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The EOQ Model under Stochastic Lead Time

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We consider a continuous deterministic-demand, stochastic lead-time inventory model such that the individual unit demands are non-interchangeable. We derive equations that define the optimal values of the two decision variables: order size and timing This model is shown to be a stochastic lead-time generalization of the EOQ model with backlogging of demand. An illustrative example is presented Finally, a lower bound, which is independent of the order size, is developed for the optimal ordering time.

A WELL-KNOWN stochastic extension of the classical economic order quantity (EOQ) inventory model bases the reorder decision on the stock level. This is the familiar (R, Q) model: When the stock level reaches R (in continuous time), Q units are ordered (see Hadley and Whitin [2], Wagner [4]). To achieve mathematical tractability, it is necessary to assume that there is at most one replenishment order outstanding at any instant.

When lead times are stochastic and independently distributed, it is not always possible to establish that there is at most one outstanding order. However, the "interval between placing orders is usually large enough so that there is essentially no interaction between orders" [2, p. 203] or the probability of crossover is small enough to be ignored.

Another approach was taken by Washburn [5], who formulated and solved a continuous-review inventory problem. Demand is assumed deterministic while lead times are stochastic. His decision variables are the amount of demand time satisfied by each order and the time to order (see Figure 1). To circumvent the crossover problem, Washburn assumes the unit demands are non-interchangeable. That is, each unit is a "special order." If demands are in fact interchangeable, errors occur to the extent that orders are likely to cross, as in the Hadley-Whitin approach.

Washburn's model formulation differs from the mainstream of tradi-

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tional EOQ modeling in several respects. First, all costs are subject to continuous time discounting, and the inventory holding costs are not stated explicitly but are merged into the time discount rate. Second, the lead-time probability distribution, $G(\tau)$ is permitted to have a "defect", $\lim_{\tau\to\infty} G(\tau) \leqq 1$. This allows the possibility that an order may never arrive. Finally, although the model has a continuous time orientation, the inventory shortage cost is expressed as dollars/unit arrived late (or never arrived) in consonance with the previous assumption. This particular set of assumptions complicates the search for the optimal values of the decision variables and may yield non-unique solutions. The purpose of this note is to formulate and solve a stochastic lead-time generalization of the EOQ model with backlogging of demand under the non-interchangeable parts of assumption. This approach unifies the EOQ literature

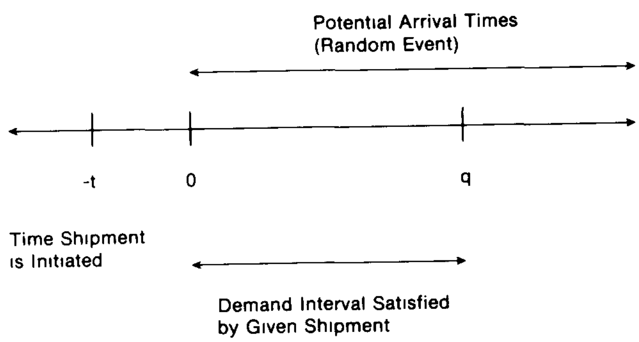

Figure 1. Time scale for one cycle in the steady-state continuous demand model.

and yields optimality equations that are easily solved and uniquely define the decision variables (see also Liberatore [3], Chapter 6).

1. THE MODEL

Define

D = constant demand rate (units/unit time demanded);

q = number of time units of demand satisfied by each order;

t = time differential between placing an order and the start of the q time units that will be satisfied by the given order (see Figure 1).

The ordering costs are $K+vq$ for $q>0$, 0 otherwise. The inventory holding and backlogging costs are c_1, c_2, respectively, expressed in dollars/unit/unit time. Define $g(\tau)$ as the lead-time probability mass or density function, with distribution function $G(\cdot)$. The expected total cost of each order is

$$ETC(t, q) = K + vq + \int_0^t \{c_1 Dq(t-\tau) + c_1 Dq^2/2\} g(\tau)\, d\tau + \int_t^{t+q} \{c_2 D(\tau - t)^2/2 + c_1 D(t+q-\tau)^2/2\} g(\tau)\, d\tau + \int_{t+q}^{\infty} \{c_2 Dq^2/2 + c_2 Dq(\tau - t - q)\} g(\tau)\, d\tau. \quad (1)$$

Our objective is to minimize $EAC(t, q) = ETC(t, q)/q$, the expected total cost per unit time.

THEOREM. *Let D, q and t be defined as previously stated. Then the following equations*

$$2K/(c_1+c_2)D + \int_{t^*}^{t^*+q^*} (\tau - t^*)^2 g(\tau)\, d\tau = (q^*)^2 \left\{ \int_0^{t^*+q^*} g(\tau)\, d\tau - c_2/(c_1+c_2) \right\} \quad (2)$$

$$\int_{t^*}^{t^*+q^*} (\tau - t^*) g(\tau)\, d\tau = q^* \left\{ \int_0^{t^*+q^*} g(\tau)\, d\tau - c_2/(c_1+c_2) \right\} \quad (3)$$

define the unique global minima for t and q.

Proof. Dividing (1) by q and taking the first partial derivatives with respect to t and q yield:

$$\partial EAC(t, q)/\partial q = -K/q^2 + (c_1 D/2) \int_0^t g(\tau)\, d\tau + \int_t^{t+q} \{(-c_2 D(\tau - t)^2/2q^2) + (c_1 D/2)(1 - [(\tau - t)^2/q^2])\} g(\tau)\, d\tau - (c_2 D/2) \int_{t+q}^{\infty} g(\tau)\, d\tau \quad (4)$$

$$\partial EAC(t, q)/\partial t = c_1 D \int_0^t g(\tau)\, d\tau + \int_t^{t+q} \{(-c_2 D(\tau - t)/q) + c_1 D(t+q-\tau)/q\} g(\tau)\, d\tau - c_2 D \int_{t+q}^{\infty} g(\tau)\, d\tau. \quad (5)$$

Setting the first derivative equations equal to 0 and solving will yield unique global minima for t and q if the EAC function is strictly convex over the relevant domains. A sufficient condition for strict convexity is positive diagonal elements and a positive determinant of the Hessian matrix (see, e.g., Zangwill [6]). These conditions can be expressed as

$$\partial^2 EAC(t, q)/\partial q^2,\ \partial^2 EAC(t, q)/\partial t^2 > 0 \tag{6a}$$

$$(\partial^2 EAC(t, q)/\partial q^2)(\partial^2 EAC(t, q)/\partial t^2) - [\partial^2 EAC(t, q)/\partial q \partial t]^2 > 0. \tag{6b}$$

The second partials are

$$\partial^2 EAC(t, q)/\partial q^2 = \{K + (c_1 + c_2) D \int_t^{t+q} (\tau - t)^2 g(\tau) d\tau\} / q^3$$

$$\partial^2 EAC(t, q)/\partial t^2 = \{(c_1 + c_2) D \int_t^{t+q} g(\tau) d\tau\} / q \text{ and}$$

$$\partial^2 EAC(t, q)/\partial q \partial t = \{(c_1 + c_2) D \int_t^{t+q} (\tau - t) g(\tau) d\tau\} / q^2.$$

Inequality (6a) follows since the integrals in both expressions are strictly positive. By using the above and simplifying, (6b) becomes

$$\left[K/(c_1+c_2)D + \int_t^{t+q} (\tau - t)^2 g(\tau) d\tau\right]\left[\int_t^{t+q} g(\tau) d\tau\right] > \left[\int_t^{t+q} (\tau - t) g(\tau) d\tau\right]^2.$$

Since $K/(c_1+c_2)D > 0$, it is sufficient to show that

$$\left[\int_t^{t+q} (\tau - t)^2 g(\tau) d\tau\right]\left[\int_t^{t+q} g(\tau) d\tau\right] \geqq \left[\int_t^{t+q} (\tau - t) g(\tau) d\tau\right]^2.$$

But the above equation is a special case of the Cauchy-Schwarz inequality. Thus, unique global minima for t and q can be found by setting (4) and (5) (the first derivative equations) equal to 0. After simplifying, they can be expressed as in (2) and (3).

2. RELATIONSHIPS WITH THE EOQ MODEL

In an attempt to unify these results and traditional inventory theory, we consider the following special case of our steady-state model. Suppose that lead time is no longer stochastic but *deterministic*. To accomplish this transition, we represent the lead-time density function as a Dirac Delta function (all probability concentrated at a single point). Equations (2) and (3) become

$$2K/(c_1+c_2)D + (\tau_0 - t^*)^2 = (q^*)^2 [c_1/(c_1+c_2)] \tag{7}$$

and $$(\tau_0-t^*)=q^*[c_1/(c_1+c_2)]. \quad (8)$$

Solving (7) and (8) yields $q^*=[2K(c_1+c_2)/c_1c_2D]^{1/2}$ and $\tau_0-t^*=[2Kc_1/c_2(c_1+c_2)D]^{1/2}$ so that the optimal order quantity is $Q^*=Dq^*=[2KD(c_1+c_2)/c_1c_2]^{1/2}$. Define $s^*=q^*-(\tau_0-t^*)$. Remembering that τ_0-t^* is the arrival time of each order, the amount of stock on hand immediately after satisfying the backlogged demand is $Ds^*=[2KDc_2/c_1(c_1+c_2)]^{1/2}$ and the fraction of time no shortages exist is $f^*=s^*/q^*=c_2/(c_1+c_2)$. But these results are identical to those obtained for the EOQ model with backlogging of demand (see, e.g., [4], p. 819, or [2], pp. 42–46, where $\Pi=0$). Thus, our steady-state model is a stochastic lead-time generalization of the EOQ model with backlogging of demand.

Example. Suppose lead times are uniformly distributed:

$$g_u(\tau;a,b)=\begin{cases}1/(b-a), & \text{if } a<\tau<b\\ 0, & \text{otherwise.}\end{cases}$$

Combining (4) and (5) leads to

$$2K/(c_1+c_2)D+\int_{t^*}^{t^*+q^*}(\tau-t^*)^2/(b-a)d\tau=q^*\int_{t^*}^{t^*+q^*}(\tau-t^*)/(b-a)d\tau$$

which easily gives $q^*=[12K(b-a)/(c_1+c_2)D]^{1/3}$. Solving (3) for t^* and remembering that $-t^*$ is the optimal ordering time yields $-t^*=(q^*/2)-[(c_1a+c_2b)/(c_1+c_2)]$.

3. SOME COMPUTATIONAL CONSIDERATIONS

If the cumulative probability distribution function cannot be expressed in closed form, numerical methods (such as Newton-Raphson) must be used to iteratively solve (2) and (3). The success of such methods depends largely on the starting value of the search. We now show that $G^{-1}(c_2/(c_1+c_2))$ provides an upper bound, and therefore a good initial guess, for t^*.

The left-hand side (LHS) of (3) is non-negative, implying that $G(t^*+q^*)$ is non-negative. Subtracting $q^*[G(t^*+q^*)-G(t^*)]$ from both sides of equation (3) and noting that the LHS of the resulting equation is non-positive, yield $G(t^*)\leqq c_2/(c_1+c_2)$, proving the assertion. For gamma-distributed lead times, Cantley's method [1] allows $G^{-1}(c_2/(c_1+c_2))$ to be computed within a tolerance of 10^{-4} in one or two iterations.

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Average Costs in a Continuous Review (*s,S*) Inventory System with Exponentially Distributed Lead Time

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We describe a very elementary direct numerical method to find the average number of backorders, costs and related quantities in a continuous review (*s,S*) inventory system with exponentially distributed lead time. This method can also be used in the study of *E/M/C* queues with state-dependent service and arrival rates.

IN 1959 Galliher, Morse and Simond [2] investigated the steady-state behavior of an inventory system with Poisson arrival and negative exponential leadtime under the assumption that an (*s,q*)-ordering policy is used. They derived explicit expressions for the steady-state probabilities. These probabilities can be used to calculate the average number of backorders, the average inventory costs, and the average ordering costs. We consider the same inventory model but allow the arrival and service to be state dependent. A very elementary direct numerical method is described to find the above-mentioned quantities. For the non-state-dependent case we compared this direct method with the calculation of the explicit expressions of [2] and found about the same computing times.

Various types of continuous time (*s,S*) inventory models have been considered by Feeney and Sherbrooke [1], Gross and Harris [3][4], Higa, Feyerherm and Machado [5], Rose [6], Sherbrooke [7], Tijms [8], and Van der Genugten [9]. The leadtime in most of these models is constant

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