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Telescopic mappings in typed lambda calculus

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1. Introduction

In Automath (see [1],[3],[4]) the implementation of mathematical functions is a simple matter if the domain of the function is a type, but becomes slightly awkward if that domain is a part of a type. For example, if the type is the type of real numbers (let us assume it has been called "real"), and if we take as the domain the interval (3,7) (the set of all x with $3 < x < 7$), then the function value at a point b can be obtained only if apart from the value of b we provide a proof for $3 < b < 7$. Let us call the class of all such proofs $P(b)$. So for a function call we have to provide two expressions, b and u, and to establish the typings b : real, $u : P(b)$. Therefore the partial function has to be implemented in Automath by means of two lambda abstractors instead of a single one. With the Automath notation for typed lambdas (see section 2), these abstractors are $[x : \text{real}][y : P(x)]$. If, for example, the function is complex-valued, then its type becomes $[x : \text{real}][y : P(x)]\text{compl}$.

In such a sequence of two or more abstractors the type of the second may depend on the variable in the first, the type of the third may depend on the variables of the first two, etc. It reminds of an old-fashioned telescope consisting of a sequence of segments of decreasing width, where each segment can be shifted into the previous one. That is why these abstractor strings are called *telescopes.*

In Automath, like in all typed theories, we have to learn to live with this trouble of partial functions, due to the fact that sets (or generally subtypes) are not automatically types. One way to cope with it is to make facilities to attach a type of its own to every subtype, either by extension of the language machinery or by means of a few axioms. The question whether the use of such facilities is efficient may depend on the question whether the subtype plays a substantial role in a long piece of text. But one might also take the point of view that we can better leave the telescopes as they are, possibly soothing the pain by means of facilities for abbreviated input and output.

If we leave the telescopes as they are, and implement sets as telescopes of length 2, we may have to study mappings of sets into a type (like "compl" in the example above), but we can also have the situation that we want to restrict the range of the mapping. In particular we get this if we have to describe the composition of two functions. So we get to mappings where both domain and range are described by telescopes instead of just by types.

It is to this kind of mappings that this paper is devoted. We develop a notational system for these telescopic mappings, and for telescopes corresponding to these mappings. Moreover, we shall show the use of these notations for the matter of composition (section 9), for mappings where the range is a product of two telescopes (section 10), and for mappings where the domain is such a product (section 11).

It is easy to imagine that these telescopic mappings might be used as an input facility for Automath languages, but we shall not materialize that in this note. Moreover, the new notations to be developed in this note will not be presented as parts of a formal language, so everything is to be considered as informal metalanguage of Automath.

2. Automath notation for typed lambda calculus

We remind here how Automath expressions (AUT-expressions for short) are built:

(i) an identifier is an AUT-expression,

(ii) an identifier followed by a sequence of AUT-expressions, where the sequence is enclosed in ()'s and, if of length > 1, separated by commas, is an AUT-expression.

This procedure of providing an identifier with subexpressions, like in $f(P, Q, R)$ (where f is the identifier, and P, Q, R stand for expressions), is called *instantiation*. Instantiation will not play an explicit role in this paper (although the expressions we discuss may contain instantiations). The notations like $A(x)$ in section 3 should not be confused with instantiations: they represent a symbolism of the metalanguage.

(iii) If x is an identifier, and P and Q are AUT-expressions, then $[x : P]Q$ is an AUT-expression.

The Automath notation $[x : P]Q$ represents what others might write as $\lambda x : P.Q$ or $\lambda_{x:P}Q$; the expression P gives the type of the bound variable x. The part $[x : P]$ is called an *abstractor*.

(iv) If P and Q are AUT-expressions, then $\langle P\rangle Q$ is an AUT-expression. The interpretation is that Q stands for a function and $\langle P\rangle Q$ stands for the value of that function at the point P.

This deviates from the standard notation in lambda calculus and in most other parts of mathematics, where the function symbol is written in front of the argument. The part $\langle Q\rangle$ is called an *applicator*. Since abstractors are written in front of the expressions they act on, it is reasonable to write the applicators on the left as well, since quite often (in beta and eta reductions) an abstractor and a corresponding applicator cancel each other.

We shall be informal about the matter of names of bound variables. If we wish, we can get rid of these names by means of a system of namefree lambda calculus, for which we refer to [2].

3. Telescopes and vectors fitting into them

A telescope is an abstractor string

$$[x_1 : A_1][x_2 : A_2(x_1)] \ldots [x_k : A_k(x_1, \ldots, x_{k-1})].$$

The number k is called its length. The $A_i(x_1, \ldots, x_{i-1})$ stands for an AUT-expression that we allow to contain the variables $x_1, \ldots, x_{i-1}$ Note that the A_i have no separate meaning, it is only the combination $A_i(x_1, \ldots, x_{i-1})$ that makes sense.

We use column vector notation

$$\mathbf{v} = \begin{pmatrix} v_1 \\ \vdots \\ v_k \end{pmatrix}, \qquad \mathbf{A}(\mathbf{x}) = \begin{pmatrix} A_1(x_1, \ldots, x_k) \\ \vdots \\ A_k(x_1, \ldots, x_k) \end{pmatrix}$$

On the right we have written the column vector $\mathbf{x}$ as a row $x_1, \ldots, x_k$, just for typographical reasons. And for simplicity we have extended the strings of variables to length k for every A_i, although we know that A_i does not contain $x_i, x_{i+1}, \ldots$.

If $v_1, \ldots, v_k$ are AUT-expressions, then $\mathbf{v}$ will be called an AUT-vector.

The above telescope will be abbreviated to

$$[\mathbf{x} : \mathbf{A}(\mathbf{x})].$$

We say that the AUT-vector $\mathbf{v}$ *fits into* the telescope, notation

$$\mathbf{v} \ \epsilon\epsilon \ [\mathbf{x} : \mathbf{A}(\mathbf{x})]$$

if the following ordinary Automath typings are valid:

$$v_1 : A_1$$

$$v_2 : A_2(v_1)$$

$$\vdots$$

$$v_k : A_k(v_1, \ldots, v_{k-1})$$

It has to be pointed out that what is meant here is not instantiation. $A_k(v_1, \ldots, v_{k-1})$ stands for the expression we get if we start from the expression that was denoted by $A_k(x_1, \ldots, x_{k-1})$, and replace all occurrences of x_1 by the expression v_1 , etc.

We might have chosen any other symbol instead of $\epsilon\epsilon$. The only reason for $\epsilon\epsilon$ is that in Automath the fitting of a vector into a telescope can implement the belonging of an element to a set.

For the fitting of $\mathbf{v}$ into the telescope $[\mathbf{x} : \mathbf{A}(\mathbf{x})]$ we shall also use the notation

$$\mathbf{v} : \mathbf{A}(\mathbf{v}).$$

Note that on the right of $\epsilon\epsilon$ we have a telescope, on the right of : we have a vector. If

$$Q = [\mathbf{x} : \mathbf{A}(\mathbf{x})]$$

then $\mathbf{v}\epsilon\epsilon Q$ and $\mathbf{v} : \mathbf{A}(\mathbf{v})$ are synonymous. Accordingly, we might even write $[\mathbf{x}\epsilon\epsilon Q]$ instead of Q itself. If $\mathbf{v}$ has length 1, the notation $\mathbf{v} : \mathbf{A}(\mathbf{v})$ describes just ordinary typing.

If the vector $\mathbf{v}$ fits into the telescope Q, and if the length of $\mathbf{v}$ is >1, then we do not have the right to speak of Q as being *the* telescope of $\mathbf{v}$. If the length is 1, the simple relation between type and telescope guarantees that Q is uniquely determined by $\mathbf{v}$ in the sense that if $\mathbf{v}$ fits both into Q and into R, then Q and R are definitionally equal. If the length is >1, this is no longer the case. J. Zucker gave the following simple example. If

$$\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \quad \epsilon\epsilon \quad [x_1 : A_1][x_2 : A_2(x_1)]$$

then we also have

$$\mathbf{v} \quad \epsilon\epsilon \quad [x_1 : A_1][x_2 : A_2(v_1)],$$

and these two telescopes are definitely not definitionally equal.

4. Further notation

If $\mathbf{v}$ and $\mathbf{w}$ are vectors, possibly of different length:

$$\mathbf{v} = \begin{pmatrix} v_1 \\ \vdots \\ v_k \end{pmatrix}, \quad \mathbf{w} = \begin{pmatrix} w_1 \\ \vdots \\ w_h \end{pmatrix},$$

then $\langle \mathbf{v} \rangle \mathbf{w}$ denotes the vector

$$\begin{pmatrix} \langle v_k \rangle \ldots \langle v_2 \rangle \langle v_1 \rangle w_1 \\ \vdots \\ \langle v_k \rangle \ldots \langle v_2 \rangle \langle v_1 \rangle w_h \end{pmatrix}$$

And $[\mathbf{x} : \mathbf{A}(\mathbf{x})]\mathbf{w}$ denotes the vector

$$\begin{pmatrix} [\mathbf{x} : \mathbf{A}(\mathbf{x})] w_1 \\ \vdots \\ [\mathbf{x} : \mathbf{A}(\mathbf{x})] w_h \end{pmatrix}$$

Concatenation of the vectors $\mathbf{v}$ and $\mathbf{w}$ is denoted by $\mathbf{v} \circ \mathbf{w}$:

$$\mathbf{v} \circ \mathbf{w} = \begin{pmatrix} \mathbf{v} \\ \mathbf{w} \end{pmatrix} = \begin{pmatrix} v_1 \\ \vdots \\ v_k \\ w_1 \\ \vdots \\ w_h \end{pmatrix}.$$

We can also concatenate two telescopes. The second one may depend on variables introduced in the first one. If

$$Q = [\mathbf{x} : \mathbf{A}(\mathbf{x})]$$

$$R(\mathbf{x}) = [\mathbf{y} : \mathbf{B}(\mathbf{x}, \mathbf{y})]$$

then the concatenation can be written as

$$[\mathbf{x} : \mathbf{A}(\mathbf{x})][\mathbf{y} : \mathbf{B}(\mathbf{x}, \mathbf{y})]$$

If we introduce $\mathbf{A}^\star(\mathbf{x} \circ \mathbf{y}) = \mathbf{A}(\mathbf{x})$, $\mathbf{B}^\star(\mathbf{x} \circ \mathbf{y}) = \mathbf{B}(\mathbf{x}, \mathbf{y})$ then the concatenation can be written as a single telescope

$$[\mathbf{x} \circ \mathbf{y} : \mathbf{A}^\star(\mathbf{x} \circ \mathbf{y}) \circ \mathbf{A}^\star(\mathbf{x} \circ \mathbf{y})]$$

Quite often we have to deal with the matter that a concatenated vector $\mathbf{z} \circ \mathbf{v}(\mathbf{z})$ (the first set of entries are variables, the last ones are expressions containing these variabes) fits into the concatenated telescope $[\mathbf{x} : \mathbf{A}(\mathbf{x})][\mathbf{y} : \mathbf{B}(\mathbf{x}, \mathbf{y})]$. In those cases we say (in accordance with Automath metalanguage) that in the context $[\mathbf{x} : \mathbf{A}(\mathbf{x})]$ the vector $\mathbf{v}(\mathbf{x})$ fits into $[\mathbf{y} : \mathbf{B}(\mathbf{x}, \mathbf{y})]$.

5. Telescopic mappings

The semantics of a *telescopic mapping* is: a function that attaches a vector $\mathbf{v}$ to every vector that fits into a telescope Q (to be called the it domain telescope). If Q=$[\mathbf{x} : \mathbf{A}(\mathbf{x})]$, the syntax is

$$[\mathbf{x} : \mathbf{A}(\mathbf{x})]\mathbf{v}(\mathbf{x})$$

which will also be written as $\lambda_{\mathbf{x}\epsilon\epsilon Q}\mathbf{v}(\mathbf{x})$. Note that there is some danger of confusion in the notation $[\mathbf{x}{:}\mathbf{A}(\mathbf{x})]\mathbf{v}(\mathbf{x})$. It does not reveal what the domain telescope is, since $\mathbf{v}(\mathbf{x})$ itself may start with abstractors.

We shall be concerned with mappings where the values $\mathbf{v}(\mathbf{x})$ fit into a second telescope. In general, the second one can depend on the variables of the first:

$$Q = [\mathbf{x} : \mathbf{A}(\mathbf{x})]$$

$$R(\mathbf{x}) = [\mathbf{y} : \mathbf{B}(\mathbf{x}, \mathbf{y})].$$

So if we say that the values of the mapping $[\mathbf{x} : \mathbf{A}(\mathbf{x})]\mathbf{v}(\mathbf{x})$ fit into the second telescope, we mean that in the context $[\mathbf{x} : \mathbf{A}(\mathbf{x})]$ we have $\mathbf{v}(\mathbf{x})\epsilon\epsilon\mathbf{R}(\mathbf{x})$, which means

$$\mathbf{x} \circ \mathbf{v}(\mathbf{x}) \ \epsilon\epsilon \ [\mathbf{x} : \mathbf{A}(\mathbf{x})][\mathbf{y} : \mathbf{B}(\mathbf{x}, \mathbf{y})].$$

We shall build a new telescope into which all these mappings $[\mathbf{x} : \mathbf{A}(\mathbf{x})]\mathbf{v}(\mathbf{x})$, and nothing but these mappings, fit. We denote it by $\mu_{\mathbf{x}\epsilon\epsilon Q} R(\mathbf{x})$. We shall refer to it as a *functional telescope*. It is defined as

$$\mu_{\mathbf{x}\epsilon\epsilon Q} R(\mathbf{x}) = [\mathbf{s} : [\mathbf{x} : \mathbf{A}(\mathbf{x})]\mathbf{B}(\mathbf{x}, \langle\mathbf{x}\rangle\mathbf{s})].$$

In section 7 it will be established to have the required properties.

If Q has length k, R length m, then we have the following lengths:

Q	$R(\mathbf{x})$	$\mathbf{s}$	$\mathbf{x}$	$\mathbf{A}(\mathbf{x})$	$\mathbf{B}(\mathbf{x}, \langle\mathbf{x}\rangle\mathbf{s})$	$\mu_{\mathbf{x}\epsilon\epsilon Q} R(\mathbf{x})$
k	m	m	k	k	m	m

6. An example

Let us describe mappings from the interior of the unit circle in the complex plane into the set of all real numbers y with $0 \leq y < 1$.

The interior of the unit circle can be related to the telescope

$$Q = [z : \text{compl}][u : P(z)]$$

(if z is a complex number then $P(z)$ represents the class of all proofs for the statement that the absolute value of z is less than 1). The range set is related to the telescope

$$R = [y : \text{real}][v : W(y)]$$

(if y is a real number, then $W(y)$ is the class of all proofs for the statement that $0 \leq y < 1$). The functional telescope becomes (note that $z \circ u$ is a column vector of length 2, with entries z and u)

$$\mu_{z\circ u\epsilon\epsilon Q} =$$

$$[p : [z : \text{compl}][u : P(z)]\text{real}][q : [z : \text{compl}][u : P(z)]W(\langle u\rangle\langle z\rangle p)].$$

Let the vector $\mathbf{f} = f \circ w$ fit into this functional telescope. So

$$f : [z : \text{compl}][u : P(z)]\text{real},$$

$$w : [z : \text{compl}][u : P(z)]W(\langle u\rangle\langle z\rangle f).$$

Furthermore, let $a \circ m$ fit into the domain telescope Q, whence a : compl, $m : P(a)$. Now

$$\langle a \circ m\rangle f = \begin{pmatrix} \langle m\rangle\langle a\rangle f \\ \langle m\rangle\langle a\rangle w \end{pmatrix}, \quad \text{typed by} \quad \begin{pmatrix} \text{real} \\ W(\langle m\rangle\langle a\rangle f) \end{pmatrix}$$

If we put $\langle m\rangle\langle a\rangle f = b$, $\langle m\rangle\langle a\rangle w = r$, we have $\langle a \circ m\rangle f = b \circ r$, and we infer that b : real and $r : Q(b)$, i.e., r is a proof for the statement that b satisfies $0 \leq b < 1$. In other words, $\langle a \circ m\rangle f$ fits into R.

We can now check that

$$[z : \text{compl}][u : P(z)]\langle z \circ u\rangle \mathbf{f}$$

is a telescopic mapping (defined on Q) whose values fit into R. It reduces to $\mathbf{f}$ by eta reduction (see section 7).

7. Beta and eta reduction

If $\mathbf{v} \epsilon\epsilon [\mathbf{x} : \mathbf{A}(\mathbf{x})]$ then

$$\langle \mathbf{v}\rangle[\mathbf{x} : \mathbf{A}(\mathbf{x})]\mathbf{w}(\mathbf{x}) \quad >_\beta \quad \mathbf{w}(\mathbf{v}).$$

The number of beta reduction steps is equal to the length of $\mathbf{v}$.

If the vector $\mathbf{f}$ does not contain the variable $\mathbf{x}$ then

$$[\mathbf{x} : \mathbf{A}(\mathbf{x})]\langle \mathbf{x}\rangle \mathbf{f} \quad >_\eta \quad \mathbf{f}.$$

The number of eta reduction steps is equal to the length of $\mathbf{x}$.

For simplicity, we shall always consider two expressions as equal if they are definitionally equal in the sense of beta and eta reduction.

8. The rules for telescopic mappings

In this section, Q and $R(\mathbf{x})$ will be as in section 5:

$$Q = [\mathbf{x} : \mathbf{A}(\mathbf{x})], \quad R(\mathbf{x}) = [\mathbf{y} : \mathbf{B}(\mathbf{x}, \mathbf{y})]$$

We formulate the rules for empty context, but they hold similarly in an arbitrary context.

As a warning we first mention that $\mathbf{f}(\mathbf{x})$ is not the value of a function $\mathbf{f}$ at a point $\mathbf{x}$. Like $\mathbf{A}(\mathbf{x}), \ldots$ in section 3, $\mathbf{f}(\mathbf{x})$ stands for a vector of expressions containing the variables $x_1, \ldots$. If in $\mathbf{f}(\mathbf{x})$ we replace all x_i by corresponding v_i 's, the result will be denoted by $\mathbf{f}(\mathbf{v})$. It will be a consequence of Rule I (below) that $\mathbf{f}(\mathbf{v})$ can be interpreted as a function value, but not as a function value of $\mathbf{f}$. The function it can be interpreted as a value of, is $\mathbf{g}$, where $\mathbf{g} = [\mathbf{x}\epsilon\epsilon\mathbf{Q}]\mathbf{f}(\mathbf{x})$.

Rule I. If $\mathbf{f}\epsilon\epsilon\mu_{\mathbf{x}\epsilon\epsilon Q}R(\mathbf{x})$ then (by eta reduction)

$$\lambda_{\mathbf{x}\epsilon\epsilon Q}\langle\mathbf{x}\rangle\mathbf{f} = \mathbf{f}.$$

If moreover $\mathbf{v}\epsilon\epsilon Q$ then we have as a consequence

$$\langle\mathbf{v}\rangle\lambda_{\mathbf{x}\epsilon\epsilon Q}\langle\mathbf{x}\rangle\mathbf{f} = \langle\mathbf{v}\rangle\mathbf{f}.$$

but here beta reduction will do, we do not need eta.

Rule II (Introduction rule).

$$\frac{\mathbf{f}(\mathbf{x})\epsilon\epsilon R(\mathbf{x}) \quad \text{in the context} \quad [\mathbf{x}\epsilon\epsilon Q]}{\lambda_{\mathbf{x}\epsilon\epsilon Q}\mathbf{f}(\mathbf{x}) \ \epsilon\epsilon \ \mu_{\mathbf{x}\epsilon\epsilon Q}R(\mathbf{x})}$$

Proof. In the context $[\mathbf{x} : \mathbf{A}(\mathbf{x})]$ we have $\mathbf{f}(\mathbf{x}) : \mathbf{B}(\mathbf{x}, \mathbf{f}(\mathbf{x}))$. If $\mathbf{g} = \lambda_{\mathbf{x}\epsilon\epsilon Q}\mathbf{f}(\mathbf{x})$ then $\langle\mathbf{x}\rangle\mathbf{g}$ is definitionally equal to $\mathbf{f}(\mathbf{x})$, so $\mathbf{f}(\mathbf{x}) : \mathbf{B}(\mathbf{x}, \langle\mathbf{x}\rangle\mathbf{g})$ in the context $[\mathbf{x} : \mathbf{A}(\mathbf{x})]$. Therefore

$$[\mathbf{x} : \mathbf{A}(\mathbf{x})]\mathbf{f}(\mathbf{x}) \quad : \quad [\mathbf{x} : \mathbf{A}(\mathbf{x})]\mathbf{B}(\mathbf{x}, \langle\mathbf{x}\rangle\mathbf{g})$$

and so

$$\mathbf{g} : [\mathbf{x} : \mathbf{A}(\mathbf{x})]\mathbf{B}(\mathbf{x}, \langle\mathbf{x}\rangle\mathbf{g}).$$

As $\mu_{\mathbf{x}\epsilon\epsilon Q}R(\mathbf{x}) = [\mathbf{s} : [\mathbf{x} : \mathbf{A}(\mathbf{x})]\mathbf{B}(\mathbf{x}, \langle\mathbf{x}\rangle\mathbf{s})]$ we now conclude $\mathbf{g}\epsilon\epsilon\mu_{\mathbf{x}\epsilon\epsilon Q}R(\mathbf{x})$.

Rule III (Elimination rule).

$$\frac{\mathbf{g} \epsilon\epsilon \mu_{\mathbf{x}\epsilon\epsilon Q} R(\mathbf{x}) \qquad \mathbf{v} \epsilon\epsilon Q}{\langle \mathbf{v} \rangle \mathbf{g} \ \epsilon\epsilon \ R(\mathbf{v})}$$

Proof. We have

$$\mathbf{g} : [\mathbf{x} : \mathbf{A}(\mathbf{x})]\mathbf{B}(\mathbf{x}, \langle \mathbf{x} \rangle \mathbf{g}).$$

$$\mathbf{v} : \mathbf{A}(\mathbf{v}),$$

and therefore $\langle \mathbf{v} \rangle \mathbf{g} : \mathbf{B}(\mathbf{v}, \langle \mathbf{v} \rangle \mathbf{g})$.

Now note that $R(\mathbf{v}) = [\mathbf{y} : \mathbf{B}(\mathbf{v}, \mathbf{y})]$, so $\langle \mathbf{v} \rangle \mathbf{g} : \mathbf{B}(\mathbf{v}, \langle \mathbf{v} \rangle \mathbf{g})$ can be interpreted as $\langle \mathbf{v} \rangle \mathbf{g} \ \epsilon\epsilon \ R(\mathbf{v})$.

9. Composition of functions

We restrict ourselves to the case of independent telescopes Q, R, S. Take

$$\mathbf{f} \ \epsilon\epsilon \ \mu_{\mathbf{x}\epsilon\epsilon Q} R, \qquad \mathbf{g} \ \epsilon\epsilon \ \mu_{\mathbf{y}\epsilon\epsilon R} S.$$

Then we have

$$\lambda_{\mathbf{x}\epsilon\epsilon Q} \langle \langle \mathbf{x} \rangle \mathbf{f} \rangle \mathbf{g} \ \epsilon\epsilon \ \mu_{\mathbf{x}\epsilon\epsilon Q} S.$$

Proof. In the context $[\mathbf{x} \epsilon\epsilon Q]$ we have $\mathbf{x} \epsilon\epsilon Q$, and therefore by Rule III $\langle \mathbf{x} \rangle \mathbf{f} \epsilon\epsilon R$. Again by Rule III $\langle \langle \mathbf{x} \rangle \mathbf{f} \rangle \mathbf{g} \epsilon\epsilon S$. Finally we apply Rule II.

10. Mappings into a product

Under this heading we generalize the idea of mappings of a set A into the cartesian product of two sets B and C. The set $A \to (B \times C)$ of all mappings of A into $B \times C$ can be seen as the cartesian product $(A \to B) \times (A \to C)$. We shall generalize this to telescopes. Taking

$$Q = [\mathbf{x} : \mathbf{A}(\mathbf{x})]$$
$$R_1(\mathbf{x}) = [\mathbf{y} : \mathbf{B}(\mathbf{x}, \mathbf{y})]$$
$$R_2(\mathbf{x}, \mathbf{y}) = [\mathbf{z} : \mathbf{C}(\mathbf{x}, \mathbf{y}, \mathbf{z})],$$

we have

$$\mu_{\mathbf{x}\epsilon\epsilon Q}([\mathbf{y} : \mathbf{B}(\mathbf{x}, \mathbf{y})][\mathbf{z} : \mathbf{C}(\mathbf{x}, \mathbf{y}, \mathbf{z})]) =$$
$$= [\mathbf{f}\epsilon\epsilon\mu_{\mathbf{x}\epsilon\epsilon Q} R_1(\mathbf{x})][\mathbf{g}\epsilon\epsilon\mu_{\mathbf{x}\epsilon\epsilon Q} R_2(\mathbf{x}, \langle\mathbf{x}\rangle\mathbf{f})].$$

Proof. Introduce $\mathbf{B}^\star$ and $\mathbf{C}^\star$ by

$$\mathbf{B}^\star(\mathbf{x}, \mathbf{y} \circ \mathbf{z}) = \mathbf{B}(\mathbf{x}, \mathbf{y}), \quad \mathbf{C}^\star(\mathbf{x}, \mathbf{y} \circ \mathbf{z}) = \mathbf{C}(\mathbf{x}, \mathbf{y}, \mathbf{z}).$$

Then the concatenation $[\mathbf{y} : \mathbf{B}(\mathbf{x}, \mathbf{y})][\mathbf{z} : \mathbf{C}(\mathbf{x}, \mathbf{y}, \mathbf{z})]$ can be written as a single telescope

$$[\mathbf{y} \circ \mathbf{z} : \mathbf{H}(\mathbf{x}, \mathbf{y} \circ \mathbf{z})], \text{where } \mathbf{H} = \mathbf{B}^\star \circ \mathbf{C}^\star.$$

Now

$$\mu_{\mathbf{x}\epsilon\epsilon Q}[\mathbf{y} \circ \mathbf{z} : \mathbf{H}(\mathbf{x}, \mathbf{y} \circ \mathbf{z})] = [\mathbf{s} : [\mathbf{x} : \mathbf{A}(\mathbf{x})]\mathbf{H}(\mathbf{x}, \langle\mathbf{x}\rangle\mathbf{s})].$$

Writing $\mathbf{s} = \mathbf{f} \circ \mathbf{g}$ (with appropriate lengths of $\mathbf{f}$ and $\mathbf{g}$) we get

$$\mathbf{H}(\mathbf{x}, \langle\mathbf{x}\rangle\mathbf{s}) = \mathbf{H}(\mathbf{x}, (\langle\mathbf{x}\rangle\mathbf{f}) \circ (\langle\mathbf{x}\rangle\mathbf{g})) =$$
$$= \mathbf{B}(\mathbf{x}, \langle\mathbf{x}\rangle\mathbf{f}) \circ \mathbf{C}(\mathbf{x}, \langle\mathbf{x}\rangle\mathbf{f}, \langle\mathbf{x}\rangle\mathbf{g}).$$

Therefore

$$[\mathbf{s} : [\mathbf{x} : \mathbf{A}(\mathbf{x})]\mathbf{H}(\mathbf{x}, \langle\mathbf{x}\rangle\mathbf{s})] =$$
$$= [\mathbf{f} : [\mathbf{x} : \mathbf{A}(\mathbf{x})]\mathbf{B}(\mathbf{x}, \langle\mathbf{x}\rangle\mathbf{f})][\mathbf{g} : [\mathbf{x} : \mathbf{A}(\mathbf{x})]\mathbf{C}(\mathbf{x}, \langle\mathbf{x}\rangle\mathbf{f}, \langle\mathbf{x}\rangle\mathbf{g})] =$$
$$= [\mathbf{f}\epsilon\epsilon\mu_{\mathbf{x}\epsilon\epsilon Q} R_1(\mathbf{x})][\mathbf{g}\epsilon\epsilon R_2(\mathbf{x}, \langle\mathbf{x}\rangle\mathbf{f})].$$

11. Mappings where the domain is a product

Under this heading we generalize the idea of mappings of a cartesian product of two sets A and B into a set C. The set of all those mappings, $(A \times B) \to C$, can be interpreted as the set $A \to (B \to C)$ of all mappings of A into the set of all mappings of B into C. We shall generalize this to telescopes.

Consider

$$[\mathbf{y} : \mathbf{B}(\mathbf{y})][\mathbf{z} : \mathbf{C}(\mathbf{y}, \mathbf{z})][\mathbf{w} : \mathbf{D}(\mathbf{y}, \mathbf{z}, \mathbf{w})]$$

and call the components R_1 , $R_2(\mathbf{y})$ and $T(\mathbf{y}, \mathbf{z})$. If the product of the first two is called S, we have

$$\mu_{\mathbf{y}\circ\mathbf{z}\epsilon\epsilon S} T(\mathbf{y}, \mathbf{z}) = \mu_{\mathbf{y}\epsilon\epsilon R}(\mu_{\mathbf{z}\epsilon\epsilon R_2(\mathbf{y})}(T(\mathbf{y}, \mathbf{z})).$$

Proof. $\mu_{\mathbf{y}\circ\mathbf{z}\epsilon\epsilon S} T(\mathbf{y}, \mathbf{z}) =$

$$[\mathbf{s} : [\mathbf{y} : \mathbf{B}(\mathbf{y})][\mathbf{z} : \mathbf{C}(\mathbf{y}, \mathbf{z})]\mathbf{D}(\mathbf{y}, \mathbf{z}, \langle \mathbf{y} \circ \mathbf{z} \rangle \mathbf{s})].$$

On the other hand

$$\mu_{\mathbf{z}\epsilon\epsilon R_2(\mathbf{y})} T(\mathbf{y}, \mathbf{z}) = [\mathbf{t} : [\mathbf{z} : \mathbf{C}(\mathbf{y}, \mathbf{z})]\mathbf{D}(\mathbf{y}, \mathbf{z}, \langle \mathbf{z} \rangle \mathbf{t})],$$

so

$$\mu_{\mathbf{y}\epsilon\epsilon R_1}(\mu_{\mathbf{z}\epsilon\epsilon R_2(\mathbf{y})} T(\mathbf{y}, \mathbf{z})) =$$

$$[\mathbf{s} : [\mathbf{y} : \mathbf{B}(\mathbf{y})][\mathbf{z} : \mathbf{C}(\mathbf{y}, \mathbf{z})]\mathbf{D}(\mathbf{y}, \mathbf{z}, \langle \mathbf{y} \rangle \langle \mathbf{z} \rangle \mathbf{s})].$$

It now suffices to note that $\langle \mathbf{y} \circ \mathbf{z} \rangle \mathbf{s} = \langle \mathbf{z} \rangle \langle \mathbf{y} \rangle \mathbf{s}$.

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References.

1. N.G. de Bruijn. The mathematical language AUTOMATH, its usage and some of its extensions, Symposium on Automatic Demonstration, IRIA,

Versailles, December 1968. (Lecture Notes in Mathematics, Vol. 125, Springer Verlag 1970, pp. 29-61).

2. N.G. de Bruijn. Lambda calculus notation with nameless dummies, a tool for automatic formula manipulation, with application to the Church-Rosser theorem. Kon. Nederl. Akad. Wetensch. Proceedings Ser. A 75 (=Indagationes Math. 34) pp.381-392 (1972).

3. N.G. de Bruijn. A survey of the project Automath. To H.B. Curry: Essays in combinatory logic, lambda calculus and formalism, Academic Press 1980, pp. 579-606.

4. D.T. van Daalen. The language theory of AUTOMATH. Doctoral Thesis, Dept. of Mathematics, Eindhoven University of Technology, 1980.