TEICHMÜLLER POLYNOMIALS OF FIBERED ALTERNATING LINKS

Robert BILLET and Livio LIECHTI

Abstract

We give an algorithm for computing the Teichmüller polynomial for a certain class of fibered alternating links associated to trees. Furthermore, we exhibit a mutant pair of such links distinguished by the Teichmüller polynomial.

1. Introduction

McMullen introduced the Teichmüller polynomial as a geometric analogue of the Alexander polynomial [16]. It is an invariant of flow equivalence classes of pseudo-Anosov homeomorphisms, where two pseudo-Anosov homeomorphisms are *flow-equivalent* if there is a homeomorphism between their mapping tori identifying the pseudo-Anosov flows induced by the respective vertical directions of the mapping tori. The Teichmüller polynomial is used to study all dilatations of elements in a flow equivalence class simultaneously.

Even though the Alexander and Teichmüller polynomials are invariants of flow equivalence classes, they can be computed from a single monodromy. Thought of as invariants of a pseudo-Anosov mapping class (S, ψ), these polynomials give subtle information that goes beyond the information contained in the characteristic polynomials of the induced action of ψ on the first homology of S or the edge space of a ψ-invariant train track $\tau \subset S$, respectively. For instance, we will give an example of two mapping classes (defining a mutant pair of fibered alternating links) that are not distinguished by the characteristic polynomial of their induced algebraic and geometric actions, but are distinguished by their Alexander and Teichmüller polynomials.

The first explicit computations of Teichmüller polynomials were done case-by-case [16, 10, 1, 14]. More recently, Sun considered a special case of pseudo-Anosov maps arising from Penner's construction [19], and Lanneau and Valdez gave an algorithm computing the Teichmüller polynomials of braid monodromies using folding automata for train tracks [15].

Our goal is to present an algorithm computing the Teichmüller polynomial for a large class of fibered alternating links. Their fiber surfaces are of genus greater than one and their monodromies are pseudo-Anosov products of multitwists, as in constructions developed by Thurston and Penner [21, 17].

1.1. The Teichmüller polynomial. Let (S, ψ) be a pseudo-Anosov mapping class with mapping torus M. Results of Thurston and Fried [20, 6] show that the flow equivalence

class of (S, ψ) determines a polygonal cone $C \subset H^1(M; \mathbb{R})$, called a *fibered cone*, with the following properties:

(i) for each primitive integral point $\alpha \in C$, the corresponding map $\alpha_* : \pi_1(M) \to \mathbb{Z}$ is induced by a fibration $\rho_\alpha : M \to S^1$,

(ii) the fibrations of M arising in this way are exactly the ones with a monodromy which is flow-equivalent to (S, ψ).

The Teichmüller polynomial Θ_C defined by McMullen in [16] is an invariant of a flow equivalence class, and is useful for understanding the behavior of the dilatation of its elements. If $n = \dim(C)$, then Θ_C is a polynomial in n variables and is well-defined up to change of coordinates and multiplication by monomials. More precisely, given a fibered cone C of a fibered hyperbolic 3-manifold M, the corresponding Teichmüller polynomial Θ_C is an element of the group ring $\mathbb{Z}[G]$, where $G = H_1(M; \mathbb{Z})/\text{torsion}$:

$$\Theta_C = \sum_{g \in G} a_g x^g \in \mathbb{Z}[G].$$

The Teichmüller polynomial is a geometric analogue of the Alexander polynomial in the following sense: for each primitive integral point $\alpha \in C$, the geometric dilatation $\lambda(\psi_\alpha)$ of the monodromy mapping class ψ_α is the largest (real) root of the specialization

$$\Theta_C^{(\alpha)}(x) = \sum_{g \in G} a_g x^{\alpha(g)}$$

by a theorem of McMullen [16]. Similarly, the largest root of the the specialized Alexander polynomial equals the largest eigenvalue of the homological action of ψ_α.

1.2. Alternating-sign Coxeter links. Our examples for the computation of the Teichmüller polynomial are alternating-sign Coxeter links. They are constructed as follows. To each finite plane tree Γ, we associate a surface S obtained by thickening vertical and horizontal annuli whose incidence graph is Γ. We define a mapping class (S, ψ) as a composition of a positive multitwist along the horizontal annuli and a negative multitwist along the vertical annuli. Such a mapping class (S, ψ) is called an alternating-sign Coxeter mapping class, since its homological action acts, up to a sign, as the Coxeter element corresponding to the tree Γ with alternating signs [11, 12].
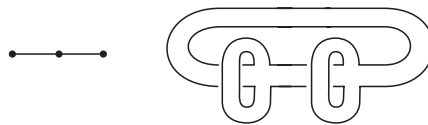

Fig. 1. The tree $\Gamma = A_3$ and its associated surface S.

Equivalently, we can think of S as the fiber surface for the alternating-sign Coxeter link corresponding to Γ and of ψ as its monodromy, see [12]. In this context, we obtain the ingredients to compute the Teichmüller polynomial of the flow-equivalence class of ψ explicitly from the tree Γ. We will see that the number of variables of the Teichmüller polynomial equals one plus the rank of the kernel of the adjacency matrix of Γ. Therefore, we from now on consider trees Γ with a singular adjacency matrix.
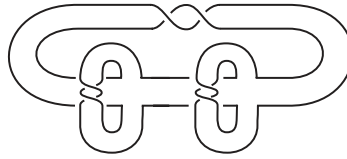
Fig. 2. The fiber surface of the alternating-sign Coxeter link associated to A_3 is obtained by adding full twists to the bands of S.

1.3. Main result. We describe an algorithm with input a plane tree Γ and output the Teichmüller polynomial Θ_{C_Γ} of the flow-equivalence class of the associated alternating-sign Coxeter mapping class (S, ψ). The description of the Teichmüller polynomial we use for our computation builds on the cover $\widetilde{S} \to S$ with the ψ-invariant homology $H_1(S; \mathbb{Z})^\psi$ as deck group, as described in [16]. In order to compute the Teichmüller polynomial, it is necessary to find matrices for the induced action of a lift $\widetilde{\psi}$ of ψ on the lift $\widetilde{\tau}$ of a ψ-invariant train track $\tau \subset S$. In our algorithm, we reduce this problem to one which can be solved explicitly using only the structure of the plane tree Γ. A sample of this reduction is given below.

Illustration of the algorithm. Let Γ be a plane tree. In the following, we will define matrices U, V, W and T, obtained explicitly using only the structure of Γ. These matrices determine the specialization to a two-dimensional slice of the Teichmüller polynomial Θ_{C_Γ} of the fibered cone C_Γ containing the monodromy of the alternating-sign Coxeter link corresponding to Γ.

Theorem 1. *The fibered cone C_Γ of the alternating-sign Coxeter link associated to Γ has a two-dimensional slice with coordinates (x, u) to which the Teichmüller polynomial Θ_{C_Γ} specializes as*

$$\Theta_{C_\Gamma}(x, u) = \frac{\det(uI - UWVT)}{\det(uI - WT)^{\frac{1}{2}}}.$$

We now show how to define the matrices U, V, W and T from the plane tree Γ. Let $\mathcal{A}$ be an arrangement of vertical and horizontal segments a_i and b_j in the plane with incidence graph Γ. Identifying top and bottom of every vertical segment a_i and identifying left and right of every horizontal segment b_j yields a graph τ with edges $e_0, \ldots e_n$, see Fig.3 for the example $\Gamma = A_3$. We define the following maps.
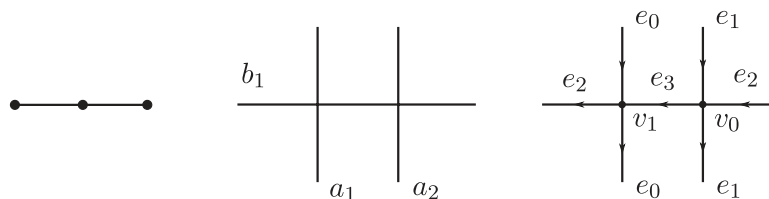

Fig. 3. The tree $\Gamma = A_3$, the arrangement $\mathcal{A}$ with incidence graph A_3 (the vertex of degree 2 in Γ corresponds to the segment b_1) and the graph τ.

(1) Let $s, t : E(\tau) \to V(\tau)$ be the maps associating to every edge of τ its starting and terminal vertex, respectively, when oriented downwards and to the left.

(2) Let $l : V(\Gamma) \to V(\tau)$ associate to every vertex of Γ the uppermost or rightmost vertex on the corresponding vertical or horizontal segment, respectively.

(3) Let $d : E(\tau) \to V(\Gamma)$ be the map associating to an edge e of τ the vertex of Γ in whose corresponding segment e is contained.

(4) Let $c : E(\tau) \times V(\tau) \times V(\tau) \to \{0, 1\}$ be the map assigning 1 to (e, v_i, v_j) if e is contained in the convex hull of v_i and v_j and 0 otherwise.

(5) Let $p : E(\tau) \to \{0, 1\}$ be the map assigning 1 to vertical edges and 0 to horizontal ones.

(6) Let $g : E(\tau) \times V(\tau) \to \{0, 1\}$ be the map assigning 1 to (e, v) if e and v lie on a straight line and 0 otherwise.

(7) Let $v_0 \in V(\tau)$ be fixed. For every vertex $v_i \in V(\tau)$, there is a unique simple oriented edge path γ_{v_i} in τ connecting v_0 to v_i without using any of the edges with top-bottom and left-right identifications. Let $\operatorname{or}(e)$ of an oriented edge e be $+1$ if the orientation matches the orientation of the edge in τ as in (1) and -1 otherwise.

(8) Let $a : E(\tau) \to V(\Gamma)$ be the map assigning to a horizontal edge e the vertex of Γ corresponding to the segment perpendicular to e at its terminal vertex, and to a vertical edge e the vertex of Γ corresponding to the segment perpendicular to e at its starting vertex.

If $k = (k_1, \ldots, k_{|V|}) \in \mathbb{Z}^{|V(\Gamma)|}$ is a given primitive element of the kernel of the adjacency matrix of Γ, define matrices $U = I + (m_{ij})$ and $V = I + (n_{ij})$ of size $|E(\tau)| \times |E(\tau)|$ by

$$m_{ij} = p(e_i)(1 - p(e_j))g(e_i, t(e_j))x^{c(e_i, l(d(e_i)), t(e_j))k_{d(e_i)}},$$

$$n_{ij} = p(e_j)(1 - p(e_i))g(e_i, s(e_j))x^{c(e_i, l(d(e_i)), s(e_j))k_{d(e_i)}}.$$

For example, $m_{ij} \neq 0$ if e_i is vertical and part of the segment perpendicular to the terminal vertex of the horizontal edge e_j. In this case, $m_{ij} = x^{k_{d(e_i)}}$ if e_i lies between the terminal vertex of e_j and the uppermost vertex of the segment perpendicular to the terminal vertex of e_j, and $m_{ij} = 1$ otherwise.

Define two diagonal matrices $W = (w_{ij})$ and $T = (t_{ij})$ of size $|E(\tau)| \times |E(\tau)|$ by

$$w_{ii} = \prod_{e \in \gamma_{s(e_i)}} x^{(1 - p(e))\operatorname{or}(e)k_{a(e)}},$$

$$t_{ii} = \prod_{e \in \gamma_{s(e_i)}} x^{p(e)\operatorname{or}(e)k_{a(e)}},$$

where the product is taken over the $e \in E(\tau)$ that appear in the path $\gamma_{s(e_i)}$. The matrices U, V, W and T we just defined are the ones to be used in the formula of Theorem 1. Note that since every vertex $v \in V(\tau)$ occurs twice as a starting point of an edge $e \in E(\tau)$, the denominator in Theorem 1 is also a polynomial.

Remark 2. Alternating-sign Coxeter mapping classes (S, ψ) preserve the orientation of the invariant train track. Equivalently, they preserve the orientation of the corresponding invariant foliations. Thus, the induced foliations on the mapping torus are orientable, and every other element in the flow equivalence class, given by a cross section to the suspension flow of (S, ψ), is also orientable. In fact, one can show that the Alexander polynomial and the Teichmüller polynomial agree for alternating-sign Coxeter links corresponding to trees.

This gives an alternative way of computing the Teichmüller polynomial for these cases. However, our focus lies on the perspective corresponding to the Teichmüller polynomial.

1.4. Application to mutant links. Let $B \subset \mathbb{S}^3$ be a ball whose boundary sphere ∂B meets a link $L \subset \mathbb{S}^3$ transversally and symmetrically in four points. The operation of exchanging the interior of B with its image under a sequence of reflections is a *mutation*. Two links are *mutant* if they are connected by a sequence of mutations. Mutants are hard to distinguish. For example, by a theorem of Wehrli [23], an invariant f that is able to distinguish a pair of mutant links cannot satisfy a skein relation of the form

$$\alpha f(L_+) + \beta f(L_-) + \gamma f(L_0) = 0,$$

where $\alpha, \beta \in R^*$ and $\gamma \in R$ are fixed elements of an arbitrary ring R. For instance, the single variable Jones and Alexander polynomials are not able to distinguish mutant links. We give an example of mutant alternating-sign Coxeter links distinguished by their Teichmüller polynomials. In particular, it follows that there is no skein relation of the above form for the Teichmüller polynomial of fibered links.

Proposition 3. *There exists a pair of fibered alternating mutant links distinguished by their Teichmüller polynomials.*

In our examples of alternating-sign Coxeter links, the Teichmüller polynomial coincides with the multivariable Alexander polynomial. In particular, Proposition 3 also applies to the multivariable Alexander polynomial. This is in contrast with the existence of more involved skein relations for the multivariable Alexander polynomial [13] and the fact that the multivariable Alexander polynomial is invariant under many mutations [24].

1.5. Organization. In Section 2, we give the necessary background on the Teichmüller polynomial. In Section 3, we recall the construction of alternating-sign Coxeter links, their fiber surfaces and their monodromies. In Section 4, we describe a general algorithm computing the Teichmüller polynomial of a fibered cone containing the alternating-sign Coxeter link associated to a tree. In particular, we will prove Theorem 1. Finally, in Section 5, we describe the Teichmüller polynomial of the alternating-sign Coxeter links associated to the trees A_n, give a detailed computation for A_5 and prove Proposition 3.

2. Dilatation, fibered faces, and the Teichmüller polynomial

In this section, we recall the necessary notions concerning the Teichmüller polynomial and describe the formula we will use to compute it.

2.1. Dilatation. Let M be a 3-manifold that fibers over the circle. Then M has the structure of a mapping torus $M = S \times [0, 1]/(x, 1) \sim (\psi(x), 0)$, where S is a surface and $\psi : S \to S$ is a homeomorphism defined up to isotopy and conjugation. If M is hyperbolic, then by a fundamental result of Thurston, the *monodromy* (S, ψ) is *pseudo-Anosov* [22] and has an associated *dilatation* (or *stretch factor*) $\lambda = \lambda(\psi) > 1$, which can be characterized in any of several equivalent ways:

(i) (*pseudo-Anosov property*) there is a pair of ψ-invariant transverse measured singular foliations $(\mathcal{F}^\pm, \nu^\pm)$ so that $\psi_*(\nu^\pm) = \lambda \nu^\pm$;

(ii) (*exponential growth rate of lengths of curves*) for any Riemannian metric ω on S, and any essential simple closed curve γ on S, we have

$$\lambda = \lim_{n \to \infty} \ell(\phi^n(\gamma))^{\frac{1}{n}};$$

(iii) (*entropy*) $\log(\lambda)$ is the minimal topological entropy of any representative of the isotopy class of homeomorphisms defined by ϕ; and

(iv) (*geodesics on moduli space*) the orbits of ϕ in Teichmüller space determine a geodesic on the moduli space of S whose Teichmüller length is $\log(\lambda)$.

See [3, 4, 5, 21] for more details and background.

2.2. Fibered faces. The mapping classes we are considering are pseudo-Anosov, so each mapping torus M is hyperbolic. Thus, the Thurston norm $\|\psi\|_T$ defined on $H^1(M; \mathbb{Z})$ is non-degenerate and has convex norm ball B_x [20]. Let F be a top-dimensional face of B_x and let $CF = \mathbb{R}^+ \cdot F$ be the cone over F. We say F is a *fibered face* of the Thurston norm ball if every integral element α in CF corresponds to a fibration $\rho_\alpha : M \to S^1$. By a theorem of Fried [7], proximity in the projectivization F of the cone CF implies closeness for the normalized dilatation $L(S_\alpha, \psi_\alpha) = \lambda(\psi_\alpha)^{|\chi(S_\alpha)|}$, where $\psi_\alpha : S_\alpha \to S_\alpha$ is the monodromy of ρ_α. In fact, L extends to a convex, real analytic function on F that goes to infinity toward the boundary of F.

2.3. The Teichmüller polynomial. Let $G = H_1(M; \mathbb{Z})/\text{torsion}$. For each fibered face F of the Thurston norm ball on $H^1(M; \mathbb{Z})$, McMullen [16] defined an element

$$\Theta_{CF} = \sum_{g \in G} a_g x^g \in \mathbb{Z}[G],$$

called the *Teichmüller polynomial* of the fibered face F (or the fibered cone CF), which comes packaged with the dilatation of each primitive integral class $\alpha_\psi \in CF$: the dilatation $\lambda(\psi_\alpha)$ is the largest (real) root of the specialization

$$\Theta_{CF}^{(\alpha)}(x) = \sum_{g \in G} a_g x^{\alpha(g)}.$$

McMullen also gives a formula for the Teichmüller polynomial Θ_C, see [16]. We quickly review the statement. Fix a fiber $\iota : S \to M$ and let $\psi : S \to S$ be the associated pseudo-Anosov monodromy with ψ-invariant cohomology $H^1(S; \mathbb{Z})^\psi$. Let $H = \text{Hom}(H^1(S; \mathbb{Z})^\psi, \mathbb{Z})$ and let $\widetilde{S}$ be the cover defined by $\iota_* : \pi_1(S) \to H$. We think of $\widetilde{S}$ as a component of the preimage of the chosen fiber S under the covering map $\pi : \widetilde{M} \to M$ where $\widetilde{M}$ is the maximal abelian cover of M. From this point of view, H is the subgroup of $\text{Deck}(\pi)$ that fixes $\widetilde{S}$. We can think of $\widetilde{M}$ as

$$\bigsqcup_{i \in \mathbb{Z}} \left(\widetilde{S}_i \times [0, 1] \right),$$

where $\widetilde{S}_i \cong \widetilde{S}$ and $(s, 1)$ in $\widetilde{S}_i$ is identified with $(\widetilde{\psi}(s), 0)$ in $\widetilde{S}_{i+1}$.

Let τ be a connected graph with an embedding $j : \tau \to S$ so that the edges are tangent at the vertices. We can partition the edges at each vertex into two sets, thought of as incoming and outgoing edges. The region created by two consecutive edges (in the cyclic order determined by j) of the same type is called a cusp. Then τ is a *train track* if no vertex has

degree 1 or 2 and the connected components of $S \setminus j(\tau)$ are either polygons with at least one cusp or annuli with one boundary contained in ∂S and the other boundary with at least one cusp. Associated to a pseudo-Anosov mapping class ψ there is a ψ-invariant train track τ. In particular, there is an action on the space of weights on the edges of τ induced by ψ, sending an edge e to the edges (counted with multiplicity) in the edge path given by $\psi(e)$. Similarly, there is an induced action on the space of weights on the vertices of τ.

Let and E and V denote the edges and vertices of a ψ-invariant train track $\tau \subset S$. Let $(t_1, ..., t_b, u)$ be a multiplicative basis for

$$H_1(M; \mathbb{Z})/\text{torsion} \cong \text{Hom}(H^1(S; \mathbb{Z})^\psi, \mathbb{Z}) \oplus \mathbb{Z}.$$

The action of H on $\widetilde{S}$ restricts to an action on the lifts of E and V. Thus, we can think of a lift $\widetilde{\psi}$ of ψ as acting on the free $\mathbb{Z}[H]$-modules generated by the lifts of E and V. Let the action of $\widetilde{\psi}$ on the edge and vertex space of the lifted train track $\widetilde{\tau}$ (thought of as $\mathbb{Z}[H]$-modules) be given by the matrices $P_E(t)$ and $P_V(t)$. Then McMullen's determinant formula states that the Teichmüller polynomial for the fibered cone C containing $[S]$ is given by

$$\Theta_C(t, u) = \frac{\det(uI - P_E(t))}{\det(uI - P_V(t))},$$

independently of any choices [16].

3. Alternating-sign Coxeter links

In this section, we associate fibered alternating links L to trees Γ and we describe the monodromy ψ of their fiber surface S. More precisely, we construct an invariant train track $\tau \subset S$ for ψ and describe the dual $\text{Hom}(H^1(S; \mathbb{Z})^\psi, \mathbb{Z})$ of the ψ-invariant cohomology $H^1(S; \mathbb{Z})^\psi$ in terms of the adjacency matrix of Γ. While we restrict our attention to trees, much of what we are about to do generalizes to a more general class of bipartite Coxeter graphs. For more details on this construction and Coxeter links and mapping classes, see [11, 12].

Let Γ be a finite *plane* tree, that is, a finite tree together with an embedding in the plane. For every vertex v of Γ, this embedding defines a circular ordering of the edges touching v. It is possible to draw horizontal and vertical segments in the plane such that their incidence graph equals Γ and respects the circular ordering of edges at every vertex. We thicken the union of all segments and attach bands with one full twist to the ends of each segment, as in Fig.4. The resulting surface S is an iterated Murasugi sum of Hopf bands, hence S is a fiber surface for the boundary link $L = \partial S$ and the monodromy is given by a product of Dehn twists along the core curves of the Hopf bands [8]. In the classical case, each Hopf band is positive. Equivalently, each Dehn twist is positive. In the mixed-sign case [11], the tree Γ comes equipped with a function $\mathfrak{s} : V(\Gamma) \to \{-1, 1\}$. Each vertex marked with a negative sign then corresponds to a negative Hopf band and hence a negative Dehn twist. In what follows, we consider signs $\mathfrak{s}$ respecting the bipartition of the tree Γ. An *alternating-sign Coxeter link* is a link arising from a tree Γ with a bipartition $V(\Gamma) = V_1 \sqcup V_2$, together with a labeling $\mathfrak{s}$ so that $\mathfrak{s}(V_1) = \pm 1$ and $\mathfrak{s}(V_2) = \mp 1$. We have already seen that alternating-sign Coxeter links are fibered. Furthermore, one can show they are alternating [12]. We regard their monodromy $\psi : S \to S$ as the product of a positive Dehn twist along all core curves of the positive Hopf bands and a negative Dehn twist along all core curves of the negative
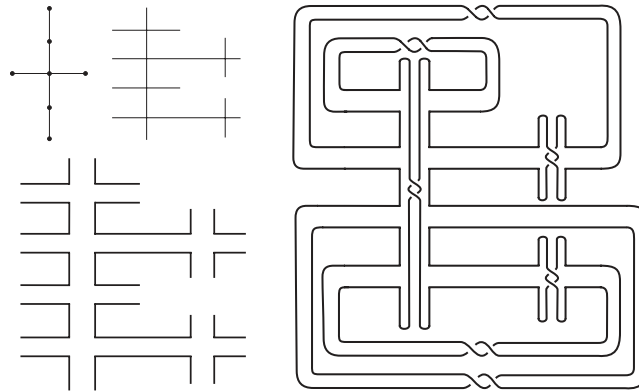
Fig. 4. A plane tree Γ, horizontal and vertical segments in the plane with Γ as incidence graph, thickened segments and the fiber surface S obtained by adding twisted bands.

Hopf bands. Indeed, for any tree Γ, the constructed mapping class does, up to conjugation, not depend on the order of Dehn twists [18].

Remark 4. It is possible for a given abstract tree Γ to have several non-equivalent planar embeddings, and for the corresponding diagrams to give mutant but distinct links. Fig. 5 shows a sphere cutting a link in four points. Rotating by π exchanges the two vertical bands shown. Using isotopies and such operations, one can arbitrarily change the circular ordering of edges at any vertex of Γ. This shows that the alternating-sign Coxeter links associated to any two planar embeddings of a given abstract tree Γ are mutants. While the single variable Alexander and Teichmüller polynomial can not distinguish these links, in Section 5 we provide an example where two distinct embeddings of an abstract tree result in a pair of links that have different Teichmüller polynomials.
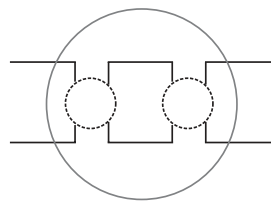

Fig. 5

By a theorem of Penner [17], the mapping class (S, ψ) constructed above is pseudo-Anosov as soon as Γ has at least two vertices. Furthermore, an orientable ψ-invariant train track $\tau \subset S$ is obtained by smoothing the union of core curves of the Hopf bands at their intersection points, as in Fig. 6. An orientation of τ is then given by orienting all horizontal edges to the left and all vertical edges downwards.

Lemma 5. *If the tree Γ has $|E|$ edges, then the ψ-invariant train track $\tau \subset S$ has $2|E|$ edges.*
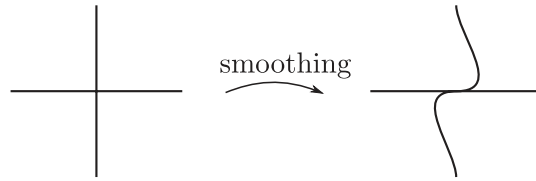
Fig. 6. The local smoothing operation used to build an invariant train track out of the core curves of the Hopf bands.

Proof. We construct τ from a collection of horizontal and vertical segments, where these collections of segments correspond to the bipartition of Γ and intersect according to their adjacency in Γ. Then, the segment corresponding to some vertex v is subdivided into $\deg(v)$ edges. Thus, the train track τ has $\sum_v \deg(v) = 2|E|$ edges. $\qquad\square$

In order to construct the covers necessary to compute the Teichmüller polynomial, we need to know the dual $\mathrm{Hom}(H^1(S;\mathbb{Z})^\psi, \mathbb{Z})$ of the ψ-invariant cohomology $H^1(S;\mathbb{Z})^\psi$ of our constructed surface S. This is the content of the following lemma. From the construction, we have $H_1(S;\mathbb{Z}) \cong \mathbb{Z}^{|V|}$, where V is the set of vertices of the original tree Γ.

Lemma 6. *Let (S, ψ) be a mapping class constructed from a plane tree Γ as above. Let A be the adjacency matrix of Γ. If we regard A as a linear map $H_1(S;\mathbb{Z}) \to H_1(S;\mathbb{Z})$, then $H_1(S;\mathbb{Z})^\psi = \ker(A)$.*

Proof. By labeling the vertices one set of the bipartition at a time, we can arrange A to have the form

$$A = \begin{pmatrix} 0 & X \\ X^\top & 0 \end{pmatrix}.$$

Then the action of ψ on $H_1(S;\mathbb{Z})$ can be expressed as the product of the two multitwists

$$T_a = \begin{pmatrix} I & X \\ 0 & I \end{pmatrix}, \quad T_b = \begin{pmatrix} I & 0 \\ X^\top & I \end{pmatrix}.$$

Let $v \in H_1(S;\mathbb{Z})$ be expressed according to the bipartition as $v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$. Then $\psi_*(v) = v$ holds if and only if

$$v_1 + XX^\top v_1 + Xv_2 = v_1,$$

$$X^\top v_1 + v_2 = v_2.$$

But this is equivalent to

$$\begin{pmatrix} 0 & X \\ X^\top & 0 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} Xv_2 \\ X^\top v_1 \end{pmatrix} = 0,$$

which is what we wanted to show. $\qquad\square$

In order to obtain the dual $\mathrm{Hom}(H^1(S;\mathbb{Z})^\psi, \mathbb{Z})$ of the ψ-invariant cohomology $H^1(S;\mathbb{Z})^\psi$, we remark that the matrix product $T_a T_b$ describing the action ψ_* of ψ induced on the first

homology is symmetric. In particular, the dual $\mathrm{Hom}(H^1(S;\mathbb{Z})^\psi,\mathbb{Z})$ of the ψ-invariant cohomology $H^1(S;\mathbb{Z})^\psi$ is simply the ψ-invariant homology $H_1(S;\mathbb{Z})^\psi$.

4. Algorithm

The aim of this section is to present a general way to compute the Teichmüller polynomial of the fibred cone C_Γ corresponding to the alternating-sign Coxeter link for any plane tree Γ. In order to apply our algorithm in practice and obtain the Teichmüller polynomial, additional choices have to be made. For explicit calculations, see Section 5.

Let Γ be a plane tree and let S and τ be the surface and the train track, respectively, constructed from Γ as in Section 3. We denote the generators for the homology $H_1(M;\mathbb{Z})$ and the ψ-invariant homology of S according to the isomorphism

$$H_1(M;\mathbb{Z})/\text{torsion} \cong H_1(S;\mathbb{Z})^\psi \oplus \mathbb{Z} \cong\; <x_0, ..., x_{n-1}> \oplus <x_n>\;.$$

The algorithm is divided into three main steps:

(1) Construct (S,ψ) together with a ψ-invariant train track $\tau \subset S$.
(2) Compute $H_1(S;\mathbb{Z})^\psi$, construct the cover $\widetilde{S} \to S$ corresponding to $\pi_1(S) \to H_1(S;\mathbb{Z})^\psi$ and lift the train track τ to obtain $\widetilde{\tau}$.
(3) Decompose the action of $\widetilde{\psi}$ on the edges and the vertices of $\widetilde{\tau}$ as a product of two multitwists on a single fundamental domain of $\widetilde{\tau}$.

Step 1. We quickly recall the construction from Section 3. Given a plane tree Γ, the surface S is constructed by suitably gluing horizontal and vertical bands, whose core curves have Γ as incidence graph. The corresponding mapping class (S,ψ) is defined to be a product of a positive and a negative multitwist along the core curves of the horizontal and vertical bands, respectively. The ψ-invariant train track $\tau \subset S$ is then obtained by smoothing the crossings of the core curves of the bands. The induced action of ψ on the train track τ can be decomposed as a product $T_a T_b$ of the induced actions T_a and T_b of the positive and negative multitwist, respectively. To make this induced action explicit, let the bipartition of Γ be given by $V(\Gamma) = V_1 \sqcup V_2$, where the vertices of V_1 and V_2 correspond to vertical and horizontal bands, respectively. Let $\mathfrak{s} : V_1 \sqcup V_2 \to \{1,-1\}$ so that $\mathfrak{s}(V_1) = -1$ and $\mathfrak{s}(V_2) = 1$ be our labeling. We choose a basis of the edge space of τ in the following way. First we take all the edges of the train track τ that correspond to V_1, that is, to bands along whose core curves we twist negatively. Then we take all the edges that correspond to V_2. The induced actions T_a and T_b of the negative and positive multitwist, respectively, now have the form

$$T_a = \begin{pmatrix} I & X \\ 0 & I \end{pmatrix}, \qquad T_b = \begin{pmatrix} I & 0 \\ Y & I \end{pmatrix}.$$

At this point, it is possible to compute the dilatation of the mapping class (S,ψ) corresponding to the fixed fiber S as the largest root of

$$\det(xI - T_a T_b).$$

Step 2. We first explain how to construct $\widetilde{S}$, the cover of S corresponding to $\pi_1(S) \to \mathrm{Hom}(H^1(S;\mathbb{Z})^\psi,\mathbb{Z})$. Let A be the adjacency matrix of Γ. In the basis induced by the vertices of Γ, the ψ-invariant homology $H_1(S;\mathbb{Z})^\psi$ is the kernel $\ker(A)$ by Lemma 6. As a submodule of a finitely generated free $\mathbb{Z}$-module, $\ker(A)$ is finitely generated and admits

a basis. Let B be a matrix whose columns form a basis for ker(A). By the correspondence of ker(A) and $H_1(S;\mathbb{Z})^\psi$, each column of B represents an element of a basis of $H_1(S;\mathbb{Z})^\psi$. On the other hand, each row of B represents a vertex v_i of Γ and hence an annulus A_i in S. A nonzero entry $B_{ij} \neq 0$ signifies that a loop winding around the core curve of the annulus A_i should lift to a path in $\widetilde{S}$ corresponding to $x_j^{B_{ij}}$ as an element of the deck transformation group $H_1(S;\mathbb{Z})^\psi$. With this correspondence in mind, we define a function f on the vertices of Γ as follows:

$$v_i \mapsto \prod_j x_j^{B_{ij}}.$$

Then the cover $\widetilde{S}$ is constructed by cutting copies of S along cohomology classes (thought of as arcs dual to bands) fixed by ψ and attaching one end of each cut band v with the other end of the equivalent band moved by what will be the deck transformation $f(v)$. Having constructed the cover $\widetilde{S} \to S$, we also obtain a natural lift $\widetilde{\tau}$ of τ.

Step 3. Choose a lift $\widetilde{\psi}$ of ψ. We can compute the action of $\widetilde{\psi}$ on the edge space of $\widetilde{\tau}$ by looking at a single fundamental domain of $\widetilde{S}$. For this, recall from Section 2 that we regard the edge space of $\widetilde{\tau}$ as a $\mathbb{Z}[H]$-module. Let $\widetilde{T}_a$ and $\widetilde{T}_b$ be the actions of the lifted multitwists on the edge space of $\widetilde{\tau}$. Then the matrix product $\widetilde{T}_a\widetilde{T}_b$ equals the action of $\widetilde{\psi}$ on the edge space of $\widetilde{\tau}$. We now explain how both $\widetilde{T}_a$ and $\widetilde{T}_b$ can be written as a product of two matrices which in turn can be obtained directly from our given data.

Let U and V be matrices that differ from T_a and T_b only in that they take into account that when a curve winds around the core of the band corresponding to v_i, it moves to the $f(v_i)$-level of $\widetilde{S}$. More precisely, the image after such a winding of an edge of $\widetilde{\tau}$ is multiplied by the coefficient $f(v_i)$. In particular, the entries of the matrices U and V are polynomials in the variables $x_0, \ldots, x_{n-1}$, and specializing every x_i to 1 yields the original matrices T_a and T_b. It is important to note that the matrices U and V might still differ from the matrices $\widetilde{T}_a$ and $\widetilde{T}_b$, respectively. Indeed, it is possible that the image of an edge of $\widetilde{\tau}$ under a lifted multitwist starts at a different level of $\widetilde{S}$. Equivalently, a lifted multitwist does not necessarily fix the vertices of $\widetilde{\tau}$ pointwise. However, this can be accounted for by multiplication of the columns of U and V corresponding to the affected edges by coefficients. This can be done by right multiplication with diagonal matrices W and T, respectively, yielding $\widetilde{T}_a = UW$ and $\widetilde{T}_b = VT$. Note that the correction process which determines the matrices W and T depends on the choice of a lift $\widetilde{\psi}$. Summarizing, we can express $P_E(x_0, \ldots, x_{n-1})$ as a product $UWVT$.

Since the action of a lifted multitwist on the vertices of the lifted train track $\widetilde{\tau}$ shifts levels as does the action on the edges starting at this vertex, we see that $P_V(x_0, \ldots, x_{n-1})$ is the diagonal matrix given by the upper (or lower) half diagonal block of the matrix product WT. Finally, according to McMullen's determinant formula, we have

$$\Theta_{C_\Gamma}(x_0, \ldots, x_n) = \frac{\det(x_n I - P_E(x_0, \ldots, x_{n-1}))}{\det(x_n I - P_V(x_0, \ldots, x_{n-1}))}.$$

Proof of Theorem 1. The matrices U, V, W and T described in Theorem 1 are obtained exactly as described in the algorithm above. We would like to highlight two things about this description. Firstly, some parts of the description determine choices in the algorithm: the map a defined in (8) in Section 1.3 determines the choice of a representative of the multitwists T_a and T_b twisting to the right and below the core curves in the thickened arrangement

of vertical and horizontal lines, the choice of a vertex $v_0 \in V(\tau)$ and a path γ_{v_i} connecting v_0 to any other vertex v_i, as in (7), determines a lift of $\widetilde{\psi}$, and the map c defined in (4) determines the choice of a fundamental domain of the lifted train track. Secondly, the matrices described in Theorem 1 take into account only the invariant homology generated by a single element of the kernel of the adjacency matrix of Γ. By the isomorphism

$$H_1(M;\mathbb{Z})/\text{torsion} \cong H_1(S;\mathbb{Z})^\psi \oplus \mathbb{Z} \cong\ < x_0, ..., x_{n-1} > \oplus < x_n >,$$

this amounts to setting all except one of the coordinates coming from ψ-invariant homology equal to zero. In other words, the matrices U, V, W, and T described in Theorem 1 determine the Teichmüller polynomial with all these coordinates specialized to zero.   □

We note that given the algorithm described in this section, it is theoretically possible to describe the full Teichmüller polynomial in the setting of Theorem 1. However, compared with the statement of Theorem 1, this would come at the cost of more cumbersome notation.

5. Examples

In this section, we perform the computation of the Teichmüller polynomial Θ_C for several examples. We first give the result for general alternating-sign A_n diagrams. Then, we proceed with a detailed account of the computation for the alternating-sign A_5 diagram. Finally, we give the Teichmüller polynomials of two mutant links related by different planar embeddings of their underlying abstract tree Γ, which distinguishes them.

5.1. Teichmüller polynomial of A_n. We now describe the Teichmüller polynomial for $\Gamma = A_n$, where n is odd. As explained in Section 3, we consider the fiber surface of the associated alternating-sign Coxeter link. It is not hard to check that adding two additional bands along a tree does not change the number of components of the link. Thus, this is a
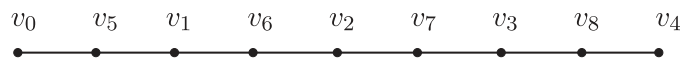

Fig. 7. The A_9 graph with the standard bipartite labeling.

family of two-component links. By picking the standard bipartite vertex labeling of A_n, as shown for $n = 9$ in Fig. 7, we can arrange for the adjacency matrix A to have the form

$$A = \begin{pmatrix} 0 & X \\ X^T & 0 \end{pmatrix},$$

where X is the matrix of size $(\frac{n+1}{2}) \times (\frac{n-1}{2})$ with form (here for $n = 9$):

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Since the links we are considering have two components, the kernel of the adjacency

matrix is always generated by a single vector. With the choice of the standard bipartite basis of $H_1(S; \mathbb{Z})$, we can write

$$H_1(S; \mathbb{Z})^\psi = \left\langle \sum_{i=0}^{\frac{n+1}{2}} (-1)^i e_i \right\rangle.$$

Notice that the invariant homology is contained in a single set of the bipartition. Thus, we can take T, the correction matrix for the lifted multitwist defined by the other set of the bipartition, to be the identity matrix I_{2n-2}. In the notation of the previous section,

$$V = I_{2n-2} + \begin{pmatrix} 0 & 0 \\ P & 0 \end{pmatrix}, \quad U = I_{2n-2} + \begin{pmatrix} 0 & Y \\ 0 & 0 \end{pmatrix},$$

where P and Y are diagonal block matrices of dimension $n - 1$ and of the form

$$P = \begin{pmatrix} J & & \\ & \ddots & \\ & & J \end{pmatrix}, \quad Y = \begin{pmatrix} 1 & & & \\ & R_1 & & \\ & & \ddots & \\ & & & R_{\frac{n-3}{2}} & \\ & & & & 1 \end{pmatrix},$$

with blocks

$$J = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad R_i = \begin{pmatrix} 1 & 1 \\ x_0^{(-1)^{i+1}} & 1 \end{pmatrix}.$$

Furthermore, the diagonal correction matrix W can be written as

$$W = \begin{pmatrix} S & 0 \\ 0 & N \end{pmatrix},$$

where S and N are diagonal matrices

$$S = \begin{pmatrix} L & & \\ & \ddots & \\ & & L \end{pmatrix}, \quad N = \begin{pmatrix} M & & \\ & \ddots & \\ & & M \end{pmatrix}.$$

If $n \equiv 3 \mod 4$, we have blocks

$$M = L = \begin{pmatrix} x_0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & x_0 \end{pmatrix},$$

and if $n \equiv 1 \mod 4$, then

$$M = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & x_0 & 0 & 0 \\ 0 & 0 & x_0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

and $\dim(S) = \dim(N)$. Then we have $P_E(x_0) = UWVT$, $P_V(x_0)$ equals the upper left

diagonal block of W of dimension $n - 1$, and

$$\Theta_{C_{A_n}}(x_0, x_1) = \frac{\det(x_1 I - P_E(x_0))}{\det(x_1 I - P_V(x_0))}.$$

REMARK 7. The tree $\Gamma' = A_{2n+1}$ is a vertex extension of Γ, where Γ consists of two disjoint copies of A_n. Hence, the spectra of the adjacency matrices $A(\Gamma)$ and $A(\Gamma')$ interlace, see, for example, [2]. In particular, each eigenvalue of $A(\Gamma)$ is also an eigenvalue of $A(\Gamma')$, since $A(\Gamma)$ has no simple eigenvalue. It follows that the single-variable Alexander polynomial of the link associated to A_n divides the single-variable Alexander polynomial of the link associated to A_{2n+1}. Indeed, these polynomials equal the characteristic polynomial of homological action of the alternating-sign Coxeter mapping class associated to A_n and A_{2n+1}, respectively. Furthermore, each eigenvalues of the adjacency matrix determines an eigenvalue of the homological action [12]. Interestingly, the same pattern of divisibility seems to hold among Teichmüller polynomials associated to the trees A_n:

$$\Theta_{A_3} = (x_0 x_1 - 2x_0 - 2x_1 + 1),$$

$$\Theta_{A_5} = (x_0^2 x_1^2 - 4x_0^2 x_1 - 4x_0 x_1^2 + 4x_0^2 + 9x_0 x_1 + 4x_1^2 - 4x_0 - 4x_1 + 1),$$

$$\Theta_{A_7} = (x_0 x_1 - 2x_0 - 2x_1 + 1)(x_0^2 x_1^2 - 4x_0^2 x_1 - 4x_0 x_1^2 + 4x_0^2 + 8x_0 x_1$$
$$+ 4x_1^2 - 4x_0 - 4x_1 + 1),$$

$$\Theta_{A_{11}} = (x_0 x_1 - 2x_0 - 2x_1 + 1)(x_0^2 x_1^2 - 4x_0^2 x_1 - 4x_0 x_1^2 + 4x_0^2 + 7x_0 x_1$$
$$+ 4x_1^2 - 4x_0 - 4x_1 + 1)(x_0^2 x_1^2 - 4x_0^2 x_1 - 4x_0 x_1^2 + 4x_0^2 + 9x_0 x_1 + 4x_1^2$$
$$- 4x_0 - 4x_1 + 1).$$

These Teichmüller polynomials can be calculated using the matrices given above. Furthermore, we changed variables so that the polynomials are given in coordinates of the first homology corresponding to the meridians of the link, as described in detail for A_5 in Section 5.2.

5.2. Computation of the Teichmüller polynomial for A_5. Let $\Gamma = A_5$ and recall the construction of the surface S, the monodromy ψ and the ψ-invariant train track τ associated to Γ, as in Section 3. The result of this construction is depicted in Fig.8.
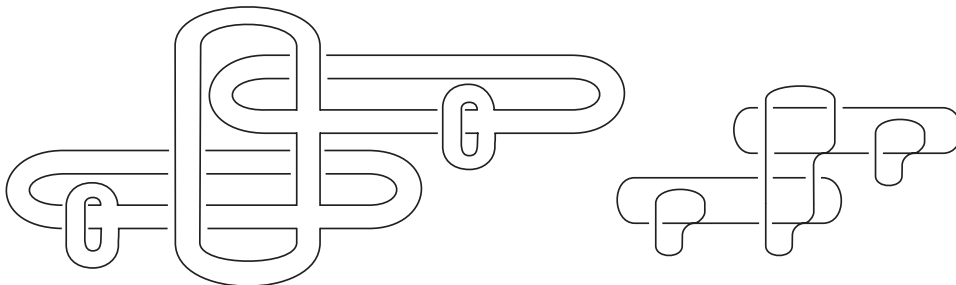

Fig.8. The abstract surface S and the ψ-invariant train track τ associated to $\Gamma = A_5$.

In order to have a cleaner picture, we consider the graph below as the core of the train track τ. Note that we omitted the smoothings at the vertices and the edges on the ends

should be identified with the edge on the opposite side, and that this graph can be naturally embedded into the surface S. The surface S actually deformation retracts to τ. We take the core curves of the Hopf bands $\{a_1, a_2, a_3, b_1, b_2\}$ as a generating set for $H_1(S; \mathbb{Z})$ and subdivide them at the intersection points to obtain a basis $\{e_0, \ldots, e_7\}$ of the edge space of τ, as indicated in Fig.9.
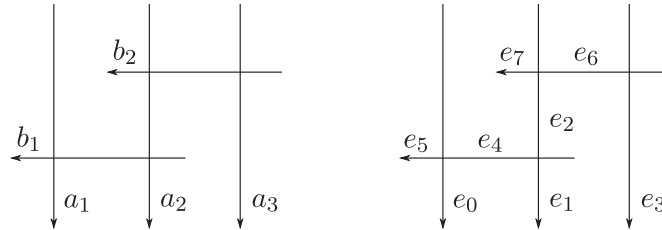

Fig. 9. Homology generators for the surface S and our edge labeling of the train track τ.

Recall that the monodromy $\psi : S \to S$ is the product of a positive multitwist along the b_i and a negative multitwist along the a_j. We also write the map $P_E : \tau \to \tau$ as a product of the actions induced by these two multitwists. Let T_b be the action on the edge space of τ induced by the positive multitwist around the loops $\{b_1, b_2\}$. Similarly, let T_a be the action induced by the negative multitwist around the loops $\{a_1, a_2, a_3\}$. We adopt the convention that if a curve is to wind around another, it does so to the right or below the curve. Then, using the edge labels as in Fig.9, we arrive at the following matrices for T_a and T_b.

$$
T_b = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 & 0 & 1
\end{pmatrix}, \quad
T_a = \begin{pmatrix}
1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}.
$$

In order to compute the Teichmüller polynomial, we need to find the induced map on the edges of the train track lifted to the cover defined by $\pi_1(S) \to H_1(S; \mathbb{Z})^\psi$. Since the surface in this example has two boundary components, we have $H_1(S; \mathbb{Z})^\psi \cong \mathbb{Z}$. The cover is then constructed by cutting the surface along the cohomology classes that are invariant under the monodromy ψ and attaching them to form a $\mathbb{Z}$-covering. It is verified directly that the invariant homology for this example is generated by $a_1 - a_2 + a_3$, for example, by calculating the kernel of the adjacency matrix for A_5. Thus, winding around a_1 or a_3 leads one layer higher in the cover. Winding around a_3 leads one layer lower. The covering surface $\widetilde{S}$ is visualized in Fig.10.

We are interested in the induced action of $\widetilde{\psi}$ on the edge space of the lifted train track $\widetilde{\tau}$, regarded as a $\mathbb{Z}[H]$-module. We factor this action as $P_E = \widetilde{T_a}\widetilde{T_b}$ where $\widetilde{T_a} = UW$ and $\widetilde{T_b} = V$, as in Section 4: the matrices U and V are obtained by analyzing a single fundamental domain in the cover $\widetilde{S}$ and W is a diagonal matrix that corrects some columns to their proper levels.
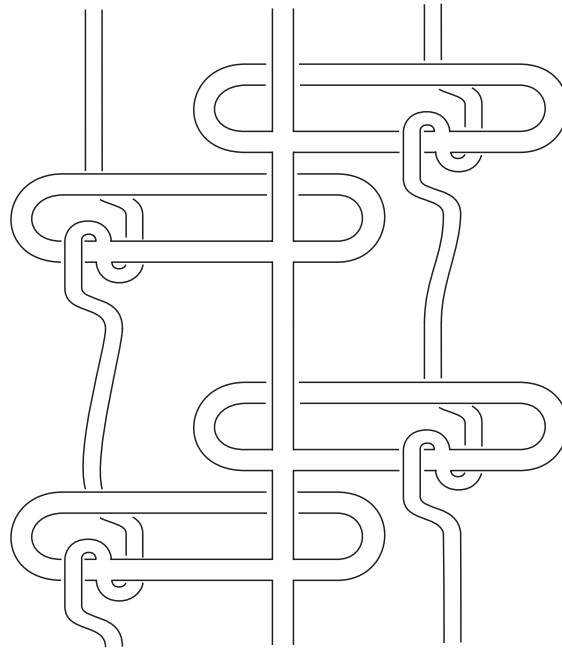
Fig. 10. Two copies of a fundamental domain of the covering surface $\widetilde{S}$ corresponding to $H_1(S;\mathbb{Z})^\psi$.

Note that no correction matrix T is needed to correct V to $\widetilde{T}_b$, since the ψ-invariant homology $H_1(S;\mathbb{Z})^\psi$ is supported entirely in the subspace of $H_1(S;\mathbb{Z})$ generated by the curves a_i. A fundamental domain of $\widetilde{\tau}$ well suited to our purposes is drawn in Fig.11. We caluclate U and V as

$$
V = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 & 0 & 1
\end{pmatrix}, \quad
U = \begin{pmatrix}
1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & x_0^{-1} & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}.
$$

The map on the train track τ induced by the negative multitwist along the a_i sends the edge e_5 to the edge path $e_5 e_1 e_2$. Tracing the lift of this edge path starting from the corresponding edge e_5 in the chosen fundamental domain, we see that we leave the fundamental domain and pass in the downward (x_0^{-1}) direction when tracing the lift of the edge e_2. This accounts for the entry x_0^{-1} in the matrix U.

The last thing to account for is the fact that when lifting the negative multitwist along the a_i to the train track $\widetilde{\tau}$, the starting point of the image of an edge from the fundamental domain might not lie in the fundamental domain. We decide to use the intersection of the curves b_1 and a_2 as basepoint for the lifts. In this case, the lift of the negative multitwist along the a_i sends the edges e_0, e_3, e_5, and e_6 to edge paths starting one level shifted by x_0 in
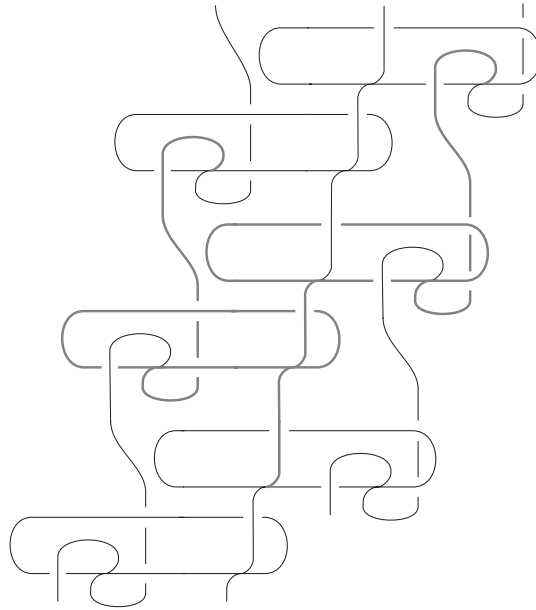
Fig. 11. The lifted train track $\widetilde{\tau}$ and the fundamental domain we use in our calculations, drawn thickly.

the lifted train track $\widetilde{\tau}$. The images of the other edges start in the fundamental domain, and so does the image of every edge under the lifted positive multitwist along the β_j. This gives the matrix W, with which we have to correct U in order to obtain $\widetilde{T}_a$:

$$W = \begin{pmatrix} x_0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & x_0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & x_0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & x_0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

The determinant formula stated in Section 2 also requires the action of the lifted monodromy on the vertices $\widetilde{\tau}$. Since the action on vertices is the same as on starting points of edges, this action is given by

$$P_V = \begin{pmatrix} x_0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & x_0 \end{pmatrix},$$

the upper diagonal block of W. All of our calculations so far have been in coordinates $H_1(M) \cong H_1(S)^\psi \oplus \mathbb{Z} \cong \langle x_0, x_1 \rangle$. The usual basis for the homology of link exteriors is the set of meridians of the link. If we push $a_1, -a_2,$ or a_3 off the embedded fiber surface S as shown in Fig.12, we can find the isomorphism needed the move to the usual coordinates. We can
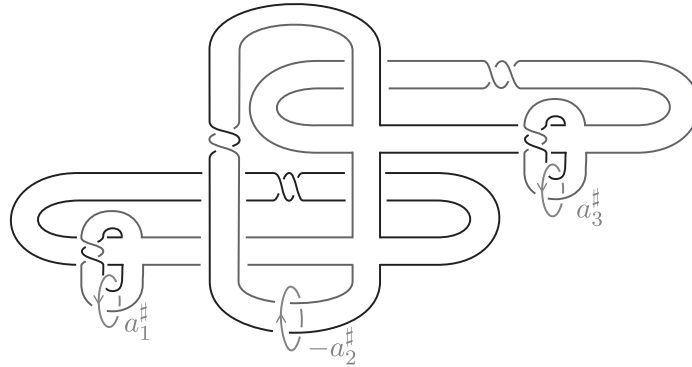
Fig. 12. The pushoffs $a_1^\sharp$, $-a_2^\sharp$ and $a_3^\sharp$ of a_1, $-a_2$ and a_3, respectively.

see that the isomorphism is

$$x_0 \mapsto y_0 y_1^{-1},$$
$$x_1 \mapsto y_1.$$

If we compute

$$\Theta_{A_5}(x_0, x_1) = \frac{\det(x_1 I - P_E(x_0))}{\det(x_1 I - P_V(x_0))}$$

and then change variables as described, we obtain

$$\Theta_{A_5}(y_0, y_1) = (y_0^2 y_1^2 - 4y_0^2 y_1 - 4y_0 y_1^2 + 4y_0^2 + 9y_0 y_1 + 4y_1^2 - 4y_0 - 4y_1 + 1).$$

Notice this is also (up to units) the Alexander polynomial for this link. The polynomial is associated to the face of the Thurston norm ball whose cone contains the homology class of S. This face is described by the hyperplane $2x_0 + 2x_1 = 1$ in the first quadrant of $H_2(M, \partial M; \mathbb{R}) \cong \mathbb{R}^2$.

5.3. Mutant links. The links obtained from distinct planar embeddings of an abstract tree Γ might be different. For example, in the classical case where all the signs are positive, this is true for a large class of trees by a theorem of Gerber [9]. In any case, different embeddings can not be distinguished by the single variable Alexander or Teichmüller polynomial. Indeed, these polynomials only depend on the abstract tree Γ and not the chosen embedding. While the train tracks corresponding to two distinct embeddings have the same incidence matrix, and the invariant homology of the monodromy maps are defined in terms of $A(\Gamma)$, the distribution of the invariant homology throughout the varying train tracks can be very different. With this in mind, it is not surprising that in some cases the multivariable Teichmüller polynomial distinguishes such links.

EXAMPLE 8. Consider the two embeddings Γ_1 and Γ_2 of the tree Γ shown in Fig. 13. By Remark 4, the two alternating-sign Coxeter links associated to Γ_1 and Γ_2 are mutant. We now give their Teichmüller polynomials and see that they are distinct. We consider the ambient manifolds of the alternating Coxeter links defined by these plane trees. In a basis of the first homology consisting of meridians of the corresponding links, the Teichmüller polynomials Θ_1 and Θ_2 of the fibered cones associated to Γ_1 and Γ_2 are given by
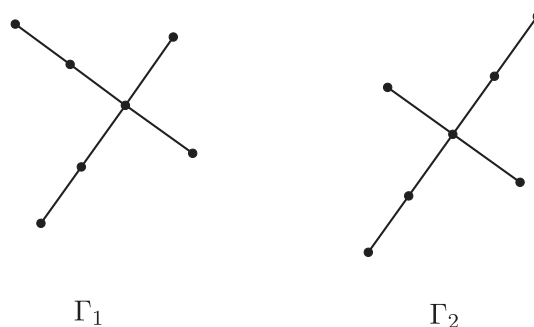
Fig. 13. Two different plane trees Γ_1 and Γ_2 with the same underlying abstract tree.

$$\Theta_1 = (x_1^2 - 3x_1 + 1)(x_0 x_1^3 - 7x_0 x_1^2 - 2x_1^3 + 9x_0 x_1 + 9x_1^2 - 2x_0 + 7x_1 + 1),$$

$$\Theta_2 = x_0^3 x_1^3 - 6x_0^3 x_1^2 - 6x_0^2 x_1^3 + 8x_0^3 x_1 + 30x_0^2 x_1^2 + 8x_0 x_1^3 - 2x_0^3 - 34x_0^2 x_1$$
$$- 34x_0 x_1^2 - 2x_1^3 + 8x_0^2 + 30x_0 x_1 + 8x_1^2 - 6x_0 - 6x_1 + 1,$$

which are distinct. The Teichmüller polynomials Θ_1 and Θ_2 can be calculated, for example, by using Theorem 1 and changing to the bases consisting of meridians. Note that since both links have two components, Theorem 1 can be used to compute the whole Teichmüller polynomial. The change of base is calculated explicitly. However, it does not change the fact that Θ_1 factors as a product of two polynomials while Θ_2 does not factor. In particular, the change of base is not needed to show that Θ_1 and Θ_2 are distinct.

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Robert Billet
Mathematical institute
Florida State University
600 W College Ave, Tallahassee
FL 32306
USA
e-mail: rbillet@math.fsu.edu

Livio Liechti
Sorbonne Université
IMJ-PRG, 75252 Paris cédex 05
France
email: livio.liechti@imj-prg.fr

[*Current Address*:]
Département de Mathématiques
Université de Fribourg
Chemin du Musée 23
1700 Fribourg
Suisse
e-mail: livio.liechti@unifr.ch