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Technical Note—On Converse Duality in Nonlinear Programming

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fore seems worthwhile to investigate whether this process of cross-fertilization could be carried further to develop more efficient algorithms for general-integer and mixed-integer programming problems.

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ON CONVERSE DUALITY IN NONLINEAR PROGRAMMING

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HUARD AND MANGASARIAN have proved the converse duality theorem in mathematical programming by using the KUHN-TUCKER theorem. However, this note points out the advantages of using FRITZ JOHN's theorem.

HUARD[3] AND MANGASARIAN[5, 6] have proved the converse duality theorem in mathematical programming by using the KUHN-TUCKER theorem.[4] How-

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ever, this note points out the advantages of using instead FRITZ JOHN's theorem,[1, 8] by which a new and simpler proof is obtained. As in reference 6 [page 118, Theorem 6.1 (i)], no KUHN-TUCKER constraint qualification is required.

NOTATION AND PREVIOUS RESULTS

R^n DENOTES Euclidean n-space. f, g, and h are differentiable functions from R^n to R, R^m, and R^k respectively. ∇f denotes the gradient vector of f with respect to x, or u as context requires. ∇g denotes the matrix whose i, jth element is $\partial g_j/\partial x_i$. ∇h is defined similarly. Note that, with $y \epsilon R^m$,

$$\nabla y^t g = (\nabla g) y. \tag{1}$$

Thus, $\nabla^2 f \equiv \nabla(\nabla f)$ is the matrix whose i, jth element is $(\partial/\partial x_i)(\partial f/\partial x_j)$.

The Fritz John theorem [1] states that a necessary condition for x_0 to be a solution of the problem: maximize $f(x)$ subject to $g(x) \geqq 0$ is that there exist $\tau \epsilon R$, $u \epsilon R^n$ such that

$$\tau \nabla f(x_0) + \nabla u^t g(x_0) = 0, \tag{2}$$

$$u^t g(x_0) = 0, \tag{3}$$

$$(\tau, u) \geqq 0, \tag{4}$$

$$(\tau, u) \neq 0. \tag{5}$$

The generalized Fritz John theorem[8] states that a necessary condition for x_0 to be a solution of the problem: maximize $f(x)$ subject to $g(x) \geqq 0$, $h(x) = 0$ is that there exist $\tau \epsilon R$, $u \epsilon R^m$, $v \epsilon R^k$ such that, in addition to (3) and (4),

$$\tau \nabla f(x_0) + \nabla u^t g(x_0) + \nabla v^t h(x_0) = 0, \tag{2'}$$

$$(\tau, u, v) \neq 0. \tag{5'}$$

CONVERSE DUALITY

CONSIDER THE following two nonlinear programs and their corresponding duals:

PRIMAL	DUAL
P1	D1
Minimize $f(x)$ subject to $g(x) \geqq 0$.	Maximize $f(u) - y^t g(u)$
	s.t. $\nabla f(u) - \nabla y^t g(u) = 0$,
	$y \geqq 0$.
P2	D2
Minimize $f(x)$	Maximize $f(u) - y^t g(u) - u^t \nabla [f(u) - y^t g(u)]$
s.t. $g(x) \geqq 0$,	s.t. $\nabla f(u) - \nabla y^t g(u) \geqq 0$,
$x \geqq 0$.	$y \geqq 0$.

Here f is assumed to be a convex twice-differentiable function from R^n to R, and g a concave twice-differentiable function from R^n to R^m.

We shall make use of the weak duality theorem,[9] i.e., the infimum of the primal is greater than or equal to the supremum of the corresponding dual.

THEOREM. *If (u_0, y_0) is an optimal solution of D1 or D2 and the matrix*

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$$\nabla^2 f(u_0) - \nabla^2 y_0{}^t g(u_0) \tag{6}$$

is nonsingular, then $x_0 = u_0$ *is an optimal solution of the corresponding primal, and the extrema are equal.*

Proof. (a) If (u_0, y_0) is optimal for D1, by the generalized Fritz John theorem, there exist $\tau\epsilon R$, $v\epsilon R^n$, $s\epsilon R^m$, such that

$$\tau[\nabla f(u_0) - \nabla y_0{}^t g(u_0)] + \nabla v^t[\nabla f(u_0) - \nabla y_0{}^t g(u_0)] = 0, \tag{7}$$

$$s^t y_0 = 0, \tag{8}$$

$$(\tau, s) \geqq 0, \tag{9}$$

$$(\tau, v, s) \neq 0, \tag{10}$$

$$-\tau g(u_0) - [\nabla g(u_0)]^t v + s = 0. \tag{11}$$

The equality constraint of D1 plus (7) gives [noting (1)]

$$[\nabla^2 f(u_0) - \nabla^2 y_0{}^t g(u_0)]v = 0.$$

The nonsingularity of (6) thus assures that $v=0$. From (11), if $\tau=0$, then $s=0$, thus $(\tau, v, s)=0$, contradicting (10); hence $\tau>0$. Eliminating v, s, and τ, we obtain from (8), (9), and (11),

$$g(u_0) = s/\tau \geqq 0, \tag{12}$$

$$[g(u_0)]^t y_0 = y_0{}^t g(u_0) = 0. \tag{13}$$

Thus $x_0 = u_0$ is feasible for the primal, and $f(x_0) = f(u_0) + y_0{}^t g(u_0)$. This, with the weak duality theorem, assures that $x_0 = u_0$ is optimal for P1.

(b) In dealing with D2 we need only the Fritz John (not the generalized Fritz John) theorem. Thus, if (u_0, y_0) is optimal for D2, there exist $\tau\epsilon R$, $v\epsilon R^n$, $s\epsilon R^m$ such that, in addition to (8) and (10),

$$[\nabla^2 f(u_0) - \nabla^2 y_0{}^t g(u_0)](\tau u_0 - v) = 0, \tag{14}$$

$$(\tau, v, s) \geqq 0, \tag{15}$$

$$v^t[\nabla f(u_0) - \nabla y_0{}^t g(u_0)] = 0, \tag{16}$$

$$\tau g(u_0) + \nabla g(u_0)(\tau u_0 - v) - s = 0, \tag{17}$$

$$s^t y_0 = 0. \tag{18}$$

The nonsingularity of (6), and (14), give $v = \tau u_0$. So, from (17), if $\tau=0$, then $s=0$, so also $v=0$, contradicting (10). Hence $\tau>0$, so (17) and (15) give (12), and (13) follows, using (18). Substituting τu_0 for v in (16) and dividing by τ give

$$u_0{}^t[\nabla f(u_0) - \nabla y_0{}^t g(u_0)] = 0. \tag{19}$$

The feasibility of $x_0 = u_0$ for the primal P2 follows from (12) and (15), since $u = \tau^{-1} v \geqq 0$. The optimality follows from (13), (19), and the weak duality theorem.

REMARKS

Huard[3] proves the same converse dual theorem for D1 assuming that the Kuhn-Tucker (K-T) qualification is satisfied. Mangasarian,[5] although he uses the K-T conditions to obtain the converse dual theorem for D1, does not, it would

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seem, assume the K-T qualification. He does, however, state clearly, when dealing with problem D2, that the K-T qualification must be assumed. By using Fritz John conditions, we eliminate this assumption in both instances. Thus, although the K-T conditions are needed for the WOLFE duality theorem,[9] they are not necessary to prove converse duality. (This is also shown in reference 6, but the proof is more complicated.)

Other proofs of converse duality with different assumptions are given in HANSON[2] AND MANGASARIAN AND PONSTEIN.[7] While it is possible to relate our assumptions to those of references 2 and 7, which also do not need the K-T constraint qualification, the use of Fritz John conditions leads to a simpler and more direct approach to the theorem obtained here.

That the Fritz John conditions, instead of K-T conditions, can, in some instances, be used to prove duality theorems was first noticed by DANTZIG, EISENBERG, AND COTTLE.[10] Here we use this idea to give a simple proof of the converse duality result of references 5 and 3.

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