

Temperature distribution in an elliptical body without internal heat sources under boundary conditions of the third kind with partial adiabatic isolation

Alexandr Kanareykin^{1*}

¹Sergo Ordzhonikidze Russian State University for Geological Prospecting, 117997, Miklouho-Maclay St. 23., Moscow, Russia

Abstract. The article deals with the distribution of the temperature field in an elliptical body without internal heat sources. In this case, the boundary conditions are boundary conditions of the third kind. The solution is found when moving to an elliptic coordinate system. The author has obtained an analytical solution for the distribution of the temperature field in a body with an elliptical cross section of infinite length at a given ambient temperature with partial adiabatic isolation in the form of a functional series using hypergeometric functions.

1 Introduction

The processes of heat exchange and the mass transfer resulting from it currently play an important role both in the technical sphere and in nature. The temperature regime of the surrounding medium directly depends on this group of processes. In addition, this group of processes determines the workflow in various technological installations. This leads to the active development of the theory of heat transfer, especially during the last 10-15 years. The active development of the theory is also due to the needs of such spheres of human activity as cosmonautics, thermal power engineering, nuclear energy.

It is not surprising, therefore, that the theory of heat transfer has developed intensively, especially in recent decades. This is due to the needs of thermal power engineering, nuclear energy, and cosmonautics. The intensification of various technological processes, as well as the creation of optimal installations from the point of view of energy consumption, is unthinkable without a deep study of the thermophysical processes that take place in these installations. In connection with the improvement of the thermal equipment of energy-consuming and producing devices, a more accurate calculation of heat transfer processes in thermal networks is required. Therefore, it seems advisable to improve the methods of calculating heat transfer in such systems. It is known that for better cooling of fuel elements (electrical conductors, rods of nuclear reactors, etc.), it is necessary to have a large heat transfer surface. An increase in the surface can be achieved either by finning, or by replacing rods of circular cross-section, which have a minimum surface of the heat sink,

* Corresponding author: kanareykins@mail.ru

with rods of another cross-section, for example, oval or elliptical. A special place is occupied by bodies with an elliptical cross-section. Their peculiarity lies in the fact that by manipulating the change in the length of the semi-axes of the ellipse, it is possible to obtain accurate analytical solutions to stationary thermal conductivity problems for a very wide range of shape changes: from a cylinder to a thin plate.

To date, an impressive layer of work has been devoted to the study of heat exchange processes. Of particular scientific interest are the works describing modern heat exchange elements of heat exchange equipment with a detailed description of the methods of their manufacture, as well as the control of a heat exchanger with a variable surface area of the heat exchanger. Several works [1-4] are devoted to the calculation of temperature fields in bodies of elliptical cross-section in the presence of internal heat sources under various conditions. This article is a continuation of the works [5, 6].

2 Main Part

The main task of this work is to find the distribution of the temperature field in a body with an elliptical cross-section of infinite length under boundary conditions of the third kind without internal heat sources. In power plants, heat transfer is often observed between two media (heat carriers) through a solid wall separating them, which is called heat transfer. In this case, heat from a more heated heat carrier is transferred to the wall due to heat transfer and thermal radiation, heat exchange occurs inside the wall due to thermal conductivity, and from the opposite surface of the wall is carried out due to heat transfer to a less heated heat carrier.

Boundary conditions of this type play an important role in the theory of heat and mass transfer, since they are a mathematical formulation of the conditions of convective heat and mass transfer.

We will look for the temperature distribution in an infinitely long body, the cross-section of which is an ellipse with semi-axes a and b (fig. 1), half of the surface of which is adiabatically isolated. The body in question is located in an environment with a given temperature T_0 .

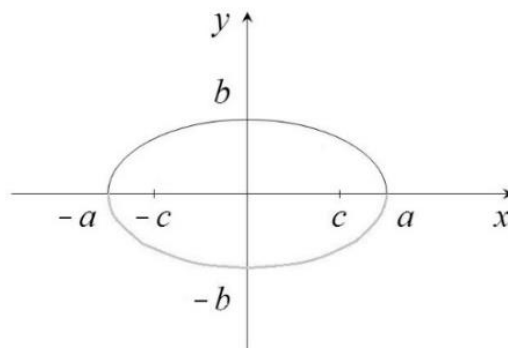


Fig. 1. Elliptical section of the body in the section.

To find the temperature distribution, it is necessary to solve the Laplace equation

$$\Delta T = 0 \quad (1)$$

which describes a stationary temperature field with boundary conditions on the body surface: the first half is the heat transfer

$$-\lambda \operatorname{grad} T = h(T - T_0) \tag{2}$$

the second heat transfer is not

$$-\lambda \operatorname{grad} T = 0 \tag{3}$$

To obtain the formula describing the temperature field, we use the elliptic coordinate system $\alpha, \beta, 0 \leq \alpha < \infty, -\pi \leq \beta \leq \pi$. If $\alpha = \alpha_0$ is the equation of the body surface, then

$$a = cch\alpha_0, b = chsh\alpha_0, c = \sqrt{a^2 + b^2} \tag{4}$$

The Laplace equation in elliptic coordinates has the form

$$\frac{\partial^2 U}{\partial \alpha^2} + \frac{\partial^2 U}{\partial \beta^2} = 0 \tag{5}$$

and the boundary conditions are given by the dependence for $\alpha = \alpha_0, 0 \leq \beta \leq \pi$

$$-\lambda \frac{1}{c\sqrt{ch^2\alpha_0 - \cos^2\beta}} \frac{\partial T}{\partial \alpha} = h(T - T_0) \tag{6}$$

for $\alpha = \alpha_0, -\pi \leq \beta \leq 0$

$$\frac{\partial T}{\partial \alpha} = 0 \tag{7}$$

The solution of equation (5) is given by the dependence

$$T(\alpha, \beta) = \sum_{n=0}^{\infty} (A_n chn\alpha \cos n\beta + B_n shn\alpha \sin n\beta) \tag{8}$$

We find the constants B_n from the boundary condition (7)

$$\sum_{n=0}^{\infty} (nA_n shn\alpha_0 \cos n\beta - nB_n chn\alpha_0 \sin n\beta) = 0 \tag{9}$$

and integrating from $-\pi$ to 0, we get

$$2B_{2n+1} ch(2n+1)\alpha_0 = 0 \tag{10}$$

from

$$B_{2n+1} = 0 \tag{11}$$

Then the solution of equation (5) will take the form

$$T(\alpha, \beta) = \sum_{n=0}^{\infty} A_n chn\alpha \cos n\beta \quad (12)$$

We find the constants A_n from the boundary condition (6)

$$\frac{1}{\sqrt{ch^2\alpha_0 - \cos^2\beta}} \sum_{n=0}^{\infty} nA_n shn\alpha_0 \cos n\beta = -Bi \left(\sum_{n=0}^{\infty} A_n chn\alpha \cos n\beta - T_0 \right) \quad (13)$$

$$Bi = \frac{hc}{\lambda} \quad (14)$$

where: Bi – the number of Bio, which characterizes the intensity of heat exchange between the surface of the body and the environment, h –the coefficient of heat transfer between the medium and the surface of the body.

The function

$$f(\beta) = \sqrt{ch^2\alpha_0 - \cos^2\beta} \quad (15)$$

is even with respect to β , we decompose it in a series of cosines

$$\sqrt{ch^2\alpha_0 - \cos^2\beta} = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\beta \quad (16)$$

where

$$a_n = \frac{2}{\pi} \int_0^{\pi} \sqrt{ch^2\alpha_0 - \cos^2\beta} \cos n\beta d\beta \quad (17)$$

Since

$$\cos n\beta = \cos^n \beta - C_n^2 \cos^{n-2} \beta \sin^2 \beta + C_n^4 \cos^{n-4} \beta \sin^4 \beta - \dots \quad (18)$$

then all with odd n vanish. Therefore

$$\sqrt{ch^2\alpha_0 - \cos^2\beta} = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_{2n} \cos 2n\beta \quad (19)$$

Rewriting (13) as

$$\begin{aligned} \sum_{n=0}^{\infty} nA_n shn\alpha_0 \cos n\beta = -Bi \left(\frac{a_0}{2} \sum_{n=0}^{\infty} A_n chn\alpha_0 \cos n\beta + \right. \\ \left. + \sum_{n=1}^{\infty} a_{2n} \cos 2n\beta \sum_{n=0}^{\infty} A_n chn\alpha_0 \cos n\beta - T_0 \sqrt{ch^2\alpha_0 - \cos^2\beta} \right) \quad (20) \end{aligned}$$

and integrating by β from 0 to π , we get

$$0 = Bi \left[T_0 \int_0^\pi \sqrt{ch^2 \alpha_0 - \cos^2 \beta} d\beta - \frac{\pi}{2} (a_0 A_0 + ch2n\alpha_0 a_{2n} A_{2n}) \right] \quad (21)$$

from where

$$A_{2n} = \frac{2T_0}{\pi} \frac{\int_0^\pi \sqrt{ch^2 \alpha_0 - \cos^2 \beta} d\beta}{ch2n\alpha_0 a_{2n}}, n = 0, 1, 2, \dots \quad (22)$$

elliptic integrals are used to calculate the integrals in the last formula

$$\begin{aligned} \int_0^\pi \sqrt{ch^2 \alpha_0 - \cos^2 \beta} d\beta &= \int_{-\pi/2}^{\pi/2} ch\alpha_0 \sqrt{1 - \frac{\sin^2 x}{ch^2 \alpha_0}} d(-x) = \\ &= 2ch\alpha_0 E\left(\frac{1}{ch\alpha_0}, \frac{\pi}{2}\right) \end{aligned} \quad (23)$$

where E is a complete elliptic integral of the second kind. Where from

$$A_{2n} = 4T_0 ch\alpha_0 E\left(\frac{1}{ch\alpha_0}, \frac{\pi}{2}\right); \pi ch2n\alpha_0 a_{2n}, n = 0, 1, 2, \dots \quad (24)$$

The coefficients a_{2n} are determined by the formula

$$\begin{aligned} a_{2n} &= \frac{2}{\pi} \int_0^\pi \sqrt{ch^2 \alpha_0 - \cos^2 \beta} (\cos^{2n} \beta - C_n^2 \cos^{2n-2} \beta \sin^2 \beta + \dots) d\beta = \\ &= \frac{4ch\alpha_0}{\pi} \int_0^{\pi/2} \sqrt{1 - \frac{\sin^2 x}{ch^2 \alpha_0}} (\sin^{2n} x - C_n^2 \sin^{2n-2} x \cos^2 x + \dots) dx = \\ &= \frac{2ch\alpha_0}{\pi} \left[B\left(\frac{2n+1}{2}, \frac{1}{2}\right) F_1\left(\frac{2n+1}{2}, -\frac{1}{2}, n+1, \frac{1}{ch^2 \alpha_0}\right) - \right. \\ &\quad \left. - C_n^2 B\left(\frac{2n-1}{2}, \frac{3}{2}\right) F_1\left(\frac{2n-1}{2}, -\frac{1}{2}, n+1, \frac{1}{ch^2 \alpha_0}\right) + \dots \right] \end{aligned} \quad (25)$$

where $B(y, z)$ is the beta function, $F_1(y, z, m; k)$ is a hypergeometric function.

The desired temperature distribution is described by the equation

$$T(\alpha, \beta) = \sum_{n=0}^{\infty} A_{2n} ch2n\alpha \cos 2n\beta \quad (26)$$

Then, using the formula (26), we calculate the coefficients A_{2n} . The coefficient A_0 is equal to

$$A_0 = 4T_0 ch\alpha_0 E\left(\frac{1}{ch\alpha_0}, \frac{\pi}{2}\right) : \pi a_0 = 2ch\alpha_0 T_0 : \pi a_0 \quad (27)$$

$$a_0 = \frac{2ch\alpha_0}{\pi} \left[B\left(\frac{1}{2}, \frac{1}{2}\right) \times F_1\left(\frac{1}{2}, -\frac{1}{2}, 1, \frac{1}{ch^2\alpha_0}\right) \right] = \frac{2ch\alpha_0}{\pi} \quad (28)$$

The remaining members of the series (26) are rapidly decreasing, so the distribution of the temperature field will take the form

$$T = T_0 \quad (29)$$

It follows from this that the temperature does not depend on either the heat transfer coefficient or the shape of the body, but is completely determined by the ambient temperature.

3 Conclusions

In this paper, we consider the solution of the stationary problem of the distribution of the temperature field in a body with an elliptical cross-section without internal heat sources under boundary conditions of the third kind for the case of adiabatic insulation of half of the wall. Since we consider the heat conduction process for a cylinder with an elliptical cross-section, which is a simply connected region. In accordance with this, the stationary solution of this problem should not depend on the heat transfer coefficient, nor on the shape of the housing, nor on the adiabatic insulation, but should depend only on the ambient temperature. Which was received. This indicates the reliability of the result obtained. If we introduce stationary internal heat sources, then the solution to this problem will depend not only on the temperature of the cooling medium, but also on the shape of the housing and the intensity of heat exchange.

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