing problem. For a sequence φ^* to be an optimal solution to problem (1) it is necessary that an interchange (transposition) of any two adjacent projects in φ^* does not result in a lowering of the discounted cost; that is, $\varphi^* \epsilon \Phi^*$ only if for each $0 \leqq n \leqq N-2$, we have

$$C_{\varphi^*(n+1)} \exp\{-rt[\psi^*(n)]\} + C_{\varphi^*(n+2)} \exp\{-rt[\psi^*(n) \cup \varphi^*(n+1)]\}$$
$$\leqq C_{\varphi^*(n+2)} \exp\{-rt[\psi^*(n)]\} + C_{\varphi^*(n+1)} \exp\{-rt[\psi^*(n) \cup \varphi^*(n+2)]\}.$$

Multiplying the above inequality by $\exp\{rt[\psi^*(n)]\}$ and rearranging yields the following necessary condition for any pair $\{\varphi^*(n+1), \varphi^*(n+2)\}$ of adjacent projects in an optimal sequence φ^*:

$$C_{\varphi^*(n+1)}/[1-\exp\{-r[t\{\psi^*(n) \cup \varphi^*(n+1)\} - t\{\psi^*(n)\}]\}]$$
$$\leqq C_{\varphi^*(n+2)}/[1-\exp\{-r[t\{\psi^*(n) \cup \varphi^*(n+2)\} - t\{\psi^*(n)\}]\}]. \tag{3}$$

Any sequence $\varphi \epsilon S$ that satisfies (3) for all $0 \leqq n \leqq N-2$ will be referred to as a *locally optimal sequence*.[3] The heuristic sequencing rule is based upon the construction of a locally optimal sequence.

For any n-project subset $\psi(n) \subset \Psi$ and for any $i \epsilon [\Psi - \psi(n)]$ define

$$\lambda_i[\psi(n)] = \begin{cases} C_i/[1-\exp\{-r[t\{\psi(n) \cup i\} - t\{\psi(n)\}]\}], \\ \text{if } t[\psi(n) \cup i] > t[\psi(n)], \\ L, \text{ otherwise,} \end{cases} \qquad (4)$$

where L is some sufficiently large real number.

A locally optimal sequence can be constructed as follows:

Step 1. Set $n=0$ and $\psi(0)=\emptyset$.

Step 2. Select the $(n+1)$st project $\varphi(n+1)$ in the sequence as the project $i \epsilon [\Psi - \psi(n)]$ with the lowest $\lambda_i[\psi(n)]$ value.

Step 3. If $n=N-2$, stop—the Nth project, $\varphi(N)$, is $[\Psi - \psi(N-1)]$. Otherwise, set $\psi(n) = \psi(n) \cup \varphi(n+1)$, $n=n+1$ and return to Step 2.

We shall refer to Steps 1 through 3 as the $\lambda_i[\psi(n)]$ *heuristic sequencing rule*. Basically, it is a multidimensional continuous version of the ingenious heuristic sequencing technique presented by Tsou, Mitten, and Russell[8] (see also references 2, 3, and 6).

2. PERFORMANCE ON SOME REAL-WORLD PROBLEMS

The $\lambda_i[\psi(n)]$ heuristic sequencing rule was coded in FORTRAN IV and implemented on Northwestern University's CDC 6400. The code was applied to a number of 2- and 3-dimensional project sequencing problems arising in water resources development in both the Ohio River Basin[9] and some of the major Texas river basins.[10] (Specific details on the sources, nature and use of these data can be found in reference 5.) The performance of the $\lambda_i[\psi(n)]$ rule is summarized in Table I.

Inspection of Table I reveals that the heuristic rule produced an optimal sequence in 16 out of the 20 problems. Furthermore, in three of the other four problems the discounted cost of the locally optimal sequence constructed by the heuristic never exceeded that of an optimal permutation sequence by as much as 1 percent The only problem (No. 15) in which the discounted cost of the locally optimal sequence significantly differed (10.62 per cent) from that of an optimal sequence was atypical in the sense that half (5) of the projects had 0 capacity to satisfy one of the demand requirements. We also note that Tsou, Mitten, and Russell[8] have reported even more favorable results with a one-dimensional discrete version of the $\lambda_i[\psi(n)]$ rule that produced an optimal solution to every one of the 24 problems to which it was applied. Furthermore, heuristic sequencing rules based upon average cost per unit of capacity performed rather poorly on the problems of Table I. (The interested reader is referred to reference 6 for an analysis of the optimality and an evaluation of the performance of these and other heuristic sequencing rules.) Finally, we note that the total execution (CPU) time required to apply the $\lambda_t[\psi(n)]$ rule to all 20 problems of Table I was less than 2 seconds, with the largest problem (No. 12) requiring only 0.253 seconds—as opposed to the 336.255

TABLE I
COMPUTATIONAL EXPERIENCE

Problem number	Number of projects N	Number of demand requirements M	PV(φ^*), discounted cost of an optimal sequence ($1,000)	Discounted cost of the locally optimal sequence produced by the $\lambda_t[\psi(n)]$ rule [% of PV(φ^*)]
1	7	3	113,504.29	100.31
2	8	3	255,291 59	100.00
3	2	3	219,515.53	100.00
4	10	3	76,739.51	100.99
5	4	3	31,595.91	100.00
6	10	3	56,014.54	100.00
7	6	3	69,405.20	100.00
8	5	3	95,852.59	100.00
9	4	3	74,246.09	100.00
10	6	3	48,503.42	100.00
11	3	3	147,502.41	100.00
12	11	3	228,600 48	100.93
13	4	3	181,456.56	100.00
14	9	3	275,855.67	100.00
15	10	3	152,751.10	110.62
16	6	2	107,762.50	100.00
17	4	2	78,346.80	100.00
18	2	2	34,606.00	100.00
19	2	2	62,020.60	100.00
20	4	2	99,721.67	100.00

seconds it took to obtain an optimal solution to Problem 12 by dynamic programming.[5]

3. DISCUSSION

The $\lambda_t[\psi(n)]$ heuristic sequencing rule has a number of advantages over exact solution techniques for multidimensional project sequencing problems. It is simple and easy to use and can be applied to practically any size problem. The computational effort is on the order of $MN(N+1)/2$, and the high-speed storage requirement is on the order of N. In contrast, the computational effort required for solution by dynamic programming is on the order of $MN2^{N-1}$, and the high-speed storage requirement is on the order of $\max_n \binom{N}{n}$. The $\lambda_i[\psi(n)]$ rule has also proved to be extremely effective on the solution of a number of real-world problems. For these reasons the $\lambda_i[\psi(n)]$ rule is probably preferable to exact solution techniques (even when they can be applied) whenever, as is often the case in preliminary planning stages, there is significant uncertainty about the magnitude of the costs, capacities, or demand projections. Similar heuristic rules may also prove to be useful in the solution of multidimensional project sequencing problems in which shortages are allowed to occur and in the solution of the more general combined multidimensional project selection and sequencing problem.

REFERENCES

1. W. S. Butcher, Y. Y. Haimes, and W. A. Hall, "Dynamic Programming for the Optimal Sequencing of Water Supply Projects," *Water Resources Res.* **5,** 1196–1204 (1969).
2. D. Erlenkotter, "Sequencing Expansion Projects," *Opns. Res.* **21,** 542–553 (1973).
3. L. G. Mitten and C. A. Tsou, "Efficient Solution Procedures for Certain Scheduling and Sequencing Problems," presented at the Symposium on Scheduling and its Applications, North Carolina State University, Raleigh, N. C., May, 1972.
4. T. L. Morin, "Pathology of a Dynamic Programming Sequencing Algorithm," *Water Resources Res.* **9,** 1178–1185 (1973).
5. ———, "Optimal Sequencing of Capacity Expansion Projects," *Proc. ASCE, J. Hydraulics Div.* **99,** 1605–1622 (1973).
6. ———, "Myopic Sequencing Rules," *Discussion Paper,* The Technological Institute, Northwestern University, Evanston, Illinois, November, 1973.
7. ——— and A. M. O. Esogbue, "The Imbedded State Space Approach to Reducing Dimensionality in Dynamic Programs of Higher Dimensions," *J. Math. Anal. and Appl.* **48,** 801–810 (1974).
8. C. A. Tsou, L. G. Mitten, and S. O. Russell, "Search Technique for Project Sequencing," *Proc. ASCE, J. Hydraulics Div.* **99,** 833–839 (1973).
9. U.S. Army Corps of Engineers, *Ohio River Basin Comprehensive Survey,* Vols. I-XIV, U.S. Army Engineer Division, Ohio River, Cincinnati, Ohio, 1969.
10. U.S. House of Representatives, U.S. Study Commission on the Neches, Trinity, Brazos, Colorado, Guadalupe, San Antonio, Nueches, and San Jacinto River Basins and Intervening Areas, *The Report of the U.S. Commission-Texas,* Parts I-IV, House Document No. 494, 87th Congress, 2nd Session, March, 1962.
11. R. V. Williams and P. K. Chiu, "Optimal Alternative Selection, Sequencing and Timing to Satisfy Multi-Demand Functions," presented at the XIX International TIMS Meeting, Houston, Texas, April, 1972.