Temporalization of Modal Logic

— Morphological Analysis for Logic —

Mitsuhiko Fujio
Computer Science and
Systems Engineering
Kyushu Institute of Technology
Kawazu 680-4, Iizuka, Fukuoka 820-8502, Japan
Email: fujio@ces.kyutech.ac.jp

***Abstract*— A temporalization of a modal logic is a temporal logic containing the modal logic and the temporalization problem is the problem of construction and classification of temporalization for a given modal logic. A temporal logic carries two dual pairs of modal operators (F, G) and (P, H). By taking each pair of them, one obtains a normal modal logic. In this situation, we refer to the temporal logic as a strict temporalization of the normal modal logic. Under mathematical morphological investigation over non-classical logics, the author showed that an adjoint pair of modal operators gives rise to a temporal logic ([1]). This result can be used to show the existence of a temporalization for a normal modal logic. By combining with canonical models, mathematical morphology illustrates the relationship between a modal logic and its temporalization. In this article, we will show how morphological analysis is applied to the temporalization problem.**

I. Introduction

Mathematical morphology was first introduced as an analyzing tool for image processing by shape ([2]) and has been developed as a systematic non-linear analysis methodology ([3]). Theoretically, it is founded on complete lattices ([4], [5], [6]) and its methodology has been also extended there ([7]). Within its wide range, the regions of logic and language seem most promising areas of application. Because the non-numeric and structural methods of mathematical morphology fit well for these regions. In particular, many lattices naturally rise and play central rolls there.

A direct application of mathematical morphology to logic causes modal logic ([8], [9], [10], [1]). The fundamental operators of *dilation* D and *erosion* E in mathematical morphology related to the modal operators $\Diamond$ and $\Box$ by

$$\|\Diamond\phi\| = D_R(\|\phi\|), \qquad \|\Box\phi\| = E_R(\|\phi\|),$$

where $\|\phi\|$ denotes the *true set* (or meaning) of a modal formula ϕ in a standard model. Moreover, D_R and E_R respectively denote the *set dilation* and *erosion* defined by the accessibility relation R of the model.

We note that the pairs of operators appearing in this case are both dual, *i.e.*, $E_R = \complement D_R \complement$ and $\Box = \neg\Diamond\neg$. Here $\complement$ denotes the complementation of sets and $\neg$ denotes the negation of formulas. On the other hand, in the extension to complete lattices, dilation D and erosion E are treated as to constitute an adjoint pair, *i.e.*, $D(x) \leq a \Leftrightarrow x \leq E(a)$ for x and a in lattices X and A respectively. D and E are respectively called an *algebraic dilation* and an *erosion*. This algebraic framework is also applied to investigate modal logic.

An adjoint pair of modal operators is considered in [10] and it is shown that giving an adjoint pair of modal operators is nothing but considering temporal logic ([1]). For precise, consider a normal modal logic Σ accompanied with an adjoint pair of modal operators $\Diamond_1$ and $\Box_2$. Adjointness for modal operators is defined by the condition that

$$\text{(Ad.)} \qquad \Diamond_1\phi \to \psi \ \text{ iff } \ \phi \to \Box_2\psi.$$

Namely, the formula $\Diamond_1\phi \to \psi$ is a theorem of Σ if and only if so is the formula $\phi \to \Box_2\psi$. We note that, since $\Diamond_1$ and $\Box_2$ are not necessarily dual to each other, we use the letter $\Box_2$ to distinguish it from the dual of $\Diamond_1$. On the other hand, we denote the dual of $\Diamond_1$ by $\Box_1$ and of $\Box_2$ by $\Diamond_2$. Thus, although we have started with a modal logic with only two modal operators ($\Diamond_1$, $\Box_2$), it naturally possesses additional two operators ($\Diamond_2$, $\Box_1$). With these four operators, it is shown that the condition (Ad.) is equivalent to the following two axioms:

$$\text{(Tmp.)} \qquad \phi \to \Box_2\Diamond_1\phi \ \text{ and } \ \phi \to \Box_1\Diamond_2\phi.$$

These axioms are known as *the converse axioms* in temporal logic ([11]). For the traditional notation of temporal logic, one of dual pair, say $(\Diamond_1, \Box_1)$, corresponds to the pair (F, G) and, the other, say $(\Diamond_2, \Box_2)$, corresponds to the pair (P, H)[1]. In this case, not only (F, H) but also (P, G) is an adjoint pair.

From semantical point of view, this mechanism of derivation of temporal logic from an adjoint pair of modal operators can be explained more explicitly. Now we begin with 2-dimensional modal language Φ^2 with the modal operators $\Diamond_i$, $\Box_i$ ($i = 1, 2$). The notion of a 2-dimensional model is obviously modified from that of 1-dimensional one by adding another accessibility relation R'. Then the notion of true set is also extended to 2-dimensional modal formulas so as for $\phi \in \Phi^2$,

$$\begin{aligned} \|\Diamond_1\phi\| &= D_R(\|\phi\|), & \|\Box_1\phi\| &= E_R(\|\phi\|), \\ \|\Diamond_2\phi\| &= D_{R'}(\|\phi\|), & \|\Box_2\phi\| &= E_{R'}(\|\phi\|). \end{aligned}$$

[1]In traditional temporal logic, the meanings of these operators are $F\phi$: "ϕ will be true at some *F*uture time", $G\phi$: "ϕ is always *G*oing to be true", $P\phi$: "ϕ was true at some *P*ast time" and $H\phi$: "ϕ *H*as always been true".

It is known that for a standard model for 2-dimensional modal language to satisfy the converse axioms (Tmp.), it is necessary and sufficient that $R' = {}^tR$ ([11]). Here, ${}^tR$ is the transpose of the accessibility relation R. Such a 2-dimensional model is called *bidirectional*. Consequently, for a temporal logic, one should consider bidirectional models and then the true sets are rewritten as

$$\|\Diamond_1\phi\| = D_R\big(\|\phi\|\big), \qquad \|\Box_1\phi\| = E_R\big(\|\phi\|\big),$$
$$\|\Diamond_2\phi\| = D_{{}^tR}\big(\|\phi\|\big), \qquad \|\Box_2\phi\| = E_{{}^tR}\big(\|\phi\|\big).$$

As we will recall in the next section, the pairs of morphological operators $(D_R, E_{{}^tR})$ and $(D_{{}^tR}, E_R)$ are adjoint. Thus, from semantical point of view, we observe that giving an adjoint pair of modal operators $(\Diamond_1, \Box_2)$ concerns with only a single accessibility relation R such that

$$\|\Diamond_1\phi\| = D_R\big(\|\phi\|\big), \qquad \|\Box_2\phi\| = E_{{}^tR}\big(\|\phi\|\big)$$

and then by considering their duals, a temporal logic can be derived. Note that it cannot be determined a priori whether the given adjoint pair of modal operators $(\Diamond_1, \Box_2)$ corresponds to (F, H) or (P, G).

At this moment, we should recall that any ordinary dual pair of modal operators also concerns with a single accessibility relation. Hence we can conclude that any normal modal logic is accompanied by a temporal logic provided that it is determined by some models. But it is true for any normal modal logic. That is, by virtue of Canonical Model Theorem, any normal modal logic is determined by its canonical model.

In the above case, the derived temporal logic includes the normal modal logic. In general, we call a temporal logic including a modal logic a *temporalization* or *temporal extension* of the modal logic. And then the temporalization problem can be posed as follows. For a given normal modal logic, how many there exist its temporalizations and how can they be classified? The above observation provides the existence of temporalization. We call the temporalization derived from the canonical model the *canonical temporalization*.

In this article, we will consider a strict temporalization in general. Namely, the case where the underlying modal logic is obtained from the temporalization by eliminating the formulas containing modal operators $\Diamond_2$ or $\Box_2$. Then mathematical morphological analysis will show that for any normal modal logic, the minimal temporal logic containing it and the canonical temporalization are strict temporalizations. Furthermore, with canonical models, it provides a structural view for temporalization procedure.

II. Mathematical Morphology

In this section, we recall minimal requisites for mathematical morphology. For precise description and general references, readers should see [4], [7]

A. Binary Relations and Correspondences

We often regard a binary relation $R \subseteq X \times A$ as a correspondence $R : X \to A$ by $R : X \ni x \mapsto \{\, a \in A \mid (x, a) \in R \,\} \subseteq A$ and vise versa. The *transpose* ${}^tR$ of R is given by ${}^tR = \{\, (a, x) \mid a \in A,\, x \in X,\, (x, a) \in R \,\} \subseteq A \times X$ or, in terms of correspondence, ${}^tR : A \ni a \mapsto \{\, x \in X \mid (x, a) \in R \,\} \subseteq X$. As well as for ordinary mappings, we denote the *image* of an element $x \in X$ under R by $R(x)$ and that of a subset $Y \subseteq X$ by $R(Y) = \bigcup_{y \in Y} R(y)$. The usual set theoretical inverse image of a subset $B \subseteq A$ can be expressed as $R^{-1}(B) = {}^tR(B)$ in our notation.

B. Dilation and Erosion

Notions of dilation and erosion were extended to complete lattices and general properties are investigated ([4], [6], [7]). For the sake of development of morphological analysis for formal systems of logic which are not complete in general, we generalize these notions to partially ordered sets. For notions introduced here, we basically follow [7] but we consider a slightly general case of non-complete lattices or more simply partially ordered sets ([1]).

Definition *:* Let X, A be partially ordered sets. A mapping $\delta : X \to A$ is called an *algebraic dilation* or for short, a *dilation* iff for any family $\{x_\lambda\} \subseteq X$ that admits the supremum $\bigvee_\lambda x_\lambda$ in X, the family $\{\delta(x_\lambda)\}$ also admits the supremum $\bigvee_\lambda \delta(x_\lambda)$ in A and

$$\bigvee_\lambda \delta(x_\lambda) = \delta\Big(\bigvee_\lambda x_\lambda\Big) \tag{1}$$

is satisfied. We say that every dilation has the *supremum preserving property* (SSP). Dually, a mapping $\varepsilon : X \to A$ is called an *algebraic erosion* or for short, an *erosion* iff for any family $\{x_\lambda\} \subseteq X$ that admits the infimum $\bigwedge_\lambda x_\lambda \in X$, the family $\{\varepsilon(x_\lambda)\}$ also admits the infimum and

$$\bigwedge_\lambda \varepsilon(x_\lambda) = \varepsilon\Big(\bigwedge_\lambda x_\lambda\Big) \tag{2}$$

is satisfied. Similarly to dilation, we say that every erosion has the *infemum preserving property* (IPP).

Proposition *1:* Every dilation or erosion is monotone.

Example *1 (Morphology of Set Lattices):* Let $R \subseteq X \times A$ be a binary relation. We define the following set operators from $\mathfrak{P}(A)$ into $\mathfrak{P}(X)$ (notice that the direction is opposed):

$$D_R(B) = \{\, x \in X \mid R(x) \cap B \neq \emptyset \,\} \ \big(=\ {}^tR(B)\big)\,, \tag{3}$$
$$E_R(B) = \{\, x \in X \mid R(x) \subseteq B \,\} \tag{4}$$

for $B \in \mathfrak{P}(A)$. Then $D_R : \mathfrak{P}(A) \to \mathfrak{P}(X)$ is a dilation and $E_R : \mathfrak{P}(A) \to \mathfrak{P}(X)$ is an erosion. We call D_R and E_R the *set dilation* and the *set erosion* defined by R respectively.

By considering the transpose ${}^tR$ of R, we also obtain operators $E_{{}^tR} : \mathfrak{P}(X) \to \mathfrak{P}(A)$, $D_{{}^tR} : \mathfrak{P}(X) \to \mathfrak{P}(A)$.

We note that any dilation and erosion between set lattices are obtained in this way. In fact, for any dilation $D : \mathfrak{P}(X) \to \mathfrak{P}(A)$, since any set lattice is "*atomic*" and D has SPP by definition, we have

$$D(Y) \;=\; D\Big(\bigcup_{y \in Y} \{y\}\Big) \;=\; \bigcup_{y \in Y} D(\{y\}).$$

Thus the effect of D on any set $Y \subseteq X$ is determined by its effect on each singleton. By taking the binary relation

$R : X \ni x \mapsto D(\{x\}) \subseteq A$, we have $D = D_{{}^tR}$. For any erosion $E : \mathfrak{P}(X) \to \mathfrak{P}(A)$, since its dual $\overline{E}(Y) = \left(E(Y^{\complement})\right)^{\complement}$ is a dilation, there exists a binary relation R such that $\overline{E} = D_{{}^tR}$. Then it can be verified that $E = \overline{D_{{}^tR}} = E_{{}^tR}$. Relationships among "duality", "transposition" and "adjoint" will be described in section II-D.

C. Adjoint

Definition : Let X, A be partially ordered sets and $f : X \to A$, $g : A \to X$ be mappings. The pair (f, g) is called an *adjoint* between X and A iff $\forall x \in X$, $\forall a \in A$

$$f(x) \leq a \Leftrightarrow x \leq g(a) \tag{5}$$

is satisfied. f is called the *lower adjoint* of g and also g is called the *upper adjoint* of f.

Note : It should be remarked that in the context of category theory (*cf.* [12]), authors use the words "right" and "left" instead of "lower" and "upper" to distinguish each member of an adjoint pair but in the context of mathematical morphology ([7], [1]), the opposite naming is used.

The following proposition gives another characterization for a pair of monotone mappings to give rise to an adjoint.

***Proposition** 2:* Let X, A be partially ordered sets and $f : X \to A$, $g : A \to X$ be monotone mappings. For the pair (f, g) to be an adjoint it is necessary and sufficient that

$$f(g(a)) \leq a, \qquad x \leq g(f(x)) \tag{6}$$

are satisfied for any $x \in X$, $a \in A$.

***Proposition** 3:* Let X, A be partially ordered sets and (f, g) be an adjoint between X and A. Then

1) f is a dilation,
2) g is an erosion.

The converse of Proposition 3 holds under some conditions:

***Proposition** 4:* Let X, A be partially ordered sets.

1) When A is a complete $\bigvee$-semi lattice, for a mapping $f : X \to A$ to be a dilation, it is necessary and sufficient that f is monotone and the pair (f, g) is an adjoint for the mapping defined by $g(a) = \bigvee f^{-1}\{b \in A \mid b \leq a\}$.
2) When X is a complete $\bigwedge$-semi lattice, for a mapping $g : A \to X$ to be an erosion, it is necessary and sufficient that g is monotone and the pair (f, g) is an adjoint for the mapping defined by $f(x) = \bigwedge g^{-1}\{y \in X \mid x \leq y\}$.

***Example** 2 (Adjoint of Set Lattices):* Let $R \subseteq X \times A$ be a binary relation. Then the pair $(D_{{}^tR}, E_R)$ is an adjoint between $\mathfrak{P}(X)$ and $\mathfrak{P}(A)$. In fact, for $Y \in \mathfrak{P}(X)$, $B \in \mathfrak{P}(A)$,

$$\begin{aligned} &D_{{}^tR}(Y) \subseteq B \\ &\quad\Leftrightarrow \forall a \in A, \forall x \in X\, (x \in Y \Rightarrow (a \in R(x) \Rightarrow a \in B)) \\ &\qquad\qquad\qquad\qquad\qquad\qquad \Leftrightarrow Y \subseteq E_R(B). \end{aligned}$$

Similarly, the pair $(D_R, E_{{}^tR})$ is an adjoint between $\mathfrak{P}(A)$ and $\mathfrak{P}(X)$.

D. Involutions in Boolean Lattices

1) Duality, transposition and adjunction: For erosions and dilations of Boolean lattices, there are three sorts of involutive transformations of operators, namely, *duality*, *transposition* and *adjunction* ([4], [7]). More precisely, if μ is a morphological operators and τ is one of these three transformations then $\tau(\tau(\mu)) = \mu$ is satisfied.

Definition : Let X, A be Boolean lattices and $\delta : X \to A$ be a dilation. Then its *dual* $\overline{\delta} : X \to A$, *transpose* ${}^t\delta : A \to X$ and *adjoint* $\delta^* : A \to X$ are respectively defined by

$$\begin{array}{rcll} \overline{\delta}(x) &=& \neg\left(\delta(\neg x)\right) & (x \in X), \\ {}^t\delta(a) \bigwedge x = 0 &\Leftrightarrow& a \bigwedge \delta(x) = 0 & (x \in X,\, a \in A), \\ x \leq \delta^*(a) &\Leftrightarrow& \delta(x) \leq a & (x \in X,\, a \in A). \end{array}$$

Although transpose and adjoint are implicitly defined, they are uniquely determined if they exist for a given dilation.

Similarly, for an erosion $\varepsilon : X \to A$, its *dual* $\overline{\varepsilon} : X \to A$, *transpose* ${}^t\varepsilon : A \to X$ and *adjoint* $\varepsilon^* : A \to X$ are respectively defined by

$$\begin{array}{rcll} \overline{\varepsilon}(x) &=& \neg\left(\varepsilon(\neg x)\right) & (x \in X), \\ {}^t\varepsilon(a) \bigvee x = 1 &\Leftrightarrow& a \bigvee \varepsilon(x) = 1 & (x \in X,\, a \in A), \\ \varepsilon^*(a) \leq x &\Leftrightarrow& a \leq \varepsilon(x) & (x \in X,\, a \in A). \end{array}$$

Note : We adopt a general form for definition of transpose so as to apply to any lattices that may not Boolean. In the literature [4], the author adopted a more restricted form as described below. At first, we note that the complementation $\neg$ of Boolean lattice satisfies

$$x \bigwedge \neg y = 0 \quad \Leftrightarrow \quad x \leq y \quad \Leftrightarrow \quad \neg x \bigvee y = 1\,.$$

By using this, the conditions for transpose are rewritten as for dilation,

$${}^t\delta(a) \leq \neg y \quad \Leftrightarrow \quad \delta(x) \leq \neg a \qquad (x \in X,\, a \in A),$$

and for erosion,

$$\neg x \leq {}^t\varepsilon(a) \quad \Leftrightarrow \quad \neg a \leq \varepsilon(x) \qquad (x \in X,\, a \in A).$$

***Proposition** 5:* Let X, A be Boolean lattices.

1) For a dilation $\delta : X \to A$ to have an adjoint δ^* is equivalent to have a transpose ${}^t\delta$.
2) For an erosion $\varepsilon : X \to A$ to have an adjoint ε^* is equivalent to have a transpose ${}^t\varepsilon$.

***Proposition** 6:* Let X, A be Boolean lattices.

1) For a dilation $\delta : X \to A$, the dual $\overline{\delta}$ and the adjoint δ^* are erosions and the transpose ${}^t\delta$ is a dilation.
2) For an erosion $\varepsilon : X \to A$, the dual $\overline{\varepsilon}$ and the adjoint ε^* are dilations and the transpose ${}^t\varepsilon$ is an erosion.

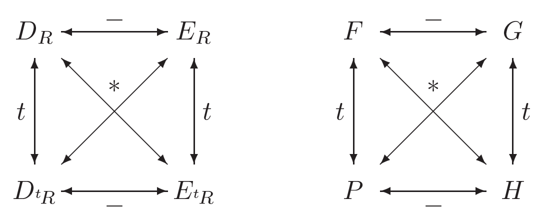
2) Interrelations among Involutions: All of the operator transformations defined above are involutive. On the other hand, successive applications of operators are independent of order. Furthermore, we have the following relations:

$$\begin{array}{lll} \overline{({}^t\delta)} = {}^t\!\left(\overline{\delta}\right) = \delta^*, & \overline{(\delta^*)} = \left(\overline{\delta}\right)^* = {}^t\delta, & {}^t(\delta^*) = ({}^t\delta)^* = \overline{\delta}, \\ \overline{({}^t\varepsilon)} = {}^t(\overline{\varepsilon}) = \varepsilon^*, & \overline{(\varepsilon^*)} = (\overline{\varepsilon})^* = {}^t\varepsilon, & {}^t(\varepsilon^*) = ({}^t\varepsilon)^* = \overline{\varepsilon}. \end{array}$$

Example 3 (Involutions of Set Lattices): In case of a binary relation R, for the operators $\delta = D_R$ and $\varepsilon = E_R$, we have more explicit relations:

$$\overline{D_R} = E_R, \qquad {}^tD_R = D_{{}^tR}, \qquad (D_R)^* = E_{{}^tR},$$
$$\overline{E_R} = D_R, \qquad {}^tE_R = E_{{}^tR}, \qquad (E_R)^* = D_{{}^tR}.$$

By virtue of these equalities, we only have to employ 4 operators among them, for example D_R, E_R, $D_{{}^tR}$ and $E_{{}^tR}$. The relations are diagrammatically represented as follows:



In this case, the diagonal pairs $(D_R, E_{{}^tR})$ and $(D_{{}^tR}, E_R)$ are adjoint. Compare with the right diagram which represents the relationships among the traditional temporal operators.

III. Modal Logics and Temporal Logics

For basic notions and results on modal logics, we refer to [13]. See also [11].

A. Language

Definition *:* Let $\Pi = \{p_1, p_2, \dots\}$ be a set of denumerable number of propositional symbols. The set of formulas generated by Π with classical connectives $\bot$, $\neg$, $\vee$ and unary modal operators $\Diamond_1, \dots, \Diamond_r$ is denoted by $\Phi(\Pi, \Diamond_1, \dots, \Diamond_r)$ or briefly Φ^r, and is called the *r-dimensional modal language*. As usual, we make use of the abbreviations $\phi \wedge \psi = \neg(\phi \vee \psi)$, $\phi \to \psi = \neg\phi \vee \psi$, $\phi \leftrightarrow \psi = (\phi \to \psi) \wedge (\psi \to \phi)$, $\top = \neg\bot$, $\Box_i\phi = \neg\Diamond_i\neg\phi$ $(i = 1, \dots, r)$.

When $r \geq s$, Φ^s can be naturally regarded as a subset of Φ^r. In particular, the language of basic modal logic is embedded in the language of temporal logic.

B. Semantics

Definition *:* A *standard model* for the r-dimensional modal language, or briefly an *r-model* is an $r+2$-tuple $\mathscr{M} = (W, R_1, \dots, R_r, P)$ consisting of a non-empty set W, binary relations $R_i \subseteq W \times W$ $(i = 1, \dots, r)$, and a mapping $P : \Pi \to \mathfrak{P}(W)$. Each element $\omega \in W$ is called a *possible world* and each binary relation R_i is called the *i-th accessibility relation*. The *assignment* P is a mapping which assigns a subset $P_i \subseteq W$ to each propositional symbol $p_i \in \Pi$. The set P_i is called the *true set* for p_i.

Let $\mathscr{M}$ be an r-model. We denote by $\vDash^{\mathscr{M}}_{\omega} \phi$ to stand for a formula $\phi \in \Phi^r$ is true at ω in $\mathscr{M}$. The *truth value* of each formula $\phi \in \Phi^r$ is recursively defined by constructions as follows. Denoting by $\|\phi\|^{\mathscr{M}} = \{\omega \in W \mid \vDash^{\mathscr{M}}_{\omega} \phi\}$ the true set for ϕ,

$$\begin{aligned}
\|p_i\|^{\mathscr{M}} &= P_i && \text{(value of } P \text{ at } p_i \in \Pi) \\
\|\bot\|^{\mathscr{M}} &= \emptyset && \text{(empty set)} \\
\|\neg\phi\|^{\mathscr{M}} &= \left(\|\phi\|^{\mathscr{M}}\right)^{\complement} && \text{(complement)} \\
\|\phi \vee \psi\|^{\mathscr{M}} &= \|\phi\|^{\mathscr{M}} \cup \|\psi\|^{\mathscr{M}} && \text{(union)} \\
\|\Diamond_i\phi\|^{\mathscr{M}} &= D_{R_i}\left(\|\phi\|^{\mathscr{M}}\right) && \text{(dilation)}
\end{aligned}$$

A formula ϕ is said to be *globally true in an r-model $\mathscr{M}$* and denoted by $\vDash^{\mathscr{M}} \phi$ when $\|\phi\|^{\mathscr{M}} = W$; ϕ is said to be *valid in a class $\mathfrak{C}$ of r-models* and denoted by $\vDash_{\mathfrak{C}} \phi$ when $\vDash^{\mathscr{M}} \phi$ for every r-model $\mathscr{M}$ in $\mathfrak{C}$. We call the set of all formulas in Φ^r that are valid in a class of r-models $\mathfrak{C}$ the *logic of* $\mathfrak{C}$ and denote it by $\mathrm{Log}_{\mathfrak{C}}$. We also need consider the formulas in Φ^s $(s \leq r)$ contained in $\mathrm{Log}_{\mathfrak{C}}$, which we denote by $\mathrm{Log}^s_{\mathfrak{C}}$:

$$\mathrm{Log}^s_{\mathfrak{C}} = \{\phi \in \Phi^s \mid \vDash_{\mathfrak{C}} \phi\}. \tag{7}$$

A special class of 2-models is of bidirectional ones. Namely, a 2-model $\mathscr{M} = (W, R_1, R_2, P)$ is called a *bidirectional model* if $R_2 = {}^tR_1$. Since any bidirectional model is determined by pointing out a single accessibility relation, we can regard 1-dimensional models as bidirectional models as follows. For a 1-dimensional model $\mathscr{M} = (W, R, P)$, we consider a bidirectional frame $\mathscr{M}^{\boldsymbol{b}} = (W, R, {}^tR, P)$ and call it the *bidirectionalization* of $\mathscr{M}$. Moreover, for any class of 1-dimensional models $\mathfrak{C}$, the bidirectionalization of $\mathfrak{C}$ is defined as the class of bidirectionalizations of models in $\mathfrak{C}$ and denoted by $\mathfrak{C}^{\boldsymbol{b}}$.

C. Modal Logics and Temporal Logics

Definition *:* A set of formulas $\Sigma \subseteq \Phi^r$ is called an *r-dimensional modal logic* if it contains all tautologies and is closed under "modus ponens": (MP) if $\phi \in \Sigma$ and $\phi \to \psi \in \Sigma$, then $\psi \in \Sigma$. Furthermore, if Σ contains the formulas (K_i) $\Box_i(\phi \to \psi) \to (\Box_i\phi \to \Box_i\psi)$, $(\mathrm{Df}\Diamond_i)$ $\Diamond_i\phi \leftrightarrow \neg\Box_i\neg\phi$ and is closed under "necessitation": (N_i) if $\Sigma \ni \phi$ then $\Sigma \ni \Box_i\phi$, it is called *normal*.

Remark We note that the notions of modal logic and hence of normal modal logic of Chellas [13] are slightly wider than the traditional ones. Indeed, many authors assumed that any modal logic is closed under "uniform substitutions" (see, for example, [14], [11]).

We recall the following propositions ([13], [11]).

Proposition *7:* Let $\mathfrak{C}$ be any class of r-models. Then for any integer $0 \leq s \leq r$, the logic $\mathrm{Log}^s_{\mathfrak{C}}$ of $\mathfrak{C}$ is an s-dimensional normal modal logic.

Proposition *8:* Let $\Sigma_\lambda \subseteq \Phi^r$ be an arbitrary family of subsets of modal language and put $\Sigma = \bigcap_\lambda \Sigma_\lambda$. Then we have the followings.

1) If every subset Σ_λ is a modal logic, then so is Σ.
2) If every subset Σ_λ is a normal modal logic, then so is Σ.

Corollary *1:* The set of all r-dimensional normal modal logics $\mathscr{N}^r$ forms a complete $\bigwedge$-semi lattice.

The minimal normal modal logic (usually $r = 1$) is denoted by **K**. **K** coincides with the logic of the class of all r-models.

Many other traditional normal modal logics are defined by adding some axioms to **K** ([13]).

Definition *:* A *temporal logic* is a normal modal logic in Φ^2 satisfying the axioms

$$\text{(Conv)} \qquad \varphi \to \Box_1 \Diamond_2 \varphi \qquad \text{and} \qquad \varphi \to \Box_2 \Diamond_1 \varphi.$$

Similarly to Proposition 7 and Proposition 8, we have the followings ([11]):

Proposition *9:* Let $\mathfrak{B}$ be any family of bidirectional models. Then the logic $\mathrm{Log}_{\mathfrak{B}} = \left\{ \phi \in \Phi^2 \mid \vDash_{\mathfrak{B}} \phi \right\}$ of $\mathfrak{B}$ is a temporal logic.

Proposition *10:* Let $T_\lambda \subseteq \Phi^2$ be an arbitrary family of temporal logics. Then $T = \bigcap_\lambda T_\lambda$ is also a temporal logic.

Corollary *2:* The set of all temporal logics $\mathscr{T}$ forms a complete $\bigwedge$-semi lattice.

The minimal temporal logic is denoted by $\mathbf{K}_t$. $\mathbf{K}_t$ coincides with the logic of the class of all bidirectional models.

D. Soundness, Completeness and Canonical Models

Definition *:* For a modal logic $\Sigma \subseteq \Phi^r$, when $\phi \in \Sigma$, we say that ϕ is a *theorem of* Σ and write $\vdash_\Sigma \phi$. Furthermore, let $\Gamma \subseteq \Phi^r$ and $\phi \in \Phi^r$. When there are formulas $\phi_1, \ldots,$ $\phi_n \in \Gamma$ $(n \geq 0)$ such that $\vdash_\Sigma (\phi_1 \wedge \cdots \wedge \phi_n) \to \phi$, we say that ϕ is deducible in Σ from Γ and write $\Gamma \vdash_\Sigma \phi$. A set of formulas Γ is *Σ-inconsistent* if $\Gamma \vdash_\Sigma \bot$, otherwise, Γ is *Σ-consistent*. Furthermore, if $\Gamma \cup \{\phi\}$ is inconsistent for any formula $\phi \notin \Gamma$, Γ is said to be *maximally Σ-consistent*.

Definition *:* Let $\mathfrak{C}$ be a class of r-models and Σ be an r-dimensional normal modal logic. Σ is *sound* with respect to $\mathfrak{C}$ if $\Sigma \subseteq \mathrm{Log}_{\mathfrak{C}}$ or equivalently, $\vdash_\Sigma \phi \Rightarrow \vDash_{\mathfrak{C}} \phi$. In this case, $\mathfrak{C}$ is called a *class of models for* Σ. Conversely, Σ is *complete* with respect to $\mathfrak{C}$ if $\mathrm{Log}_{\mathfrak{C}} \subseteq \Sigma$ or equivalently, $\vDash_{\mathfrak{C}} \phi \Rightarrow \vdash_\Sigma \phi$. Finally, Σ is *determined* by $\mathfrak{C}$ if $\mathrm{Log}_{\mathfrak{C}} = \Sigma$ or equivalently, $\vDash_{\mathfrak{C}} \phi \Leftrightarrow \vdash_\Sigma \phi$.

Proposition *11:* If $\Gamma \subseteq \Phi^r$ is a maximal Σ-consistent set for an r-dimensional modal logic Σ, then

1) $\phi \in \Gamma$ iff $\Gamma \vdash_\Sigma \phi$.
2) Γ is a modal logic containing Σ.
3) $\bot \notin \Gamma$.
4) $\neg\phi \in \Gamma$ iff $\phi \notin \Gamma$.
5) $\phi \vee \psi \in \Gamma$ iff $\phi \in \Gamma$ or $\psi \in \Gamma$.
6) $\Diamond_i \phi \in \Gamma$ iff $\phi \in \Delta$ for some maximal Σ-consistent set Δ satisfying $\Diamond_i \Delta \subseteq \Gamma$.

Theorem *1 (Lindenbaum's Lemma):* If $\Gamma \subseteq \Phi^r$ is a Σ-consistent set for an r-dimensional modal logic Σ, then there exists a maximal Σ-consistent set $\Delta \subseteq \Phi^r$ such that $\Gamma \subseteq \Delta$.

Definition *:* The *canonical model* for $\Sigma \in \mathscr{N}^r$ is the r-model $\mathscr{M}_\Sigma = (W_\Sigma, R_{\Sigma,1}, \ldots, R_{\Sigma,r}, P_\Sigma)$ with

1) W_Σ is the set of all maximal Σ-consistent sets.
2) $R_{\Sigma,i}$ is the binary relation on W_Σ defined by $\zeta \in R_{\Sigma,i}(\omega)$ iff $\Diamond_i \zeta \subseteq \omega$.
3) P_Σ is the assignment defined by $P_\Sigma(p_i) = \{\omega \in W_\Sigma |$ $p_i \in \omega\}$.

Theorem *2 (Truth Lemma):* Let $\mathscr{M}_\Sigma$ be the canonical model for $\Sigma \in \mathscr{N}^r$ and let $\phi \in \Phi^r$. Then for any ω in $\mathscr{M}$, $\vDash_\omega^{\mathscr{M}} \phi$ iff $\phi \in \omega$.

From this lemma, soundness with respect to any canonical model follows. In fact, $\vdash_\Sigma \phi \Rightarrow \phi \in \omega \, (\forall \omega \in W) \Rightarrow \vDash_{\mathscr{M}} \phi$.

Conversely, completeness is known as the following theorem.

Theorem *3 (Canonical Model Theorem):* Let $\mathscr{M}_\Sigma$ be the canonical model for $\Sigma \in \mathscr{N}^r$. Then Σ is complete with respect to $\mathscr{M}_\Sigma$.

Thus any normal logic Σ is determined by its canonical model $\mathscr{M}_\Sigma$, namely, $\mathrm{Log}_{\mathscr{M}_\Sigma} = \Sigma$.

For temporal logics the following lemma holds.

Lemma *1:* For any $T \in \mathscr{T}$, the canonical frame $\mathscr{M}_T = (W_T, R_{T,1}, R_{T,2}, P)$ is bidirectional.

IV. TEMPORALIZATION

A. Temporalization

Throughout this section, we consider over the 1-dimesional modal language Φ^1 embedded in the 2-dimensional language Φ^2. Then we can define the restriction mapping by

$$\rho : \mathfrak{P}\left(\Phi^2\right) \ni \Lambda \mapsto \Lambda \cap \Phi^1 \in \mathfrak{P}\left(\Phi^1\right).$$

Definition *:* Let Σ be a 1-dimensional modal logic and $T \in \mathscr{T}$. When $\Sigma \subseteq T$, we call T as a *temporalization* of Σ. Furthermore, if $\rho(T) = \Sigma$, T is called a *strict temporalization* of Σ.

When $\rho(T) = \Sigma$, Σ is automatically normal. Thus, for a modal logic Σ to have a strict temporalization, it is necessarily normal. It will be shown that it is also sufficient later. Then we make use of the following proposition.

Proposition *12:* Let $\mathfrak{C}$ be a class of 1-models and $\mathfrak{C}^b$ be its bidirectionalization. Then $\mathrm{Log}_{\mathfrak{C}^b}$ is a strict temporalization of $\mathrm{Log}_{\mathfrak{C}}$.

Proof. By Proposition 7, $\mathrm{Log}_{\mathfrak{C}}$ is a 1-dimensional normal modal logic. Similarly, by Proposition 9, $\mathrm{Log}_{\mathfrak{C}^b}$ is a temporal logic. It is clear that $\mathrm{Log}_{\mathfrak{C}} \subseteq \mathrm{Log}_{\mathfrak{C}^b}$. Thus all we have to show is that $\mathrm{Log}^1_{\mathfrak{C}^b} = \rho\big(\mathrm{Log}_{\mathfrak{C}^b}\big)$ coincide with $\mathrm{Log}_{\mathfrak{C}}$. But this comes from the fact that for $\phi \in \Phi^1$, $\vDash_{\mathfrak{C}} \phi$ iff $\vDash_{\mathfrak{C}^b} \phi$.

q.e.d.

B. Existence

1) Minimal Temporalization: Let Σ be a modal logic. If there is no special requirement, its temporalization always exists. In fact, the minimal temporal logic $\tau(\Sigma)$ including Σ becomes a temporalization by virtue of Proposition 10. Existence of any temporal logic including Σ is guaranteed by the full language Φ^2. We call $\tau(\Sigma)$ the *minimal temporalization* of Σ. In this case, $\Sigma \subseteq \rho(\tau(\Sigma))$.

For a strict temporalization, the answer is affirmative for normal modal logics. To show this, we first note that the operator $\tau : \mathfrak{P}\left(\Phi^1\right) \to \mathscr{T}$ is the lower adjoint of the restriction mapping (restricted to $\mathscr{T}$) $\rho : \mathscr{T} \to \mathfrak{P}\left(\Phi^1\right)$. In fact, it can be easily verified that the following equivalence is satisfied:

$$\tau(\Sigma) \subseteq T \quad \Leftrightarrow \quad \Sigma \subseteq \rho(T)$$

for $\Sigma \in \mathfrak{P}\left(\Phi^1\right)$ and $T \in \mathscr{T}$. Moreover, since ρ is monotone by Proposition 1, we have $\Sigma \subseteq \rho(\tau(\Sigma)) \subseteq \rho(T)$. Thus the minimal temporalization $\tau(\Sigma)$ is strict provided that there exists a strict temporalization T of Σ. Such a temporalization will be constructed for any normal logic in the next section.

2) Canonical Temporalization: Now suppose that $\Sigma \in \mathscr{N}^2$. Then by Canonical Model Theorem (Theorem 3), Σ can be represented as the logic of $\mathscr{M}_\Sigma$. Furthermore, considering the bidirectionalization of $\mathscr{M}_\Sigma$, $\Sigma^{\boldsymbol{b}} = \mathrm{Log}_{(\mathscr{M}_\Sigma)^{\boldsymbol{b}}}$ is a strict temporalization of $\Sigma = \mathrm{Log}_{\mathscr{M}_\Sigma}$ by virtue of Proposition 12. We call $\Sigma^{\boldsymbol{b}}$ the *canonical temporalization* of Σ.

***Theorem** 4:* For an arbitrary normal modal logic Σ, the canonical temporalization $\Sigma^{\boldsymbol{b}}$ is a strict temporalization.

***Corollary** 3:* For a normal modal logic Σ, the minimal temporalization $\tau(\Sigma)$ is a strict temporalization of Σ.

C. Structure of Temporalization

To make use of canonical models to illustrate the temporalization procedure, we refer to the following lemma ([15]):

***Lemma** 2:* Let $s \leq r$ be non-negative integers and $\rho_s : \Phi^r \to \Phi^s$ be the restriction mapping. Furthermore, let Σ be an s-dimensional modal logic included in an r-dimensional modal logic Λ.

1) For any maximal Λ-consistent set $\Gamma \subseteq \Phi^r$, $\rho_s(\Gamma)$ is maximally Σ-consistent.
2) If $\rho_s(\Lambda) = \Sigma$, for any Σ-consistent set $\Delta \subseteq \Phi^1$, it is also Λ-consistent.

We apply this lemma for a modal logic Σ and its temporalization T. Since each possible world is consisting of maximal consistency sets, by virtue of 1) of Lemma 2, we have a mapping

$$W_T \ni \omega \quad \mapsto \quad \rho_1(\omega) \in W_\Sigma. \tag{8}$$

Furthermore, we can insert subsets $\tau(\Sigma)$ and $\rho_1(T)$ between Σ and T as

$$\Sigma \subseteq \tau(\Sigma) \subseteq T \quad \text{and} \quad \Sigma \subseteq \rho_1(T) \subseteq T.$$

Thus the mapping (8) splits in two ways:

$$W_T \xrightarrow{\iota_T} W_{\tau(\Sigma)} \xrightarrow{\rho_1} W_\Sigma, \tag{9}$$

$$W_T \xrightarrow{\rho_1} W_{\rho_1(T)} \xrightarrow{\iota_{\rho_1(T)}} W_\Sigma. \tag{10}$$

Since both of (9) and (10) are splittings for the same mapping (8), we have the following commutative diagram:

$$\begin{array}{ccc} W_T & \xrightarrow{\iota_T} & W_{\tau(\Sigma)} \\ {\scriptstyle\rho_1}\downarrow & & \downarrow{\scriptstyle\rho_1} \\ W_{\rho_1(T)} & \xrightarrow[\iota_{\rho_1(T)}]{} & W_\Sigma \end{array} \tag{11}$$

By combining 2) of Lemma 2 with Lindenbaum's lemma, it follows that both of the vertical arrows are surjective. On the other hand, it follows that both of the horizontal arrows are injective from 1) of Lemma 2 with dimensional argument.

V. Conclusion

A direct application of existence of a temporalization is that one may assume that the modal operators $\Diamond$ has SPP and $\Box$ has IPP by virtue of their imlicit adjoints in any normal modal logic. But if one wants to employ the adjoints explicitly, one should indicate which temporalization is considered.

To resolve this ambiguity, one must go ahead. At this moment, the diagram (11) does not tell us any more information. But practiced geometers or categorists notice that our approach seems like the scheme of classifying space. In fact, if one can show the following conjecture, many information can be extracted from the diagram.

***Conjecture** 1:* In the diagram (11), W_T is a fibre product for $\rho_1 : W_{\tau(\Sigma)} \to W_\Sigma$ and $\iota_{\rho_1(T)} : W_{\rho_1(T)} \to W_\Sigma$.

From this conjecture, it follows that the vertical surjection $\rho_1 : \mathscr{M}_{\tau(\Sigma)} \twoheadrightarrow \mathscr{M}_\Sigma$ classifies the temporalization of Σ, that is, any temporalization of Σ can be induced from a horizontal injection $\iota : W \hookrightarrow W_\Sigma$. As a consequent, one can conclude that the uniqueness of strict temporalization because of it is the unique temporalization induced from the identity mapping $id_{W_\Sigma} : W_\Sigma \to W_\Sigma$.

Finally, we note that this scheme of mathematical morphological analysis also works for normal extension of modal logics ([15]).

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