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Technical Note—Strengthened Dantzig Cuts for Integer Programming

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Strengthened Dantzig Cuts for Integer Programming

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In 1959, Dantzig proposed a particularly simple cut for integer programming. However, in 1963, Gomory and Hoffman showed that, in general, this cut does not provide a finite algorithm. In 1968, Bowman and Nemhauser showed that a slightly modified version of the Dantzig cut does provide a finite procedure. We show how this latter cut can be strengthened through the use of group-theoretic techniques.

CONSIDER the integer program IP: max $z=c'x$, subject to $Ax=b$, $x\geqq 0$ and integer. We assume that all the components of A, b, and c are themselves integers. Dropping the integrality requirements on x leaves the associated linear program LP. Let B be the LP optimal basis. Partition A as (B, N), and c and x accordingly. Then Tucker's optimal tableau is

$$\begin{array}{ll|l} & & -x'_N \\ \hline z=c_B B^{-1}b & & c_B' B^{-1}N-c_N', \\ x_B=B^{-1}b & & B^{-1}N, \\ x_N=0 & & -I. \end{array}$$

Denote the typical element of the tableau by y_{ij}, $i=0, 1, \cdots, m+n; j=0, 1, \cdots, n$. Let f_{ij} $(=y_{ij}-[y_{ij}])$ be its fractional part. Let y_j be the jth column of the tableau, and f_j its fractional part. The corresponding quantities in the new tableau resulting from a single pivot are denoted y'_{ij}, f'_{ij}, y_j', and f_j'.

If this tableau is not feasible for IP, i.e., if at least one of the basic variables is noninteger, then at least one of the nonbasic variables must be positive in any IP optimal solution. Accordingly, the sum of the nonbasic variables must be at least unity in such a solution. Thus, in 1959, DANTZIG proposed using the cut

$$\sum_j x_{Nj} \geqq 1, \tag{1}$$

where x_{Nj} is the jth nonbasic variable.[4] We refer to (1) as the Dantzig cut. Reference 4 gives no proof that an algorithm based solely on this cut would converge to the IP optimum. In 1963, GOMORY AND HOFFMAN proved that, in general, the Dantzig cuts would not converge.[5]

In (1), the summation is taken over all j. In 1961, CHARNES AND COOPER[3] noted that, for any choice of i, the sum could be restricted to those j such that $y_{ij} \neq 0$, and in 1962 BEN-ISRAEL AND CHARNES[1] remarked that the sum could be further restricted to those j such that $f_{ij} \neq 0$. Neither reference 1 nor reference 3 gave any results concerning the convergence of these modified Dantzig cuts and the nonconvergence proof of reference 5 could not be applied here. In 1968, BOWMAN AND NEMHAUSER[2] showed that the cuts of references 1 and 3 did yield convergent algorithms. In this note we present two other modified Dantzig cuts that also guarantee convergence, and that are stronger than the cut of reference 1.

Choose as 'source row' for the cut the first row such that $f_{i0} \neq 0$. Once a source row has been chosen, the elements of this row can be used to deduce cuts similar to, but stronger than (1). Let

$$\delta_{ij} = \begin{cases} 1, & \text{if } f_{ij} \neq 0, \\ 0, & \text{if } f_{ij} = 0. \end{cases}$$

Then the cut of reference 1, which we refer to as the "MD1 cut," is defined by

$$\sum_j \delta_{ij} x_{Nj} \geqq 1. \tag{2}$$

In reference 2, Bowman and Nemhauser show that the MD1 cut can be expressed as the sum of two of Gomory's Method-of-Integer-Forms (MIF) cuts. The MD1 cut is also related to the Gomory All-Integer (AI) cut. If the source row is used to generate an MIF cut, and that cut is used to generate an AI cut with $\lambda = 1$, the resulting AI cut is the MD1 cut; the details of this may be found in references 6 and 9. While Gomory used the AI cuts only in an all-integer tableau, which is certainly not the case here, the construction is, nonetheless, identical.

For an exact statement of the MD1 algorithm and a proof of its convergence, we refer the reader to reference 2 Although convergent, the algorithm is quite slow. To remedy this, we now develop two methods by which the MD1 cuts can be strengthened.

It is well known that the slack in an MIF cut must also be an integer variable. Thus the condition

$$\sum_j f_{ij} x_{Nj} \equiv f_{i0} (\text{mod } 1) \tag{3}$$

must hold for all $i = 0, 1, \cdots, m+n$. Suppose that row i is the first row with $f_{i0} \neq 0$. Suppose further that no $f_{ij} = f_{i0}$ for $j = 1, \cdots, n$. Then $\sum_{j=1}^{j=n} \delta_{ij} x_{Nj} \geqq 2$ is a valid cut, because if only one x_{Nj} (for $f_{ij} \neq 0$) is positive and if that $x_{Nj} = 1$, the left-hand side of (3) is f_{ij}, and $f_{ij} \neq f_{i0}$, so (3) is not satisfied. This new cut we call the "MD2 cut." It is clearly stronger than the MD1 cut.

Suppose, on the other hand, that $f_{ij_1}=f_{i0}$, and that no other $f_{ij}=f_{i0}$. Scan the 0th and j_1th columns. If there is some k such that $f_{k0}\neq f_{kj_1}$, we may again conclude that the MD2 cut is valid, since the nonbasic variables must sum to at least 2 in order to satisfy (3) in both rows i and k. If there are several columns with $f_{ij}=f_{i0}$, scan them all. If no one satisfies (3) for all $i=0,\cdots,m+n$, again we conclude that the MD2 cut is valid.

The use of this cut gives us the MD2 algorithm, which differs from the MD1 algorithm in one respect: whenever the MD1 cut can be so strengthened, we use the stronger MD2 cut in its place. A repetition of the proofs in reference 2 suffices to show this result:

LEMMA 1. *With the usual boundedness assumptions, the MD2 algorithm is finite.*

The MD2 algorithm may still be quite slow. This is easily shown. Suppose that $y_{00}=98/100$ and that each $y_{0j}=1/100$. Then it will require exactly 49 MD2 cuts to drive y_{00} to 0, whereas exactly one MIF cut would have sufficed to do the same thing. On the other hand, if $y_{00}=2/100$ and each $y_{0j}=98/100$, one MD2 cut will drive y_{00} to $-194/100$, whereas one MIF cut would drive y_{00} only to 0. It would thus seem desirable to have a mixed MIF-MD2 algorithm, which always considered each of the two cuts and actually added the one that produced the greatest lexicographic decrease in the vector y_0. This can be determined very easily. Ties may be broken arbitrarily, or else preference may be given to the MIF cut, since it will reduce D, the determinant of the current basis.

We cannot show that the mixed MIF-MD2 algorithm is uniformly stronger (in terms of the number of cuts required before convergence) than either pure algorithm, but we suspect that it is a reasonable computational heuristic. Moreover, minor revisions to the proof of Lemma 1 show that the mixed algorithm is finite.

For a further strengthening of the modified Dantzig cuts, we consider Gomory's asymptotic algorithm[7] with the Dantzig cut as objective. Here we solve the problem

$$\min \textstyle\sum_j \delta_{ij}x_{Nj}, \quad \text{subject to} \quad \sum_j f_{rj}x_{Nj}\equiv f_{r0}(\bmod 1), \qquad r=0,1,\cdots,m+n,$$

and $x_N\geqq 0$ and integer.

Let k be this minimum value. Then the "MDk cut" is $\sum_j \delta_{ij}x_{Nj}\geqq k$. The MD$k$ algorithm is identical to the MD1 algorithm, except that it uses the MDk cut. In addition, this algorithm may locate feasible solutions to IP in the course of solving the asymptotic problems that yield the cuts. Keeping track of the best integer solution found to date will allow for termination at any point where it is felt that possible improvement is not sufficient to justify further calculations.

In general, we expect the MDk cuts to be stronger than the MD1 cuts, i.e., we expect to have $k>1$. Of course, some of the MDk cuts may be identical to the MD1 cuts. The smaller the values of k, the weaker the cuts and hence the slower the algorithm. We have the following negative result in that direction.

Let row i be the source row, j^* be the pivot column, and $\Delta=\{j|f_{ij}\neq 0\}$. After the pivot, let row i' be the next source row and $\Delta'=\{j|f'_{i'j}\neq 0\}$. It will often be the case that $\Delta'\subset\Delta$. In particular, this will happen if rows $0,1,\cdots,i-1$ are all integer, and if the new tableau is primal feasible.

THEOREM 1. *Let the MDk cut be* $\sum_{j\in\Delta} x_{Nj}\geqq k$. *Let* x_N^0 *be the solution to the group problem that produced the number* k. *If* $\Delta'\subset\Delta$, *then the next MDk cut will be* $\sum_{j\in\Delta'} x_{Nj}\geqq k'$, *where* k' *does not exceed* $k-x^0_{Nj^*}-\sum_{j\in\Delta-\Delta',j\neq j^*} x^0_{Nj}$.

Proof. The tableau after the pivot is given by: $y_0'=y_0-ky_{j^*}$; $y_j'=y_j-y_{j^*}$, $j\in\Delta$, $j\neq j^*$; $y_j'=y_j$, $j\notin\Delta$ or $j=j^*$. Taking fractional parts we get: $f_0'\equiv f_0-kf_{j^*}$; $f_j'\equiv f_j-f_{j^*}$, $j\in\Delta$, $j\neq j^*$; $f_j'\equiv f_j$, $j\notin\Delta$ or $j=j^*$, where all congruences are modulo 1. To get the next cut, we must solve

$$\min \textstyle\sum_{j\in\Delta'} x_{Nj} \quad \text{subject to} \quad f_0'\equiv \sum_{j=1}^{j=n} f_j' x_{Nj}, \qquad x_N\geqq 0 \text{ and integer.} \quad (4)$$

To prove the theorem, it suffices to show that x_N^1, defined by

$$x_{Nj}^1=\begin{cases} x_{Nj}^0, & \text{if } j\neq j^*,\\ 0, & \text{if } j=j^*,\end{cases}$$

is feasible for (4).

Now x_N^1 is certainly a nonnegative integer vector. Also

$$\begin{aligned}\textstyle\sum_{j=1}^{j=n} f_j' x_{Nj}^1 &\equiv \textstyle\sum_{j\in\Delta,\, j\neq j^*} (f_j-f_{j^*}) x_{Nj}^1+\sum_{j\notin\Delta} f_j x_{Nj}^1+f_{j^*}x_{Nj^*}^1\\ &= \textstyle\sum_{j=1}^{j=n} f_j x_{Nj}^1 - f_{j^*}\sum_{j\in\Delta,\, j\neq j^*} x_{Nj}^1\\ &= \textstyle\sum_{j=1}^{j=n} f_j x_{Nj}^0 - f_{j^*}x_{Nj^*}^0 - f_{j^*}\sum_{j\in\Delta,\, j\neq j^*} x_{Nj}^0\\ &= \textstyle\sum_{j=1}^{j=n} f_j x_{Nj}^0 - f_{j^*}\sum_{j\in\Delta} x_{Nj}^0 \equiv f_0-kf_{j^*}\equiv f_0'.\end{aligned}$$

The thrust of the theorem is that, the longer we retain row i as the source row and hence have $\Delta'\subset\Delta$, the weaker the MDk cuts will tend to get in the sense that the constants k will tend to decrease. Thus, if D is large and f_{00} is close to one, we might expect to require many cuts to drive f_{00} to zero, and, in general, we cannot expect to get a strong MD type cut by repeatedly using the asymptotic algorithm. Analogous to the mixed MIF-MD2 algorithm, we propose a mixed MIF-MDk algorithm in which we compute the MDk cut, compare it to the MIF cut, and use whichever one is stronger. This guarantees that i will remain the source row only if at least one of the variables $y_{00}, y_{10}, \cdots, y_{i-1,0}$ decreases below its current integer part. In this fashion we will be able to avoid the difficulty established in Theorem 1.

We have had no large-scale computational experience with these algorithms. We do know, for example, that the problem $\max(5x_1+8x_2+6x_3)$ subject to $9x_1+6x_2+10x_3\leqq 14$, $20x_1+63x_2+10x_3\leqq 110$, $x\geqq 0$ and integer, requires 52 MIF cuts; but only 9 mixed MIF-MDk cuts before converging. However, what is gained in terms of the number of cuts may be lost in terms of the additional calculation necessary to obtain the stronger cuts. The schemes presently available for solving the asymptotic problem[7,8] are computationally impractical when the determinant of the basis matrix is large. The development of schemes that can solve large asymptotic problems efficiently will greatly enhance the attractiveness of the algorithms we have presented.

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Theorem 1, as stated above, is stronger than our original version. We would like to thank the referee who called this strengthened version to our attention.

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A Note on Zero-One Integer and Concave Programming

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It has been shown that the zero-one linear integer programming problem can be formulated as a minimization of a concave quadratic objective function subject to linear constraints. This note extends the concept to a broad class of concave objective functions.

A RECENT paper by RAGHAVACHARI[1] (who attributes this also to Whinston) shows that the zero-one integer programming problem:

$$\min z=cx, \quad \text{subject to} \quad Ax\leqq b, \qquad x_j=0,1 \qquad (j=1,\cdots,n) \quad (P1)$$

can be reformulated as minimizing the following problem consisting of a concave quadratic objective function subject to linear constraints:

$$\min z=cx+Mx'(e-x), \quad \text{subject to} \quad Ax\leqq b \qquad 0\leqq x\leqq e,$$

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