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T. C. Hu, M. L. Lenard,

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Optimality of a Heuristic Solution for a Class of Knapsack Problems

T. C. HU
University of California, La Jolla, California
M. L. LENARD
University of Wisconsin, Madison, Wisconsin

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This paper presents a simpler proof for a result of Magazine, Nemhauser, and Trotter, which states recursive necessary and sufficient conditions for the optimality of a heuristic solution for a class of knapsack problems.

A RECENT paper by Magazine, Nemhauser, and Trotter[3] gives recursive necessary and sufficient conditions for the optimality of the greedy solution to a class of knapsack problems. We present a simpler proof of one of their main theorems, using an argument we developed in reference 1 for the special case where all cost coefficients are equal to one. We also give some useful sufficient conditions and pose a related problem.

We consider the problem of finding the minimum weight of coins needed to make an amount of change b, given n kinds of coins, where the value of the ith coin is a_i and its weight is c_i. Thus, we wish to solve the knapsack problem

$$\min \sum_{i=1}^{i=n} c_i x_i,\ \sum_{i=1}^{i=n} a_i x_i = b,\ x_i \geqq 0,\quad \text{integers } i=1,\cdots,n. \tag{1}$$

We assume throughout that

(i) a_i and c_i are integers, $1 \leqq i \leqq n$,

(ii) $c_1/a_1 \geqq \cdots \geqq c_n/a_n$, and

(iii) $1 = a_1 < a_i$, $2 \leqq i \leqq n$.

Define the function $F_k(y)$ as follows: $(1 \leqq k \leqq n,\ 0 \leqq y \leqq b)$,

$$F_k(y) = \min \sum_{i=1}^{i=k} c_i x_i,\ \sum_{i=1}^{i=k} a_i x_i = y,\ x_i \geqq 0,\quad \text{integers, } i=1,\cdots,k. \tag{2}$$

Note that $F_n(b)$ is the solution to (1). The function $F_k(y)$ is the minimum weight of coins necessary when the amount of change is y and the first k kinds of coins $(a_1, a_2, \cdots, a_k)$ are allowed. The problem (2) has a trivial solution when $k=1$, i.e., $F_1(y)=c_1\lfloor y/a_1\rfloor=c_1y$, where $\lfloor x\rfloor$ is the integer part of x. In general we have the recursive definition of $F_k(y)$:

$$F_k(y)=\min_{x_k=0,1,\cdots,\lfloor y/a_k\rfloor}[F_{k-1}(y-a_kx_k)+c_kx_k] \quad k=2,3,\cdots,n, \tag{3}$$

and

$$F_1(y)=c_1\lfloor y/a_1\rfloor=c_1y.$$

A heuristic algorithm for solving (1), also known as the 'greedy' algorithm, would use the largest possible number of the nth kind of coin (that is, the one with least weight per unit value), then the largest possible number of coin $n-1$, and so on.

Define $H_k(y)$ to be the weight of coins required in the heuristic algorithm when the amount of change is y and only the first k kinds of coins are used. It is easy to see that $H_1(y)=F_1(y)$. Now we want to know: For what values of a_{k+1} would the heuristic algorithm work for all y? This question is answered by the following theorem.

THEOREM 1 (Also Theorem 1 in reference 3). *Suppose $H_k(y)=F_k(y)$ for all positive integers y and some fixed k. If $a_{k+1}>a_k$ and p and δ are the unique integers for which $a_{k+1}=pa_k-\delta$ and $0\leqq\delta<a_k$, then the following are equivalent.*

(a′) $H_{k+1}(y) \leqq H_k(y)$ *for all positive integers y,*

(a) $H_{k+1}(y) = F_{k+1}(y)$ *for all positive integers y,*

(b) $H_{k+1}(pa_k) = F_{k+1}(pa_k)$,

(c) $c_{k+1}+H_k(\delta)\leqq pc_k$.

Proof. The method of proof will be (a′)⇒(a)⇒(b)⇒(c)⇒(a′). (It is the inclusion of statement (a′) that simplifies the proof.)

From (a′) and the optimality of F_{k+1} we have

$$H_k(y)\geqq H_{k+1}(y)\geqq F_{k+1}(y) \quad \text{for all positive integers } y. \tag{4}$$

Note that the number of the $(k+1)$st kind of coin used in $H_{k+1}(y)$ is always equal to or greater than x_{k+1}, where x_{k+1} is the number of the $(k+1)$st kind of coin used in $F_{k+1}(y)$.

Let $y'=y-x_{k+1}a_{k+1}$. Clearly,

$$F_{k+1}(y')=F_k(y')=H_k(y'). \tag{5}$$

Since (4) applies to y', it follows from (5) that $H_{k+1}(y')=F_{k+1}(y')$. Adding $a_{k+1}x_{k+1}$ to both sides, we have $H_{k+1}(y)=F_{k+1}(y)$, which is (a). Then, choosing $y=pa_k$, we have (b).

Next, since the optimal value with $k+1$ coins can be no larger than the optimal value with k coins, $F_{k+1}(y)\leqq F_k(y)=H_k(y)$ for all positive integers

y. As a consequence, (b) implies

$$H_{k+1}(pa_k) \leqq H_k(pa_k). \tag{6}$$

Evaluating both sides of inequality (6), we have $c_{k+1}+H_k(\delta) \leqq pc_k$, which is (c).

It remains to prove that (c)$\Rightarrow$(a′). We shall prove [not (a′)]$\Rightarrow$[not (c)]. Suppose $\bar{y}$ is the *smallest* integer for which (a′) fails. Obviously $\bar{y} > a_{k+1}$; hence $H_k(\bar{y}) < H_{k+1}(\bar{y}) = c_{k+1}+H_{k+1}(\bar{y}-a_{k+1})$. Adding $H_k(\delta)$ to both sides of the above inequality yields

$$c_{k+1}+H_k(\delta)+H_{k+1}(\bar{y}-a_{k+1}) > H_k(\delta)+H_k(\bar{y}). \tag{7}$$

Since the heuristic algorithm is optimal for k coins,

$$H_k(\delta)+H_k(\bar{y}) \geqq H_k(\bar{y}+\delta). \tag{8}$$

Since $\bar{y}+\delta = (a_{k+1}+\delta)+(\bar{y}-a_{k+1}) = pa_k+(\bar{y}-a_{k+1})$,

$$H_k(\bar{y}+\delta) = pc_k+H_k(\bar{y}-a_{k+1}). \tag{9}$$

Combining (7), (8), and (9), we have

$$c_{k+1}+H_k(\delta) > pc_k+H_k(\bar{y}-a_{k+1})-H_{k+1}(\bar{y}-a_{k+1}). \tag{10}$$

By assumption, (a′) holds for $y<\bar{y}$; in particular, (a′) holds for $y=\bar{y}-a_{k+1}$. Thus, it follows from (10) that $c_{k+1}+H_k(\delta) > pc_k$. This is the negation of (c), and the proof is complete.

Next we prove a corollary stating sufficient conditions for the optimality of the heuristic algorithm.

COROLLARY 1. *Under the hypotheses of Theorem* 1, *if*

$$c_{k+1} \leqq pc_k-\delta c_1, \tag{11}$$

then $H_{k+1}(y)=F_{k+1}(y)$ *for all positive integers* y.

Proof. Because of the optimality of H_k, $H_k(\delta) \leqq \delta H_k(1) = \delta c_1$, and condition (c) of Theorem 1 is satisfied.

We note that since $c_k/a_k \leqq c_1$, (11) implies

$$c_{k+1} \leqq [pa_k-\delta c_1(a_k/c_k)][c_k/a_k] \leqq [pa_k-\delta][c_k/a_k] = a_{k+1}(c_k/a_k).$$

That is, (11) is also sufficient to maintain the ordering of the variables required by assumption (ii).

A special case of Corollary 1 where $c_i=1$, $i=1, \cdots, k$ and $c_{k+1} \leqq 1$ appears as Corollary 4 in the paper by Magazine et al.[3] This result for the same special case was also obtained by Kernighan and Johnson.[2] (The major thrust of the Kernighan and Johnson paper is somewhat different, however, in that they consider conditions for the optimality of H_{k+1} without requiring the optimality of H_k.)

Our second corollary establishes a lower limit on a_{k+1} sufficient for optimality of the heuristic algorithm. (A related result appears as Corollary 5 in Magazine et al.[3])

COROLLARY 2. *If* $c_1 = c_k = c_{k+1} = 1$, *then for any* $a_{k+1} \geqq m_k$, *where* $m_k = (a_k - 1)^2 + 1$, $H_{k+1}(y) = F_{k+1}(y)$ *for all positive integers* y.

Proof. For this special case, the hypothesis of Corollary 1 reduces to $\delta \leqq p-1$. First, we see that the value of m_k may be written as $m_k = (a_k - 1)a_k - (a_k - 2)$. Thus, for values of a_{k+1} in the interval from m_k to $m_k + (a_k - 2)$, the hypothesis of Corollary 1 is satisfied with $p = a_k - 1$. Beyond this point—that is, for values of a_{k+1} satisfying $a_{k+1} > m_k + (a_k - 2) = (a_k - 1)a_k$—we will have $p \geqq a_k$. Since by definition $\delta < a_k$, $p \geqq a_k$ implies that $\delta \leqq p - 1$; and again, the hypothesis of Corollary 1 is satisfied.

Finally, we pose an open question related to the change-making problem. Suppose there are N amounts of change to be made b_j, $j = 1, \cdots, N$. If one is allowed to use only m kinds of coins (where $m < N$), what values of a_i, $i = 1, \cdots, m$ should be chosen such that the total *number* (i.e., $c_i = 1$, $i = 1, \cdots, m$) of coins used in all N transactions is a minimum? The question should have a neat answer when $b_j = j$, $j = 1, \cdots, N$.

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