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Tensor Complementarity Problem and Semi-Positive Tensors

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Abstract In this paper, we prove that a real tensor is strictly semi-positive if and only if the corresponding tensor complementarity problem has a unique solution for any nonnegative vector and a real tensor is semi-positive if and only if the corresponding tensor complementarity problem has a unique solution for any positive vector. It is showed that a real symmetric tensor is a (strictly) semi-positive tensor if and only if it is (strictly) copositive.

Keywords Tensor complementarity · strictly semi-positive · strictly copositive · unique solution

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1 Introduction

It is well-known that the linear complementarity problem (LCP) is the first-order optimality conditions of quadratic programming, which has wide applications in applied science and technology such as optimization and physical or economic equilibrium problems. By means of the linear complementarity problem, properties of (strictly) semi-monotone matrices were considered by Cottle and Dantzig [1], Eaves [2] and Karamardian [3], see also Han, Xiu and Qi [4], Facchinei and Pang [5] and Cottle, Pang and Stone [6].

Pang [7,8] and Gowda [9] presented that some relations between the solution of the LCP $(\mathbf{q}, A)$ and (strictly) semi-monotone. Cottle [10] showed that each completely Q-matrix is a strictly semi-monotone matrix. Eaves [2] gave an equivalent definition of strictly semi-monotone matrices using the linear complementarity problem. The concept of (strictly) copositive matrices is one of the most important concept in applied mathematics and graph theory, which was introduced by Motzkin [11] in 1952. In the literature, there are extensive discussions on such matrices [12–14].

The nonlinear complementarity problem has been systematically studied in the mid-1960s and has developed into a very fruitful discipline in the field of mathematical programming, that included a multitude of interesting connections to numerous disciplines and a widely important applications in engineering and economics. The notion of the tensor complementarity problem, a special structured nonlinear complementarity problem, is used firstly by Song and Qi [15], and they studied the existence of solution for the tensor complementarity problem with some classes of structured tensors. In particular, they showed that the tensor complementarity problem with a nonnegative tensor has a solution if and only if all principal diagonal entries of such a tensor are positive. Che, Qi, Wei [16] showed the existence of solution for the tensor complementarity problem with symmetric positive definite tensors and copositive tensors. Luo, Qi, Xiu [17] studied the sparsest solutions to Z-tensor complementarity problems.

In this paper, we will study some relationships between the unique solution of the tensor complementarity problem and (strictly) semi-positive tensors. We will prove that a symmetric m-order n-dimensional tensor is (strictly) semi-positive if and only if it is (strictly) copositive.

In Section 2, we will give some definitions and basic conclusions. We will show that all diagonal entries of a semi-positive tensor are nonnegative, and all diagonal entries of a strictly semi-positive tensor are positive.

In Section 3, we will prove that a real tensor is a semi-positive tensor if and only if the corresponding tensor complementarity problem has no non-zero vector solution for any positive vector and a real tensor is a strictly semi-positive tensor if and only if the corresponding tensor complementarity problem has no non-zero vector solution for any nonnegative vector. We show that a symmetric real tensor is semi-positive if and only if it is copositive and a symmetric real tensor is a strictly semi-positive if and only if it is strictly copositive.

2 Preliminaries

Throughout this paper, we use small letters $x, y, v, \alpha, \cdots$, for scalars, small bold letters $\mathbf{x}, \mathbf{y}, \cdots$, for vectors, capital letters $A, B, \cdots$, for matrices, calligraphic letters $\mathcal{A}, \mathcal{B}, \cdots$, for tensors. All the tensors discussed in this paper are real. Let $I_n := \{1, 2, \cdots, n\}$, and $\mathbb{R}^n := \{(x_1, x_2, \cdots, x_n)^\top; x_i \in \mathbb{R}, i \in I_n\}$, $\mathbb{R}^n_+ := \{x \in \mathbb{R}^n; x \geq \mathbf{0}\}$, $\mathbb{R}^n_- := \{\mathbf{x} \in \mathbb{R}^n; x \leq \mathbf{0}\}$, $\mathbb{R}^n_{++} := \{\mathbf{x} \in \mathbb{R}^n; x > \mathbf{0}\}$, where $\mathbb{R}$ is the set of real numbers, $\mathbf{x}^\top$ is the transposition of a vector $\mathbf{x}$, and $\mathbf{x} \geq \mathbf{0}$ ($\mathbf{x} > \mathbf{0}$) means $x_i \geq 0$ ($x_i > 0$) for all $i \in I_n$.

Let $A = (a_{ij})$ be an $n \times n$ real matrix. The linear complementarity problem, denoted by LCP $(\mathbf{q}, A)$, is to find $\mathbf{z} \in \mathbb{R}^n$ such that

$$\mathbf{z} \geq \mathbf{0}, \mathbf{q} + A\mathbf{z} \geq \mathbf{0}, \text{ and } \mathbf{z}^\top(\mathbf{q} + A\mathbf{z}) = 0, \tag{1}$$

or to show that no such vector exists. A real matrix A is said to be

(i) **semi-monotone (or semi-positive)** iff for each $\mathbf{x} \geq 0$ and $\mathbf{x} \neq \mathbf{0}$, there exists an index $k \in I_n$ such that $x_k > 0$ and $(A\mathbf{x})_k \geq 0$;

(ii) **strictly semi-monotone (or strictly semi-positive)** iff for each $\mathbf{x} \geq \mathbf{0}$ and $\mathbf{x} \neq \mathbf{0}$, there exists an index $k \in I_n$ such that $x_k > 0$ and $(A\mathbf{x})_k > 0$;

(iii) **copositive** iff $\mathbf{x}^\top A\mathbf{x} \geq 0$ for all $\mathbf{x} \in \mathbb{R}^n_+$;

(iv) **strictly copositive** iff $\mathbf{x}^\top A\mathbf{x} > 0$ for all $\mathbf{x} \in \mathbb{R}^n_+ \setminus \{\mathbf{0}\}$;

(v) **Q-matrix** iff LCP$(A, \mathbf{q})$ has a solution for all $\mathbf{q} \in \mathbb{R}^n$;

(vi) **completely Q-matrix** iff A and all its principal sub-matrices are Q-matrices.

In 2005, Qi [18] introduced the concept of positive (semi-)definite symmetric tensors. A real mth order n-dimensional tensor (hypermatrix) $\mathcal{A} = (a_{i_1\cdots i_m})$ is a multi-array of real entries $a_{i_1\cdots i_m}$, where $i_j \in I_n$ for $j \in I_m$. Denote the set of all real mth order n-dimensional tensors by $T_{m,n}$. Then $T_{m,n}$ is a linear space of dimension n^m. Let $\mathcal{A} = (a_{i_1\cdots i_m}) \in T_{m,n}$. If the entries $a_{i_1\cdots i_m}$ are invariant under any permutation of their indices, then $\mathcal{A}$ is called a **symmetric tensor**. Denote the set of all real mth order n-dimensional symmetric tensors by $S_{m,n}$. Then $S_{m,n}$ is a linear subspace of $T_{m,n}$. We denote the zero tensor in $T_{m,n}$ by $\mathcal{O}$. Let $\mathcal{A} = (a_{i_1\cdots i_m}) \in T_{m,n}$ and $\mathbf{x} \in \mathbb{R}^n$. Then $\mathcal{A}\mathbf{x}^{m-1}$ is a vector in $\mathbb{R}^n$ with its ith component as

$$\left(\mathcal{A}\mathbf{x}^{m-1}\right)_i := \sum_{i_2,\cdots,i_m=1}^{n} a_{ii_2\cdots i_m} x_{i_2}\cdots x_{i_m}$$

for $i \in I_n$. Then $\mathcal{A}\mathbf{x}^m$ is a homogeneous polynomial of degree m, defined by

$$\mathcal{A}\mathbf{x}^m := \mathbf{x}^\top(\mathcal{A}\mathbf{x}^{m-1}) = \sum_{i_1,\cdots,i_m=1}^{n} a_{i_1\cdots i_m} x_{i_1}\cdots x_{i_m}.$$

A tensor $\mathcal{A} \in T_{m,n}$ is called **positive semi-definite** if for any vector $\mathbf{x} \in \mathbb{R}^n$, $\mathcal{A}\mathbf{x}^m \geq 0$, and is called **positive definite** if for any nonzero vector $\mathbf{x} \in \mathbb{R}^n$, $\mathcal{A}\mathbf{x}^m > 0$. Recently, miscellaneous structured tensors are widely studied, for example, Zhang, Qi and Zhou [19] and Ding, Qi and Wei [20] for M-tensors, Song and Qi [21] for P-(P_0)tensors and B-(B_0)tensors, Qi and Song [22] for B-(B_0)tensors, Song and Qi [23] for infinite and finite dimensional Hilbert tensors, Song and Qi [24] for E-eigenvalues of weakly symmetric nonnegative tensors and so on.

Recently, Song and Qi [15] extended the concepts of (strictly) semi-positive matrices and the linear complementarity problem to (strictly) semi-positive tensors and the tensor complementarity problem, respectively. Moreover, some nice properties of those concepts were obtained.

Definition 2.1 *Let $\mathcal{A} = (a_{i_1 \cdots i_m}) \in T_{m,n}$. The tensor complementarity problem, denoted by TCP $(\mathbf{q}, \mathcal{A})$, is to find $\mathbf{x} \in \mathbb{R}^n$ such that*

$$\mathbf{x} \geq \mathbf{0}, \mathbf{q} + \mathcal{A}\mathbf{x}^{m-1} \geq \mathbf{0}, \ \textit{and}\ \mathbf{x}^\top (\mathbf{q} + \mathcal{A}\mathbf{x}^{m-1}) = 0, \tag{2}$$

or to show that no such vector exists.

Clearly, the tensor complementarity problem is the first-order optimality conditions of the homogeneous polynomial optimization problem, which may be referred to as a direct and natural extension of the linear complementarity problem. The tensor complementarity problem TCP $(\mathbf{q}, \mathcal{A})$ is a specially structured nonlinear complementarity problem, and so, the TCP $(\mathbf{q}, \mathcal{A})$ has its particular and nice properties other than ones of the classical nonlinear complementarity problem.

Definition 2.2 *Let $\mathcal{A} = (a_{i_1 \cdots i_m}) \in T_{m,n}$. $\mathcal{A}$ is said to be*

(i) **semi-positive** *iff for each $\mathbf{x} \geq 0$ and $\mathbf{x} \neq \mathbf{0}$, there exists an index $k \in I_n$ such that*

$$x_k > 0 \ \textit{and}\ \left(\mathcal{A}\mathbf{x}^{m-1}\right)_k \geq 0;$$

(ii) **strictly semi-positive** *iff for each* $\mathbf{x} \geq \mathbf{0}$ *and* $\mathbf{x} \neq \mathbf{0}$*, there exists an index* $k \in I_n$ *such that*

$$x_k > 0 \ \textit{and} \ \left(\mathcal{A}\mathbf{x}^{m-1}\right)_k > 0;$$

(iii) **Q-tensor** *iff the TCP* $(\mathbf{q}, \mathcal{A})$ *has a solution for all* $\mathbf{q} \in \mathbb{R}^n$*.*

Lemma 2.1 (Song and Qi [15, Corollary 3.3, Theorem 3.4]) *Each strictly semi-positive tensor must be a Q-tensor.*

Proposition 2.1 *Let* $\mathcal{A} \in T_{m,n}$*. Then*

(i) $a_{ii\cdots i} \geq 0$ *for all* $i \in I_n$ *if* $\mathcal{A}$ *is semi-positive;*

(ii) $a_{ii\cdots i} > 0$ *for all* $i \in I_n$ *if* $\mathcal{A}$ *is strictly semi-positive;*

(iii) there exists $k \in I_n$ *such that* $\sum\limits_{i_2,\cdots,i_m=1}^{n} a_{ki_2\cdots i_m} \geq 0$ *if* $\mathcal{A}$ *is semi-positive;*

(iv) there exists $k \in I_n$ *such that* $\sum\limits_{i_2,\cdots,i_m=1}^{n} a_{ki_2\cdots i_m} > 0$ *if* $\mathcal{A}$ *is strictly semi-positive.*

Proof It follows from Definition 2.2 that we can obtain (i) and (ii) by taking

$$\mathbf{x}^{(i)} = (0, \cdots, 1, \cdots, 0)^\top, \ i \in I_n$$

where 1 is the ith component x_i. Similarly, choose $\mathbf{x} = \mathbf{e} = (1, 1, \cdots, 1)^\top$, then we obtain (iii) and (vi) by Definition 2.2. $\square$

Definition 2.3 *Let* $\mathcal{A} = (a_{i_1\cdots i_m}) \in T_{m,n}$*.* $\mathcal{A}$ *is said to be*

(i) **copositive** *if* $\mathcal{A}\mathbf{x}^m \geq 0$ *for all* $\mathbf{x} \in \mathbb{R}^n_+$*;*

(ii) **strictly copositive** *if* $\mathcal{A}\mathbf{x}^m > 0$ *for all* $\mathbf{x} \in \mathbb{R}^n_+ \setminus \{\mathbf{0}\}$.

The concept of (strictly) copositive tensors was first introduced and used by Qi in [25]. Song and Qi [26] showed their equivalent definition and some special structures. The following lemma is one of the structure conclusions of (strictly) copositive tensors in [26].

Lemma 2.2 ([26, Proposition 3.1]) *Let $\mathcal{A}$ be a symmetric tensor of order m and dimension n. Then*

(i) $\mathcal{A}$ is copositive if and only if $\mathcal{A}x^m \geq 0$ for all $x \in \mathbb{R}^n_+$ with $\|x\| = 1$;

(ii) $\mathcal{A}$ is strictly copositive if and only if $\mathcal{A}x^m > 0$ for all $x \in \mathbb{R}^n_+$ with $\|x\| = 1$.

Definition 2.4 *Let $\mathcal{A} = (a_{i_1\cdots i_m}) \in T_{m,n}$. In homogeneous polynomial $\mathcal{A}\mathbf{x}^m$, if we let some (but not all) x_i be zero, then we have a less variable homogeneous polynomial, which defines a lower dimensional tensor. We call such a lower dimensional tensor a* **principal sub-tensor** *of $\mathcal{A}$, i.e., an m-order r-dimensional principal sub-tensor $\mathcal{B}$ of an m-order n-dimensional tensor $\mathcal{A}$ consists of r^m entries in $\mathcal{A} = (a_{i_1\cdots i_m})$: for any set $\mathcal{N}$ that composed of r elements in $\{1, 2, \cdots, n\}$,*

$$\mathcal{B} = (a_{i_1\cdots i_m}), \text{ for all } i_1, i_2, \cdots, i_m \in \mathcal{N}.$$

The concept were first introduced and used by Qi [18] to the higher order symmetric tensor. It follows from Definition 2.2 that the following Proposition is obvious.

Proposition 2.2 Let $\mathcal{A} = (a_{i_1\cdots i_m}) \in T_{m,n}$. Then

(i) each principal sub-tensor of a semi-positive tensor is semi-positive;

(ii) each principal sub-tensor of a strictly semi-positive tensor is strictly semi-positive.

Let $N \subset I_n = \{1, 2, \cdots, n\}$. We denote the principal sub-tensor of $\mathcal{A}$ by $\mathcal{A}^{|N|}$, where $|N|$ is the cardinality of N. So, $\mathcal{A}^{|N|}$ is a tensor of order m and dimension $|N|$ and the principal sub-tensor $\mathcal{A}^{|N|}$ is just $\mathcal{A}$ itself when $N = I_n = \{1, 2, \cdots, n\}$.

3 Main results

In this section, we will prove that a real tensor $\mathcal{A}$ is a (strictly) semi-positive tensor if and only if the tensor complementarity problem $(\mathbf{q}, \mathcal{A})$ has a unique solution for $\mathbf{q} > \mathbf{0}$ ($\mathbf{q} \geq \mathbf{0}$).

Theorem 3.1 *Let $\mathcal{A} = (a_{i_1 \cdots i_m}) \in T_{m,n}$. The following statements are equivalent:*

(i) $\mathcal{A}$ is semi-positive.

(ii) The TCP $(\mathbf{q}, \mathcal{A})$ has a unique solution for every $\mathbf{q} > \mathbf{0}$.

(iii) For every index set $N \subset I_n$, the system

$$\mathcal{A}^{|N|}(\mathbf{x}^N)^{m-1} < \mathbf{0},\ \mathbf{x}^N \geq \mathbf{0} \tag{3}$$

has no solution, where $\mathbf{x}^N \in \mathbb{R}^{|N|}$.

Proof (i) $\Rightarrow$ (ii). Since $\mathbf{q} > \mathbf{0}$, it is obvious that $\mathbf{0}$ is a solution of TCP $(\mathbf{q}, \mathcal{A})$. Suppose that there exists a vector $\mathbf{q}' > \mathbf{0}$ such that TCP $(\mathbf{q}', \mathcal{A})$ has non-zero vector solution $\mathbf{x}$. Since $\mathcal{A}$ is semi-positive, there is an index $k \in I_n$ such that

$$x_k > 0 \text{ and } \left(\mathcal{A}\mathbf{x}^{m-1}\right)_k \geq 0.$$

Then $q'_k + \left(\mathcal{A}\mathbf{x}^{m-1}\right)_k > 0$, and so

$$\mathbf{x}^\top(\mathbf{q}' + \mathcal{A}\mathbf{x}^{m-1}) > 0.$$

This contradicts the assumption that $\mathbf{x}$ solves TCP $(\mathbf{q}', \mathcal{A})$. So the TCP $(\mathbf{q}, \mathcal{A})$ has a unique solution $\mathbf{0}$ for every $\mathbf{q} > \mathbf{0}$.

(ii) $\Rightarrow$ (iii). Suppose that there is an index set N such that the system (3) has a solution $\bar{\mathbf{x}}^N$. Clearly, $\bar{\mathbf{x}}^N \neq \mathbf{0}$. Let $\bar{\mathbf{x}} = (\bar{x}_1, \bar{x}_2, \cdots, \bar{x}_n)^\top$ with

$$\bar{x}_i = \begin{cases} \bar{\mathbf{x}}_i^N, i \in N \\ 0, \quad i \in I_n \setminus N. \end{cases}$$

Choose $\mathbf{q} = (q_1, q_2, \cdots, q_n)^\top$ with

$$\begin{cases} q_i = -\left(\mathcal{A}^{|N|}(\bar{\mathbf{x}}^N)^{m-1}\right)_i = -\left(\mathcal{A}\bar{\mathbf{x}}^{m-1}\right)_i, i \in N \\ q_i > \max\{0, -\left(\mathcal{A}\bar{\mathbf{x}}^{m-1}\right)_i\} \quad i \in I_n \setminus N. \end{cases}$$

So, $\mathbf{q} > \mathbf{0}$ and $\bar{\mathbf{x}} \neq \mathbf{0}$. Then $\bar{\mathbf{x}}$ solves the TCP $(\mathbf{q}, \mathcal{A})$. This contradicts (ii).

(iii) $\Rightarrow$ (i). For each $\mathbf{x} \in \mathbb{R}^n_+$ and $\mathbf{x} \neq \mathbf{0}$, we may assume that $\mathbf{x} = (x_1, x_2, \cdots, x_n)^\top$ with for some $N \subset I_n$,

$$\begin{cases} x_i > 0, i \in N \\ x_i = 0, i \in I_n \setminus N. \end{cases}$$

Since the system (3) has no solution, there exists an index $k \in N \subset I_n$ such that

$$x_k > 0 \text{ and } \left(\mathcal{A}\mathbf{x}^{m-1}\right)_k \geq 0,$$

and hence $\mathcal{A}$ is semi-positive. □

Using the same proof as that of Theorem 3.1 with appropriate changes in the inequalities. We can obtain the following conclusions about the strictly semi-positive tensor.

Theorem 3.2 *Let $\mathcal{A} = (a_{i_1\cdots i_m}) \in T_{m,n}$. The following statements are equivalent:*

(i) $\mathcal{A}$ is strictly semi-positive.

(ii) The TCP $(\mathbf{q}, \mathcal{A})$ has a unique solution for every $\mathbf{q} \geq \mathbf{0}$.

(iii) For every index set $N \subset I_n$, the system

$$\mathcal{A}^{|N|}(\mathbf{x}^N)^{m-1} \leq \mathbf{0},\ \mathbf{x}^N \geq \mathbf{0},\ \mathbf{x}^N \neq \mathbf{0} \tag{4}$$

has no solution.

Now we give the following main results by means of the concept of principal sub-tensor.

Theorem 3.3 *Let $\mathcal{A}$ be a symmetric tensor of order m and dimension n. Then $\mathcal{A}$ is semi-positive if and only if it is copositive.*

Proof If $\mathcal{A}$ is copositive, then

$$\mathcal{A}\mathbf{x}^m = \mathbf{x}^\top \mathcal{A}\mathbf{x}^{m-1} \geq 0 \text{ for all } \mathbf{x} \in \mathbb{R}^n_+. \tag{5}$$

So $\mathcal{A}$ must be semi-positive. In fact, suppose not. Then there is a vector $\mathbf{x} \in \mathbb{R}^n$ such that for all $k \in I_n$

$$\left(\mathcal{A}\mathbf{x}^{m-1}\right)_k < 0 \text{ when } x_k > 0.$$

Then we have

$$\mathcal{A}\mathbf{x}^m = \mathbf{x}^\top \mathcal{A}\mathbf{x}^{m-1} = \sum_{k=1}^n x_k \left(\mathcal{A}\mathbf{x}^{m-1}\right)_k < 0,$$

which contradicts (5).

Now we show the necessity. Let

$$S = \{\mathbf{x} \in \mathbb{R}^n_+; \sum_{i=1}^n x_i = 1\} \text{ and } F(\mathbf{x}) = \mathcal{A}\mathbf{x}^m = \mathbf{x}^\top \mathcal{A}\mathbf{x}^{m-1}.$$

Obviously, $F : S \to \mathbb{R}$ is continuous on the set S. Then there exists $\tilde{\mathbf{y}} \in S$ such that

$$\mathcal{A}\tilde{\mathbf{y}}^m = \tilde{\mathbf{y}}^\top \mathcal{A}\tilde{\mathbf{y}}^{m-1} = F(\tilde{\mathbf{y}}) = \min_{\mathbf{x}\in S} F(\mathbf{x}) = \min_{\mathbf{x}\in S} \mathbf{x}^\top \mathcal{A}\mathbf{x}^{m-1} = \min_{\mathbf{x}\in S} \mathcal{A}\mathbf{x}^m. \quad (6)$$

Since $\tilde{\mathbf{y}} \geq 0$ with $\tilde{\mathbf{y}} \neq 0$, we may assume that

$$\tilde{\mathbf{y}} = (\tilde{y}_1, \tilde{y}_2, \cdots, \tilde{y}_l, 0, \cdots, 0)^T \ (\tilde{y}_i > 0 \text{ for } i = 1, \cdots, l, 1 \leq l \leq n).$$

Let $\tilde{\mathbf{w}} = (\tilde{y}_1, \tilde{y}_2, \cdots, \tilde{y}_l)^T$ and let $\mathcal{B}$ be a principal sub-tensor that obtained from $\mathcal{A}$ by the polynomial $\mathcal{A}x^m$ for $\mathbf{x} = (x_1, x_2, \cdots, x_l, 0, \cdots, 0)^T$. Then

$$\tilde{\mathbf{w}} \in \mathbb{R}^l_{++}, \ \sum_{i=1}^l \tilde{y}_i = 1 \text{ and } F(\tilde{\mathbf{y}}) = \mathcal{A}\tilde{\mathbf{y}}^m = \mathcal{B}\tilde{\mathbf{w}}^m = \min_{\mathbf{x}\in S} \mathcal{A}\mathbf{x}^m. \quad (7)$$

Let $\mathbf{x} = (z_1, z_2, \cdots, z_l, 0, \cdots, 0)^T \in \mathbb{R}^n_+$ for all $\mathbf{z} = (z_1, z_2, \cdots, z_l)^T \in \mathbb{R}^l_+$ with $\sum\limits_{i=1}^l z_i = 1$. Clearly, $\mathbf{x} \in S$, and hence, by (7), we have

$$F(x) = \mathcal{A}\mathbf{x}^m = \mathcal{B}\mathbf{z}^m \geq F(\tilde{\mathbf{y}}) = \mathcal{A}\tilde{\mathbf{y}}^m = \mathcal{B}\tilde{\mathbf{w}}^m.$$

Since $\tilde{\mathbf{w}} \in \mathbb{R}^l_{++}$, $\tilde{\mathbf{w}}$ is a local minimizer of the following optimization problem

$$\min_{\mathbf{z}\in\mathbb{R}^l} \mathcal{B}\mathbf{z}^m \quad s.t. \quad \sum_{i=1}^l z_i = 1.$$

So, the standard KKT conditions implies that there exists $\mu \in \mathbb{R}$ such that

$$\nabla(\mathcal{B}\mathbf{z}^m - \mu(\sum_{i=1}^{l} z_i - 1))|_{\mathbf{z}=\tilde{\mathbf{w}}} = m\mathcal{B}\tilde{\mathbf{w}}^{m-1} - \mu\mathbf{e} = 0,$$

where $\mathbf{e} = (1, 1, \cdots, 1)^\top$, and hence

$$\mathcal{B}\tilde{\mathbf{w}}^{m-1} = \frac{\mu}{m}\mathbf{e}.$$

Let $\lambda = \frac{\mu}{m}$. Then

$$\mathcal{B}\tilde{\mathbf{w}}^{m-1} = (\lambda, \lambda, \cdots, \lambda)^\top \in \mathbb{R}^l,$$

and so

$$\mathcal{B}\tilde{\mathbf{w}}^m = \tilde{\mathbf{w}}^\top \mathcal{B}\tilde{\mathbf{w}}^{m-1} = \lambda \sum_{i=1}^{l} \tilde{y}_i = \lambda.$$

It follows from (7) that

$$\mathcal{A}\tilde{\mathbf{y}}^m = \tilde{\mathbf{y}}^\top \mathcal{A}\tilde{\mathbf{y}}^{m-1} = \mathcal{B}\tilde{\mathbf{w}}^m = \min_{\mathbf{x}\in S} \mathcal{A}\mathbf{x}^m = \lambda.$$

Thus, for all $\tilde{y}_k > 0$, we have

$$\left(\mathcal{A}\tilde{\mathbf{y}}^{m-1}\right)_k = \left(\mathcal{B}\tilde{\mathbf{w}}^{m-1}\right)_k = \lambda.$$

Since $\mathcal{A}$ is semi-positive, for $\tilde{\mathbf{y}} \geq \mathbf{0}$ and $\tilde{\mathbf{y}} \neq \mathbf{0}$, there exists an index $k \in I_n$ such that

$$\tilde{\mathbf{y}}_k > 0 \text{ and } \left(\mathcal{A}\tilde{\mathbf{y}}^{m-1}\right)_k \geq 0.$$

and hence, $\lambda \geq 0$. Consequently, we have

$$\min_{\mathbf{x}\in S} \mathcal{A}\mathbf{x}^m = \mathcal{A}\tilde{\mathbf{y}}^m = \lambda \geq 0.$$

It follows from Lemma 2.2 that $\mathcal{A}$ is copositive. The theorem is proved. $\square$

Using the same proof as that of Theorem 3.3 with appropriate changes in the inequalities, we can obtain the following conclusions about the strictly copositive tensor.

Theorem 3.4 *Let $\mathcal{A} = (a_{i_1\cdots i_m}) \in S_{m,n}$. Then $\mathcal{A}$ is strictly semi-positive if and only if it is strictly copositive.*

By Lemma 2.1 and Theorem 3.4, the following conclusion is obvious.

Corollary 3.1 *Let $\mathcal{A} = (a_{i_1\cdots i_m}) \in S_{m,n}$ be strictly copositive. Then the tensor complementarity problem TCP $(\mathbf{q}, \mathcal{A})$,*

$$\textit{finding } \mathbf{x} \in \mathbb{R}^n \textit{ such that } \mathbf{x} \geq \mathbf{0}, \mathbf{q} + \mathcal{A}\mathbf{x}^{m-1} \geq \mathbf{0}, \textit{ and } \mathbf{x}^\top(\mathbf{q} + \mathcal{A}\mathbf{x}^{m-1}) = 0$$

has a solution for all $\mathbf{q} \in \mathbb{R}^n$.

4 Perspectives

There are more research topics on the tensor complementarity problem for further research.

It is known that there are many efficient algorithms for computing a solution of (non-) linear complementarity problem. Then, whether or not may these algorithms be applied to tensor complementarity problem? If not, can one construct an efficient algorithm to compute the solution of the tensor complementarity problem with a special structured tensor?

A real m-order n-dimensional tensor is said to be **completely Q-tensor** iff it and all its principal sub-tensors are Q-tensors. Clearly, each strictly semi-positive tensor must be a completely Q-tensor. Naturally, we would like to ask whether each completely Q-tensor is strictly semi-positive or not.

5 Conclusions

In this paper, we discuss some relationships between the unique solution of the tensor complementarity problem and (strictly) semi-positive tensors. Furthermore, we establish the equivalence between (strictly) symmetric semi-positive tensors and (strictly) copositive tensors.

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