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TEMPERATURE OF A SEMI-INFINITE ROD WHICH RADIATES BOTH LINEARLY AND NONLINEARLY*

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1. Introduction. Let $\theta(x, t)$ represent the temperature distribution in a semi-infinite rod from which energy (heat) is radiated along the entire length. In addition, suppose that at the end $x = 0$ energy is radiated at a rate proportional to the nth power of the temperature there ($n > 1$) and is radiated or absorbed linearly. Finally, suppose that energy is applied to the end at a rate proportional to some given function $f(t)$. If the temperature is initially zero then $\theta(x, t)$ may be described by the initial-boundary-value problem

$$\theta_t(x, t) = \theta_{xx}(x, t) - h\theta(x, t), \qquad x > 0, \quad t > 0, \tag{1.1}$$

$$\theta_x(0, t) = \alpha\theta^n(0, t) + b\theta(0, t) - f(t), \qquad t > 0, \tag{1.2}$$

$$\theta(x, 0) = 0, \qquad x \geq 0, \tag{1.3}$$

$$\theta(x, t) \to 0 \quad \text{as} \quad x \to \infty, \qquad t \geq 0. \tag{1.4}$$

Here $\alpha \geq 0$, $h \geq 0$, and b are given constants. Also $f(t)$ is bounded, non-negative, at least piecewise continuous, and becomes positive in a neighborhood of the origin. Moreover $f(t) = 0$ for $t_c < t < \infty$.

This problem is a generalization of the one considered by Keller and Olmstead [1] and by Handelsman and Olmstead [2]. Specifically, they considered this problem with $b = h = 0$. Friedman [3] discussed the existence, uniqueness, and certain other properties of the solutions of similar problems but with the requirement that an outward derivative at the surface be a strictly decreasing function of the dependent variable.

An alternative physical interpretation of (1.1)–(1.4) is that $\theta(x, t)$ may represent the concentration of a diffusant in an absorbent occupying the halfspace $x \geq 0$ between which a first-order chemical reaction takes place. In this case the boundary condition (1.2) is taken to describe the evaporation or absorbtion of the diffusant through the surface.

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In the following work a constructive proof of the existence of a positive solution is presented and it is demonstrated that there can be only one positive solution to the nonlinear problem. Linear problems whose solutions bound $\theta(x, t)$ are presented. These bounds are then used to describe the large-time behavior of $\theta(x, t)$. Specifically, $\theta(x, t)$ is bounded by quadratures which yield the asymptotic results that for $b > 0$, $h \geq 0$

$$\theta(x, t) = C_1 t^{-3/2} \exp(-ht) \quad \text{as} \quad t \to \infty,$$

and for $b = 0$, $h > 0$

$$\theta(x, t) = C_2 t^{-1/2} \exp(-ht) \quad \text{as} \quad t \to \infty.$$

Even for $b < 0$, i.e. linear absorbtion, $\theta(x, t)$ must approach zero if $b^2 - h < 0$. Hence

$$C_3 t^{-1/2} \exp(-ht) \leq \theta(0, t) \leq C_4 \exp[(b^2 - h)t] \quad \text{as} \quad t \to \infty.$$

On the other hand, if $b^2 - h \geq 0$, there is no assurance that the concentration approaches a limiting value in time. However, if it does, then

$$|b| - h^{1/2} \leq \lim_{t\to\infty} \alpha\theta^{n-1}(0, t) \leq |b|.$$

2. Existence and uniqueness. The existence and uniqueness of solutions of general problems similar to (1.1)–(1.4) have been discussed in detail by many authors. Straightforward proofs are provided here for completeness and because certain points will be raised which will be useful in later discussions.

It is not difficult to show that any solution of (1.1)–(1.4) must satisfy the integral solution

$$\theta(x, t) = \int_0^t f(s)K_\rho(x, t; s)\, ds + \int_0^t \{\rho(s) - [b + \alpha\theta^{n-1}(0, s)]\}\theta(0, s)K_\rho(x, t; s)\, ds \tag{2.1}$$

where $K_\rho(x, t; s)$ is defined by the system

$$K_{\rho,t}(x, t; s) = K_{\rho,xx}(x, t; s) - hK_\rho(x, t; s), \qquad x > 0, \quad t \geq s \geq 0, \tag{2.2}$$

$$K_{\rho,x}(0, t; s) = \rho(t)K_\rho(0, t; s) - \delta(t - s), \qquad t \geq s \geq 0, \tag{2.3}$$

$$K_\rho(x, t; s) = 0, \qquad t < s, \tag{2.4}$$

$$K_\rho(x, t; s) \to 0 \quad \text{as} \quad x \to \infty. \tag{2.5}$$

Here $\rho(t)$ is non-negative but otherwise arbitrary. The desirability of allowing the coefficient ρ to be a function of time rather than a constant is suggested by the observation that (1.1)–(1.4) is analogous to a linear problem in which the "evaporation coefficient" $[b + \alpha\theta^{n-1}(0, t)]$ varies with time. Notice that the solution of (2.1) is reduced to a quadrature if $\theta(0, t)$ is known. For this reason the existence of a solution is discussed in the context of finding $\theta(0, t)$.

Certain properties of the Green's function $K_\rho(x, t; s)$ are now presented so that they will be available when they are needed for subsequent calculations. If we let $K_\rho(x, t; s) = G_\rho(x, t; s) \exp[-h(t - s)]$ Eqs. (2.2)–(2.5) become

$$G_{\rho,t}(x, t; s) = G_{\rho,xx}(x, t; s), \qquad x > 0, \qquad t \geq s \geq 0, \tag{2.6}$$

$$G_{\rho,x}(0, t; s) = \rho(t)G_\rho(0, t; s) - \delta(t - s), \qquad t \geq s \geq 0, \tag{2.7}$$

$$G_\rho(x, t; s) = 0, \qquad t < s, \tag{2.8}$$

$$G_\rho(x, t; s) \to 0 \quad \text{as} \quad x \to \infty. \tag{2.9}$$

These equations are exactly those which define the Green's function used by Keller and Olmstead, and many useful results are obtained as obvious extensions of their work. In particular, $K_\rho \geq 0$ because G_ρ is. Moreover, $K_\rho \leq G_\rho$ so $\int_0^t K_\eta(x, t; s)\, ds \leq \eta^{-1}$, where $\rho(t) = \eta =$ constant. A result implied by Keller and Olmstead, although not explicitly stated by them, is that, for $\rho(t) = \rho^*(t) \equiv \alpha B t^{-1}$, as $t \to \infty$

$$\int_{t_1}^{t_2} G_{\rho^*}(x, t; s)\, ds = \pi^{-1/2} t^{-1/2} \int_0^\infty \exp\,[-2\alpha B \xi^{1/2}] \int_{t_1}^{t_2} s \exp\,(-\xi s)\, ds\, d\xi + o(t^{-1/2}).$$

Hence as $t \to \infty$

$$\int_{t_1}^{t_2} K_{\rho^*}(x, t; s)\, ds \geq \pi^{-1/2} \exp\,(ht_1)\left[\int_0^\infty \exp\,[-2\alpha B\xi^{1/2}] \int_{t_1}^{t_2} s \exp\,(-\xi s)\, ds\, d\xi\right] t^{-1/2}$$
$$\cdot \exp\,(-ht) + o(t^{-1/2} \exp\,(-ht)). \tag{2.10}$$

Finally, it is useful to have bounds on $K_\rho(x, t; s)$ for $\rho(t) = \eta =$ constant. In this case K_η may be calculated explicitly and is

$$K_\eta(x, t; s) = H(t - s) \exp\,[-h(t - s)]\left\{\frac{x \exp\,[-x^2/4(t - s)]}{\pi^{1/2}(t - s)^{1/2}[x + 2\eta(t - s)]}\right.$$
$$\left. + \frac{\eta}{2\pi^{1/2}} \exp\,[\eta x + \eta^2(t - s)]\Gamma\left(-\tfrac{1}{2}, \eta^2(t - s) + \eta x + \frac{x^2}{4(t - s)}\right)\right\}, \tag{2.11}$$

where $H(t - s)$ is the Heaviside function and $\Gamma(-\frac{1}{2}, r)$ is an incomplete gamma function. Various representations for $\Gamma(-\frac{1}{2}, r)$ (see e.g. Erdelyi [4]) are used in deriving the desired bounds. In particular,

$$\Gamma(-\tfrac{1}{2}, r) = \frac{2e^{-r} r^{-1/2}}{\pi^{1/2}} \int_0^\infty \frac{e^{-t} t^{1/2}}{r + t}\, dt = \frac{e^{-r} r^{-1/2}}{r + q(r)}, \qquad r > 0, \tag{2.12}$$

where $q(r)$ is a certain continued fraction. From (2.12)

$$\frac{dq(r)}{dr} = \frac{\pi^{1/2}}{2} \int_0^\infty \frac{e^{-t} t^{1/2}}{(r + t)^2}\, dt \left[\int_0^\infty \frac{e^{-t} t^{1/2}}{r + t}\, dt\right]^{-2} - 1, \tag{2.13}$$

and by the Schwarz inequality

$$\left[\int_0^\infty \frac{e^{-t} t^{1/2}}{r + t}\, dt\right]^2 \leq \left(\int_0^\infty \frac{e^{-t} t^{1/2}}{(r + t)^2}\, dt\right)\left(\int_0^\infty e^{-t} t^{1/2}\, dt\right) = \frac{\pi^{1/2}}{2} \int_0^\infty \frac{e^{-t} t^{1/2}}{(r + t)^2}\, dt. \tag{2.14}$$

Thus $dq(r)/dr \geq 0$, $r > 0$, i.e. $q(r)$ is nondecreasing. As $r \to 0$, Eq. (2.12) yields

$$q(0) = \frac{\pi^{1/2}}{2} \left[\int_0^\infty e^{-t} t^{-1/2}\, dt\right]^{-1} = \frac{1}{2} \tag{2.15}$$

or

$$\Gamma(-\tfrac{1}{2}, r) \leq e^{-r} r^{-1/2}/(r + \tfrac{1}{2}). \tag{2.16}$$

As $r \to \infty$, $\Gamma(-\frac{1}{2}, r)$ has the asymptotic expansion

$$\Gamma(-\tfrac{1}{2}, r) = e^{-r}r^{-3/2}\left[\sum_{j=0}^{N-1} \frac{\Gamma(3/2 + j)}{\Gamma(3/2)} \frac{1}{(-r)^j} + O(|r|^{-N})\right]. \tag{2.17}$$

This and (2.12) yield that $q(r) \to 3/2$ as $r \to \infty$, or

$$\Gamma(-\tfrac{1}{2}, r) \geq e^{-r}r^{-1/2}/(r + 3/2). \tag{2.18}$$

In conclusion, then,

$$K_\eta(x, t; s) \geq H(t - s)\pi^{-1/2} \exp\left[-h(t - s) - \frac{x^2}{4(t - s)}\right]\left\{\frac{x}{(t - s)^{1/2}[x + 2\eta(t - s)]} + \frac{\eta}{2} \cdot \left(\left[\frac{x + 2\eta(t - s)}{2(t - s)^{1/2}}\right]\left\{\left[\frac{x + 2\eta(t - s)}{2(t - s)^{1/2}}\right]^2 + \frac{3}{2}\right\}\right)^{-1}\right\} \tag{2.19}$$

and

$$K_\eta(x, t; s) \leq H(t - s)\pi^{-1/2} \exp\left[-h(t - s) - \frac{x^2}{4(t - s)}\right]\left\{\frac{x}{(t - s)^{1/2}[x + 2\eta(t - s)]} + \frac{\eta}{2} \cdot \left(\left[\frac{x + 2\eta(t - s)}{2(t - s)^{1/2}}\right]\left\{\left[\frac{x + 2\eta(t - s)}{2(t - s)^{1/2}}\right]^2 + \frac{1}{2}\right\}\right)^{-1}\right\}. \tag{2.20}$$

Recall that (2.1) is solved if a $\theta(0, t)$ can be found to satisfy the nonlinear integral equation

$$\theta(0, t) = \int_0^t f(s)K_\rho(0, t; s)\, ds + \int_0^t \{\rho(s) - [b + \alpha\theta^{n-1}(0, s)]\}\theta(0, s)K_\rho(0, t; s)\, ds. \tag{2.21}$$

Let $M = \max f(t)$, pick $R > 0$ such that $R(\alpha R^{n-1} + b) = M$ and choose $\rho(t) = \eta^* = \alpha n R^{n-1} + b$. Finally, define the elements of the sequence $\{\varphi_i(t)\}$ by the equations

$$\varphi_0(t) = \int_0^t f(s)K_{\eta^*}(0, t; s)\, ds, \tag{2.22}$$

$$\varphi_i(t) = \varphi_0(t) + \alpha \int_0^t [nR^{n-1} - \varphi_{i-1}{}^{n-1}(s)]\varphi_{i-1}(s)K_{\eta^*}(0, t; s), \quad i = 1, 2, 3, \cdots. \tag{2.23}$$

$\varphi_0(t)$ is certainly positive. Moreover, it is bounded because

$$\varphi_0(t) \leq M \int_0^t K_{\eta^*}(0, t; s)\, ds \leq M/\eta^* \tag{2.24}$$

or

$$\varphi_0(t) \leq \frac{R(\alpha R^{n-1} + b)}{\alpha n R^{n-1} + b} < R. \tag{2.25}$$

If (2.23) is rewritten as

$$\varphi_i(t) = \varphi_0(t) + \alpha(n - 1)R^n \int_0^t K_{\eta^*}(0, t; s)\, ds - \alpha \int_0^t [(n - 1)R^n + \varphi_{i-1}{}^n(s) - nR^{n-1}\varphi_{i-1}(s)]K_{\eta^*}(0, t; s)\, ds, \tag{2.26}$$

then it follows from the easily-proven inequality

$$F(x, y) \equiv x^n + (n-1)y^n - nxy \geq 0 \quad \text{for} \quad x \geq 0, y > 0, n > 1$$

and $F(x, y) = 0$ only for $x = y$ that

$$\varphi_i(t) < \varphi_0(t) + \alpha(n-1)R^n \int_0^t K_{\eta*}(0, t; s)\, ds. \tag{2.27}$$

Hence

$$\varphi_i(t) < \frac{R(\alpha R^{n-1} + b)}{\alpha n R^{n-1} + b} + \frac{\alpha(n-1)R^n}{\alpha n R^{n-1} + b} = R, \qquad i = 1, 2, \cdots. \tag{2.28}$$

Assume now that $\varphi_{i+1}(t) > \varphi_i(t)$ for some i and consider the difference

$$\varphi_{i+2}(t) - \varphi_{i+1}(t) = \alpha \int_0^t [nR^{n-1}\varphi_{i+1}(s) - \varphi_{i+1}{}^n(s) - nR^{n-1}\varphi_i(s) + \varphi_i{}^n(s)]K_{\eta*}(0, t; s)\, ds. \tag{2.29}$$

Again, it is a simple matter to prove that the integrand is positive and since $\varphi_1(t) > \varphi_0(t) > 0$ the uniformly-bounded sequence $\{\varphi_i(t)\}$ is monotonically increasing. The sequence therefore must converge to a limiting function, and it can be shown that this function satisfies (2.21). It follows that $\theta(x, t)$ exists.

Although uniqueness may be examined within the context of the integral equation, it follows directly from the initial-boundary-value problem by applying certain maximum principles to the system of equations

$$y_t(x, t) = y_{xx}(x, t) + 2by_x(x, t) \qquad x > 0, \quad t > 0, \tag{2.30}$$

$$y_x(0, t) = \alpha \exp[(n-1)(b^2 - h)t]y^n(0, t) - \exp[-(b^2 - h)t]f(t), \qquad t > 0, \tag{2.31}$$

$$y(x, 0) = 0, \qquad x \geq 0, \tag{2.32}$$

$$y(x, t) \to 0 \quad \text{as} \quad x \to \infty, \qquad t \geq 0 \tag{2.33}$$

which is obtained by making a simple transformation of (1.1)–(1.4). Then if $w(x, t) \equiv y_1(x, t) - y_2(x, t)$, where y_1 and y_2 are two positive solutions to the above problem, $w(x, t)$ must satisfy the system

$$w_t(x, t) = w_{xx}(x, t) + 2bw_x(x, t), \qquad x > 0, \quad t > 0, \tag{2.34}$$

$$w_x(0, t) = \alpha \exp[(n-1)(b^2 - h)t]\left\{\frac{y_1{}^n(0, t) - y_2{}^n(0, t)}{y_1(0, t) - y_2(0, t)}\right\}w(0, t), \tag{2.35}$$

$$w(x, 0) = 0, \qquad x \geq 0, \tag{2.36}$$

$$w(x, t) \to 0 \quad \text{as} \quad x \to \infty, \qquad t \geq 0. \tag{2.37}$$

Standard maximum principles (see e.g. Protter and Weinberger [5]) for operators like (2.34) and subject to a condition like (2.36) indicate that if $w(x, t)$ has a positive maximum it must occur along the line $x = 0, t > 0$. Moreover, a derivative in an outward direction, in particular in the $-x$ direction, must be positive where the maximum occurs. For $y_1 \geq 0, y_2 \geq 0$, this contradicts (2.35); hence $w(x, t)$ cannot have a positive maximum, or $w(x, t) \leq 0$. Exactly the same arguments apply to $w'(x, t) \equiv y_2(x, t) - y_1(x, t)$ and thus imply $w'(x, t) \leq 0$. The conclusion, of course, is that $y_1(x, t) = y_2(x, t)$. Notice that the positivity of the coefficient of $w(0, t)$ in (2.35) is crucial to this proof and this situation can be assured only if $y_1(x, t)$ and $y_2(x, t)$ are positive. This proof therefore verifies that there is only one positive solution of the problem but does not preclude the possibility of negative solutions. These, however, would not be physically significant and the possibility is not considered further.

3. Bounds on $\theta(x, t)(b \geq 0)$. Because $\rho(t)$ is arbitrary but non-neagtive, when $b \geq 0$ we may set $\rho(t) = b$ in (2.1) to get the upper bound

$$\theta(x, t) \leq \int_0^t f(s)K_b(x, t; s)\, ds. \tag{3.1}$$

Furthermore $\theta(0, t)$ is bounded, so there exists a $\rho(t) = \eta_1 =$ constant such that $\eta_1 - b - \alpha\theta^{n-1}(0, t) \geq 0$, $0 < t < \infty$, or

$$\theta(x, t) \geq \int_0^t f(s)K_{\eta_1}(x, t; s)\, ds. \tag{3.2}$$

These bounds may be improved by using them as a basis for choosing new $\rho(t)$ with the idea of making the integrand in the second integral of Eq. (2.1) progressively smaller. Keller and Olmstead, in fact, do this to get a sequence of upper and lower bounds which converge to the solution of their special case of (2.1).

Recall that for $t > t_c$, $f(t) = 0$, so that

$$\int_0^{t_c} f(s)K_{\eta_1}(x, t; s)\, ds \leq \theta(x, t) \leq \int_0^{t_c} f(s)K_b(x, t; s)\, ds, \qquad t > t_c\,. \tag{3.3}$$

These bounds can be bounded, with the result

$$m \int_{t_1}^{t_2} K_{\eta_1}(x, t; s)\, ds \leq \theta(x, t) \leq M \int_0^{t_c} K_b(x, t; s)\, ds \tag{3.4}$$

because $f(t)$ is bounded and there exists a number $m > 0$ such that

$$\begin{aligned} f(t) &\geq 0, \quad 0 \leq t \leq t_1\,, \\ &\geq m, \quad t_1 < t < t_2\,, \\ &\geq 0, \quad t_2 \leq t \leq t_c\,. \end{aligned} \tag{3.5}$$

Inequalities (2.19) and (2.20) may be employed to yield the bounds

$$\begin{aligned} &m \exp\left[-h(t - t_1) - \frac{x^2}{4(t - t_2)}\right] \int_{t_1}^{t_2} \left\{\frac{x}{\pi^{1/2}(t - s)^{1/2}[x + 2\eta_1(t - s)]} + \frac{\eta}{2} \right. \\ &\quad \left. \cdot \left(\left[\frac{x + 2\eta_1(t - s)}{2(t - s)^{1/2}}\right]\left\{\left[\frac{x + 2\eta_1(t - s)}{2(t - s)^{1/2}}\right]^2 + \frac{3}{2}\right\}\right)^{-1}\right\} ds \\ &\leq \theta(x, t) \leq M \exp\left[-h(t - t_c) - \frac{x^2}{4t}\right] \int_0^{t_c} \left\{\frac{x}{\pi^{1/2}(t - s)^{1/2}[x + 2b(t - s)]} + \frac{b}{2} \right. \\ &\quad \left. \cdot \left(\left[\frac{x + 2b(t - s)}{2(t - s)^{1/2}}\right]\left\{\left[\frac{x + 2b(t - s)}{2(t - s)^{1/2}}\right]^2 + \frac{1}{2}\right\}\right)^{-1}\right\} ds. \end{aligned} \tag{3.6}$$

These integrals can be evaluated explicitly, but the results are too complicated to be very instructive. Asymptotically, however, these bounds become

$$\frac{m(t_2 - t_1)}{2\eta_1\pi^{1/2}}\left[\frac{1}{\eta_1} + x\right]\exp\left[ht_1 - \frac{x^2}{4(t - t_2)}\right]t^{-3/2}e^{-ht} + o(t^{-3/2}e^{-ht}) \leq \theta(x, t)$$

$$\leq \frac{Mt_c}{2b\pi^{1/2}}\left[\frac{1}{b} + x\right]\exp\left[ht_c - \frac{x^2}{4t}\right]t^{-3/2}e^{-ht} + o(t^{-3/2}e^{-ht}) \quad \text{as} \quad t \to \infty. \tag{3.7}$$

Notice that the asymptotic behavior of the upper bound is meaningless for $b = 0$. We must therefore examine this case more closely. Eq. (2.11) shows that $K_0(x, t; s)$ has a particularly simple form. The upper bound in (3.4) then becomes

$$\theta(x, t) \leq M \int_0^t \frac{\exp[-h(t - s) - x^2/4(t - s)]}{\pi^{1/2}(t - s)^{1/2}}\,ds \tag{3.8}$$

$$\leq M \exp\left[-\frac{x^2}{4t}\right]\int_0^{t_c} \frac{\exp[-h(t - s)]}{\pi^{1/2}(t - s)^{1/2}}\,ds. \tag{3.9}$$

The integral is easily evaluated with the result that

$$\theta(x, t) \leq \frac{M}{h^{1/2}}\exp\left[-\frac{x^2}{4t}\right]\{\operatorname{erf}[ht]^{1/2} - \operatorname{erf}[h(t - t_c)]^{1/2}\} \tag{3.10}$$

or

$$\theta(x, t) \leq \frac{M}{\pi^{1/2}h}(\exp(ht_c) - 1)\exp\left[-\frac{x^2}{4t}\right]t^{-1/2}e^{-ht} + o(t^{-1/2}e^{-ht}) \quad \text{as} \quad t \to \infty. \tag{3.11}$$

There is still an $\eta_1 > 0$ which provides a lower bound on $\theta(x, t)$, and the asymptotic behavior of this lower bound is known from previous calculations. Unfortunately, these results do not indicate the form of the asymptotic behavior of $\theta(x, t)$ nearly as specifically as the results for $b > 0$. Observe, however, that for $h > 0$ a constant B can be chosen sufficiently large so that $Bt^{-1} - \theta^{n-1}(0, t) \geq 0$, $t > 0$, $n > 1$. This can be done for $h > 0$ because $\theta(0, t)$ is bounded and decays no more slowly than $O(e^{-ht}t^{-1/2})$ as $t \to \infty$. If $h = 0$, such a constant can be chosen only for $n \geq 3$. With $\rho(t) = \rho^*(t) \equiv \alpha Bt^{-1}$, Eq. (2.1) becomes

$$\theta(x, t) = \int_0^t f(s)K_{\rho*}(x, t; s)\,ds + \alpha\int_0^t [Bs^{-1} - \theta^{n-1}(0, s)]\theta(0, s)K_{\rho*}(x, t; s)\,ds \tag{3.12}$$

or

$$\theta(x, t) \geq \int_0^t f(s)K_{\rho*}(x, t; s)\,ds. \tag{3.13}$$

As before,

$$\theta(x, t) \geq m\int_{t_1}^{t_2} K_{\rho*}(x, t; s)\,ds \quad \text{for} \quad t > t_2 \tag{3.14}$$

and from (2.10) as $t \to \infty$

$$\theta(x, t) \geq \frac{m\exp(ht_1)}{\pi^{1/2}}\left[\int_0^\infty \exp[-2\alpha B\xi^{1/2}]\int_{t_1}^{t_2} s\exp(-\xi s)\,ds\,d\xi\right]t^{-1/2}e^{-ht} + o(t^{-1/2}e^{-ht}). \tag{3.15}$$

Physically, it is almost obvious that $\theta(x, t)$ must approach zero for $b > 0$, $h \geq 0$

or for $b \geq 0$, $h > 0$ because $f(t)$ is the only source of the diffusant while both b and h represent losses. In fact, even for $b = h = 0$

$$\theta(x, t) \leq \int_0^t f(s) \frac{\exp[-x^2/4(t-s)]}{\pi^{1/2}(t-s)^{1/2}} ds \to 0 \quad \text{as} \quad t \to \infty.$$

What is more surprising is that the nonlinearity has no effect on the form of the large-time asymptotic behavior. This is evident from the fact that none of the previous analyses is changed by setting $\alpha = 0$ in Eq. (2.1), and thus

$$\theta(x, t) = \int_0^t f(s) K_b(x, t; s) ds.$$

Apparently h affects only the exponential decay and the form of the algebraic decay depends only on the value of b. Physically, this is reasonable because $\alpha\theta^{n-1}(0, t)$, the nonlinear radiation coefficient, becomes negligible relative to b as $t \to \infty$. In summary, then, conduction must account for the $t^{-1/2}$ decay, linear radiation changes the algebraic rate of decay to $t^{-3/2}$, and the nonlinear radiation ($n > 1$) has no affect on the large-time decay rate of the temperature.

4. Bounds on $\theta(x, t)(b < 0)$. It is more difficult to get very specific bounds on $\theta(x, t)$ for $b < 0$. In this case, setting $\rho(t) = 0$ in (2.1) yields only that

$$\theta(x, t) \leq \int_0^t f(s) K_0(x, t; s) ds - b \int_0^t \theta(0, s) K_0(x, t; s) ds.$$

Physically, $b < 0$ implies that energy is being absorbed at a rate proportional to the surface temperature, so there is no reason to expect an upper bound on the solution to depend only on the source $f(t)$.

Let $-b = \beta > 0$ and rewrite (2.1) as

$$\theta(x, t) = \int_0^t f(s) K_\rho(x, t; s) ds + \int_0^t \{\rho(s) + [\beta - \alpha\theta^{n-1}(0, s)]\} \theta(0, s) K_\rho(x, t; s) ds. \quad (4.1)$$

In this form we see that the difficulty lies in the fact that the quantity $[\beta - \alpha\theta^{n-1}(0, s)]$ may change sign. For example, if $\beta < \alpha\theta^{n-1}(0, t)$ there is a $\rho(t) = \eta_2 =$ constant such that $\eta_2 + \beta - \alpha\theta^{n-1}(0, t) \geq 0$, $0 < t < \infty$ and $\theta(x, t)$ is bounded from below by

$$\theta(x, t) \geq \int_0^t f(s) K_{\eta_2}(x, t; s) ds.$$

On the other hand, if $\beta \geq \alpha\theta^{n-1}(0, t)$ then $\rho(t) = 0$ will do and

$$\theta(x, t) \geq \int_0^t f(s) K_0(x, t; s) ds.$$

The behavior of these integrals has been investigated, but there is no indication as to which lower bound is appropriate. The situation, of course, is not hopeless; it just requires a slightly different approach.

Define $u(x, t)$ by the equation

$$u(x, t) = \int_0^t af(s) K_0(x, t; s) ds + \int_0^t \beta u(0, s) K_0(x, t; s) ds. \quad (4.2)$$

Let $\rho(t) = 0$ in (4.1) and consider the difference

$$u(x, t) - \theta(x, t) = \int_0^t \alpha\theta^n(0, s)K_0(x, t; s)\, ds + \beta \int_0^t [u(0, s) - \theta(0, s)K_0(x, t; s)\, ds \quad (4.3)$$

or, for $w(x, t) \equiv u(x, t) - \theta(x, t)$,

$$w(x, t) = \int_0^t \alpha\theta^n(0, s)K_0(x, t; s)\, ds + \beta \int_0^t w(0, s)K_0(x, t; s)\, ds. \quad (4.4)$$

It is a simple matter to solve for $u(x, t)$ and incidentally to prove that $w(x, t) \geq 0$. Thus

$$\theta(x,t) \leq u(x, t) = \int_0^t f(s)\left\{\frac{\exp[-h(t - s) - x^2/4(t - s)]}{\pi^{1/2}(t - s)^{1/2}} + \beta \exp[-\beta x + (\beta^2 - h)(t - s)] \operatorname{erf}\left[\frac{x}{2(t - s)^{1/2}} - \beta(t - s)^{1/2}\right]\right\} ds. \quad (4.5)$$

The integral is a decreasing function of x, so for the sake of getting more specific information look at the concentration at $x = 0$ for $t > t_c$. Here

$$\theta(x, t) \leq \exp[(\beta^2 - h)t]\left\{\int_0^{t_c} \beta f(s) \exp[-(\beta^2 - h)s](1 + \operatorname{erf}[\beta(t - s)^{1/2}])\, ds\right\} + \int_0^{t_c} f(s) \frac{\exp[-h(t - s)]}{\pi^{1/2}(t - s)^{1/2}}\, ds. \quad (4.6)$$

As before, the right-hand side of this inequality can be bounded to yield the result that as $t \to \infty$

$$\theta(0, t) \leq \exp[(\beta^2 - h)t]\left\{2 \int_0^{t_c} \beta f(s) \exp[-(\beta^2 - h)s]\, ds + o(t^{-1/2} \exp[-\beta^2 t])\right\}. \quad (4.7)$$

Since $\theta(0, t)$ is bounded this offers no new information for $\beta^2 - h \geq 0$. If $\beta^2 - h < 0$, however, (4.7) proves the physically obvious fact that the temperature must approach zero if energy is being lost along the length of the rod faster than it can be absorbed at the end.

Because of this bound and the fact that $\theta(0, t)$ is bounded, there is certainly a constant M_1 such that $\theta(0, t) \leq M_1 \exp[(\beta^2 - h)t]$, $0 < t < \infty$, $\beta^2 - h < 0$ and

$$\theta(0, t) \geq \int_0^t f(s)K_\rho(0, t; s)\, ds + \int_0^t \{\rho(s) + \beta - \alpha M_1^{n-1} \exp[(n - 1)(\beta^2 - h)s]\} \cdot \theta(0, s)K_\rho(0, t; s)\, ds. \quad (4.8)$$

With the choice of $\rho(t) = \rho'(t) \equiv \alpha M_1^{n-1}/((n - 1)\, |\beta^2 - c|\, t)$, this inequality becomes

$$\theta(0, t) \geq \int_0^t f(s)K_{\rho'}(0, t; s)\, ds + \int_0^t \{\beta + \alpha M_1^{n-1}([(n - 1)\, |\beta^2 - h|\, s]^{-1} - \exp[(n - 1)(\beta^2 - h)s])\}\, \theta(0, s)K_{\rho'}(0, t; s)\, ds. \quad (4.9)$$

The integrand is obviously positive, so, as before,

$$\theta(0, t) \geq \frac{m \exp(ht_1)}{\pi^{1/2}} \left[\int_0^\infty \exp\left[-\frac{2\alpha M_1^{n-1}}{(n - 1)\, |\beta^2 - h|}\, \xi^{1/2}\right] \int_{t_1}^{t_2} s \exp(-\xi s)\, ds\, d\xi\right] e^{-ht} t^{-1/2} + o(e^{-ht} t^{-1/2}) \quad (4.10)$$

as $t \to \infty$.

For $\beta^2 - h \geq 0$ we still do not have a well-defined upper bound and there are two choices for a lower bound depending on whether $\beta > \alpha\theta^{n-1}(0, t)$ or $\beta \leq \alpha\theta^{n-1}(0, t)$. Let us investigate the consequence of assuming that $\beta > \alpha\theta^{n-1}(0, t)$; in particular, let

$$\beta - \alpha\theta^{n-1}(0, t) = \mu + \Delta(t) \tag{4.11}$$

where $\mu > 0$, $\Delta(t) \geq 0$ and there is a time t' such that $\Delta(t') = 0$. Now (4.1) with $\rho(t) = 0$ can be rewritten as

$$\theta(x, t) = \int_0^t f(s)K_0(x, t; s)\, ds + \int_0^t [\mu + \Delta(s)]\theta(0, s)K_0(x, t; s)\, ds. \tag{4.12}$$

As before, define $u(x, t)$ to satisfy the linear problem

$$u(x, t) = \int_0^t f(s)K_0(x, t; s)\, ds + \int_0^t \mu u(0, s)K_0(x, t; s)\, ds, \tag{4.13}$$

and then $w(x, t) \equiv \theta(x, t) - u(x, t)$ must satisfy the equation

$$w(x, t) = \int_0^t \Delta(s)\theta(0, s)K_0(x, t; s)\, ds + \mu \int_0^t w(0, s)K_0(x, t; s)\, ds. \tag{4.14}$$

Notice that (4.13) and (4.14) are of exactly the same form as (4.2) and (4.4), so we immediately have the result for $t > t_c$

$$\theta(0, \mathrm{t}) \geq \exp\,[(\mu^2 - h)t] \cdot \int_0^{t_c} \mu f(s) \exp\,[-(\mu^2 - h)s]\{1 + \operatorname{erf}\,[\mu(t - s)^{1/2}]\}\, ds + e^{-ht} \int_0^{t_c} f(s) \frac{e^{hs}}{\pi^{1/2}(t - s)^{1/2}}\, ds.$$

If $\mu^2 - h > 0$, the fact that $\theta(0, t)$ is bounded is contradicted; hence $\mu^2 \leq h$. Therefore from (4.11)

$$\alpha\theta^{n-1}(0, t) \geq \beta - h^{1/2} - \Delta(t) \tag{4.15}$$

and at $t = t'$

$$\alpha\theta^{n-1}(0, t') \geq \beta - h^{1/2} \geq 0. \tag{4.16}$$

Since it is certainly possible for $f(t)$ to cut off before $\theta(0, t)$ achieves the value implied above, the linear absorbtion must be responsible for forcing the temperature to reach this value. It is therefore reasonable to expect that there is no time beyond which the temperature can always remain below the value $[\alpha^{-1}(\beta - h^{1/2})]^{1/n-1}$.

That this is, in fact, the case is easily proven by restating (1.1)–(1.4) as

$$\theta_\tau(x, \tau) = \theta_{xx}(x, \tau) - h\theta(x, \tau), \qquad x > 0, \quad \tau = t - t_0 > 0, \tag{4.17}$$

$$\theta_x(0, \tau) = \alpha\theta^n(0, \tau) - \beta\theta(0, \tau), \qquad \tau > 0, \tag{4.18}$$

$$\theta(x, 0) = \psi(x), \qquad x \geq 0, \tag{4.19}$$

$$\theta(x, \tau) \to 0 \quad \text{as} \quad x \to \infty, \tag{4.20}$$

where $t_0 > t_c$ but otherwise arbitrary. This problem may be expressed as an integral equation in the form

$$\theta(x, \tau) = \int_0^\infty \psi(y)K_\rho(x, \tau; y, 0)\, dy + \int_0^t [\rho(s) + \beta - \alpha\theta^{n-1}(0, s)]\theta(0, s)K_\rho(x, \tau; 0, s)\, ds \tag{4.21}$$

where $K_\rho(x, \tau; y, s) = K_\rho(x - y, \tau; s)$. If it is again supposed that $\beta - \alpha\theta^{n-1}(0, \tau) = \mu + \Delta(\tau)$, $\mu > 0$, $\Delta(\tau) \geq 0$ and $\Delta(\tau') = 0$, an analysis similar to that just used follows to yield the same contradiction unless $\mu^2 \leq h$. Since t_0 is arbitrary, the desired result is proven.

The converse—that there is no finite time beyond which $\alpha\theta^{n-1}(0, t) \geq \beta$—is very easily proven. If there were such a time, say $t = t^*$, then (4.1) could be rewritten as

$$\theta(0, t) = \int_0^{t_c} f(s)K_0(0, t; s)\, ds + \int_0^{t^*} [\beta - \alpha\theta^{n-1}(0, s)]\theta(0, s)K_0(0, t; s)\, ds$$
$$- \int_{t^*}^t [\alpha\theta^{n-1}(0, s) - \beta]\theta(0, s)K_0(0, t; s)\, ds.$$

This, however, implies that $\theta(0, s) \to 0$ as $t \to \infty$, which contradicts the original hypothesis.

In summary, then, if the limit exists when $\beta^2 - h \geq 0$,

$$\beta - h^{1/2} \leq \lim_{t\to\infty} \alpha\theta^{n-1}(0, t) \leq \beta.$$

References

[1] J. B. Keller and W. E. Olmstead, *Temperature of a nonlinearly radiating semi-infinite solid,* Quart. Appl. Math. **29**, 559–566 (1972)

[2] R. A. Handelsman and W. E. Olmstead, *Asymptotic solution to a class of nonlinear Volterra integral equations,* Report No. 71–2, Northwestern University Series in Applied Mathematics, April 1971

[3] A. Friedman, *Generalized heat transfer between solids and gases under nonlinear boundary conditions,* J. Math and Mech. **8**, 161–183 (1959)

[4] A. Erdelyiy *et al., Higher transcendental functions II,* Bateman Manuscript Project, McGraw-Hill, New York, 1953

[5] M. H. Protter and H. F. Weinberger, *Maximum principles in differential equations,* Prentice-Hall, Englewood Cliffs, 1967