

TENSOR FIELDS AND CONNECTIONS IN CROSS-SECTIONS IN THE TANGENT BUNDLE OF ORDER 2

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The prolongations of tensor fields and connections given in a differentiable manifold M to its tangent bundle $T(M)$ have been studied in [1], [2], [5], [7]. If a vector field V is given in M , V determines a cross-section in $T(M)$ which is as an n -dimensional submanifold in $T(M)$. Yano [3] has recently studied the behavior of the prolongations of tensor fields and connections to $T(M)$ on the cross-sections determined by a vector field in M . On the other hand, the prolongations of tensor fields and connections in M to its tangent bundle $T_2(M)$ of order 2 are studied in [6]. If a vector field V is given in M , V determines a cross-section in $T_2(M)$. The main purpose of the present paper is to study the behavior of the prolongations of tensor fields and connections in M to $T_2(M)$ on the cross-section determined by a vector field in M .

In §1 we first recall properties of the prolongations of tensor fields and connections in M to $T_2(M)$. In §2 we study the cross-sections determined in $T_2(M)$ by vector fields given in M . §3 will be devoted to the study of the prolongations of tensor fields given in M to $T_2(M)$ along the cross-sections and §4 will be devoted to the study of the prolongations of connections given in M to $T_2(M)$ along the cross-sections.

§1. Prolongations of tensor fields and linear connections to the tangent bundle of order 2.

We shall recall, for the later use, some properties of the tangent bundle $T_2(M)$ of order 2 over a differentiable manifold M of dimension n , and those of prolongations of tensor fields and linear connections in M to $T_2(M)$ (cf. [6]).

The tangent bundle $T_2(M)$ of order 2 is the space of equivalence classes of mappings from the real line R into M , the equivalence relation being defined as follows: we say that two mappings F and G are equivalent to each other if, in a coordinate neighborhood U , they satisfy the conditions

$$F(0)=G(0)=p, \quad \frac{dF^h}{dt}(0)=\frac{dG^h}{dt}(0), \quad \frac{d^2F^h}{dt^2}(0)=\frac{d^2G^h}{dt^2}(0),$$

where $F^h(t)$ and $G^h(t)$ are the coordinates of $F(t)$ and $G(t)$ in U respectively. This

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definition of the equivalence does not depend on the choice of the local coordinates. We call this equivalence class containing F a 2-jet and denote it by $j_p^2(F)$. Namely the tangent bundle of order 2 over M is the space of all 2-jets of M and its bundle projection $\pi_2: T_2(M) \rightarrow M$ is defined by

$$\pi_2(j_p^2(F)) = p.$$

Let (U, x^h) be a coordinate neighborhood with the local coordinate system (x^h) . A system of local coordinates (x^h, y^h, z^h) can be introduced in $\pi_2^{-1}(U)$ in such a way that a 2-jet $j_p^2(F)$ ($p \in U$) has coordinates as

$$x^h = F^h(0), \quad y^h = \frac{dF^h}{dt}(0), \quad z^h = \frac{d^2F^h}{dt^2}(0).$$

We call the local coordinate system (x^h, y^h, z^h) thus introduced in $\pi_2^{-1}(U)$ the *induced coordinate system* and sometimes denote them by $(\xi^A)^{1)}$ i.e.,

$$(1.1) \quad \xi^i = x^i, \quad \xi^{n+i} = y^i, \quad \xi^{2n+i} = z^i.$$

Let (U, x^h) and $(U', x^{h'})$ be two coordinates neighborhoods of M related by coordinate transformation

$$x^{h'} = x^{h'}(x^h)$$

in $U \cap U'$. If we denote by (x^h, y^h, z^h) and $(x^{h'}, y^{h'}, z^{h'})$ the induced coordinates in $\pi_2^{-1}(U)$ and $\pi_2^{-1}(U')$ respectively, the coordinate transformation in $\pi_2^{-1}(U) \cap \pi_2^{-1}(U')$ is given by

$$\begin{aligned} x^{h'} &= x^{h'}(x^h), & y^{h'} &= \frac{\partial x^{h'}}{\partial x^h} y^h, \\ z^{h'} &= \frac{\partial x^{h'}}{\partial x^h} z^h + \frac{\partial^2 x^{h'}}{\partial x^j \partial x^i} y^j y^i \end{aligned}$$

and its Jacobian matrix by

$$(1.2) \quad \begin{pmatrix} \frac{\partial x^{h'}}{\partial x^h}, & 0, & 0 \\ \frac{\partial^2 x^{h'}}{\partial x^h \partial x^s} y^s, & \frac{\partial x^{h'}}{\partial x^h}, & 0 \\ \frac{\partial^2 x^{h'}}{\partial x^h \partial x^s} z^s + \frac{\partial^3 x^{h'}}{\partial x^h \partial x^t \partial x^s} y^t y^s, & 2 \frac{\partial^2 x^{h'}}{\partial x^h \partial x^s} y^s, & \frac{\partial x^{h'}}{\partial x^h} \end{pmatrix}.$$

1) The indices A, B, C, D, \dots and i, j, k, \dots run over the ranges $1, \dots, 3n$ and $1, \dots, n$, respectively.

We denote by $\mathcal{T}_s^r(M)$ the space of all tensor fields of type (r, s) in M . Especially, $\mathcal{T}_s^0(M)$, $\mathcal{T}_0^1(M)$ and $\mathcal{T}_0^0(M)$ are respectively the spaces of all functions, of all vector fields and of all 1-forms all defined in M . We denote also by $\mathcal{T}_s^r(T_2(M))$ the space of all tensor fields of type (r, s) in $T_2(M)$.

Prolongations of tensor fields. For any element f of $\mathcal{T}_0^0(M)$, its prolongations f^0 , f^I and f^{II} to $T_2(M)$ are elements of $\mathcal{T}_0^0(T_2(M))$ and have respectively local expressions of the form

$$(1.3) \quad f^0: f(x^h), \quad f^I: y^i \partial_i f(x^h), \quad f^{II}: z^i \partial_i f(x^h) + y^j y^i \partial_j \partial_i f(x^h)$$

in the induced coordinate system (ξ^A) , $f(x^h)$ being the local expression of f in (x^h) , where $\partial_i = \partial/\partial x^i$.

For any element X of $\mathcal{T}_0^1(M)$, its prolongations X^0 , X^I and X^{II} are elements of $\mathcal{T}_0^1(T_2(M))$ and have the following properties:

$$(1.4) \quad \begin{aligned} X^0 f^0 &= 0, & X^0 f^I &= 0, & X^0 f^{II} &= (Xf)^0, \\ X^I f^0 &= 0, & X^I f^I &= \frac{1}{2}(Xf)^0, & X^I f^{II} &= (Xf)^I, \\ X^{II} f^0 &= (Xf)^0, & X^{II} f^I &= (Xf)^I, & X^{II} f^{II} &= (Xf)^{II}, \end{aligned}$$

f being an arbitrary element of $\mathcal{T}_0^0(M)$.

For any element ω of $\mathcal{T}_0^1(M)$, its prolongations ω^0 , ω^I and ω^{II} are elements of $\mathcal{T}_0^1(T_2(M))$ and have the following properties:

$$(1.5) \quad \begin{aligned} \omega^0(X^0) &= 0, & \omega^0(X^I) &= 0, & \omega^0(X^{II}) &= (\omega(X))^0, \\ \omega^I(X^0) &= 0, & \omega^I(X^I) &= \frac{1}{2}(\omega(X))^0, & \omega^I(X^{II}) &= (\omega(X))^I, \\ \omega^{II}(X^0) &= (\omega(X))^0, & \omega^{II}(X^I) &= (\omega(X))^I, & \omega^{II}(X^{II}) &= (\omega(X))^{II}, \end{aligned}$$

X being an arbitrary element of $\mathcal{T}_0^1(M)$.

Taking arbitrarily two tensor fields P and Q in M , we have the following formulas:

$$(1.6) \quad \begin{aligned} (P \otimes Q)^0 &= P^0 \otimes Q^0, \\ (P \otimes Q)^I &= P^I \otimes Q^0 + P^0 \otimes Q^I, \\ (P \otimes Q)^{II} &= P^{II} \otimes Q^0 + 2P^I \otimes Q^I + P^0 \otimes Q^{II}. \end{aligned}$$

The prolongations P^0 , P^I and P^{II} are called respectively the 0-th, the 1-st and the 2-nd lifts of P , P being an arbitrary tensor field in M .

REMARK. Let \tilde{X} and \tilde{Y} be two vector fields in $T_2(M)$. If we have $\tilde{X}f^{II} = \tilde{Y}f^{II}$ for any element f of $\mathcal{T}_0^0(M)$, then we have $\tilde{X} = \tilde{Y}$. Generally speaking, any tensor field in $T_2(M)$ is completely determined by giving its values for the 2-nd lifts of vector fields arbitrarily given in M .

Let F be an element of $\mathcal{T}_1^1(M)$ and $P(t)$ a polynomial of t . Then we have

$$(1.7) \quad (P(F))^{II} = P(F^{II}).$$

We now note that the 2-nd lift of the identity tensor field I of type $(1,1)$ is also the identity tensor field in $T_2(M)$, which is also denoted by I in $T_2(M)$, that is to say, $I^{II} = I$. For example, if $F^2 + I = 0$, we have $(F^{II})^2 + I = 0$. Thus, we obtain

PROPOSITION. *If F is an almost complex structure in M , so is F^{II} in $T_2(M)$.*

We denote by N_F the Nijenhuis tensor of an element F of $\mathcal{T}_1^1(M)$. We have then

$$(1.8) \quad (N_F)^{II} = N_{F^{II}}$$

for $F \in \mathcal{T}_1^1(M)$.

Prolongations of linear connections. Let there be given a linear connection \mathcal{V} in M . Then there exists a unique linear connection \mathcal{V}^{II} in $T_2(M)$ characterized by the equation

$$(1.9) \quad \mathcal{V}^{II}_{Y^{II}} X^{II} = (\mathcal{V}_Y X)^{II},$$

X and Y being arbitrary elements of $\mathcal{T}_1^1(M)$. The connection \mathcal{V}^{II} is called the lift of the given connection \mathcal{V} . If we denote by T and R respectively the torsion and the curvature tensors of \mathcal{V} , we have

$$(1.10) \quad \tilde{T} = T^{II}, \quad \tilde{R} = R^{II},$$

where \tilde{T} and \tilde{R} are the torsion and the curvature tensors of \mathcal{V}^{II} respectively.

We have the following formulas:

$$(1.11) \quad \mathcal{V}^{II}_{Y^{II}} X^0 = (\mathcal{V}_Y X)^0, \quad \mathcal{V}^{II}_{Y^{II}} X^I = (\mathcal{V}_Y X)^I$$

for $X, Y \in \mathcal{T}_1^1(M)$.

Let there be given a pseudo-Riemannian metric g in M . Then g^{II} is a pseudo-Riemannian metric in $T_2(M)$. If we denote by \mathcal{V} the Riemannian connection

determined by g , then its lift ∇^{Π} is the Riemannian connection determined by g^{Π} in $T_2(M)$.

§ 2. Cross-sections determined by vector fields.

Let there be given a vector field V in M . Denote by $\varphi_P: I \rightarrow M$ the orbit of V passing through a point P of M in such a way that $\varphi_P(0) = P$, where I is an interval $(-\varepsilon, \varepsilon)$, ε being a certain positive number. If we denote by $\gamma_V(P)$ the 2-jet $j_P^2(\varphi_P)$, we set that the correspondence $P \rightarrow \gamma_V(P)$ defines a mapping $\gamma_V: M \rightarrow T_2(M)$ such that $\pi_2 \circ \gamma_V$ is the identity mapping, i.e., that $\gamma_V: M \rightarrow T_2(M)$ is a cross-section in $T_2(M)$. The submanifold $\gamma_V(M)$ imbedded in $T_2(M)$ is called the *cross-section* determined by the vector field V . If U is a coordinate neighborhood in M the cross-section $\gamma_V(M)$ is expressed locally in $\pi_2^{-1}(U)$ by equations

$$(2.1) \quad x^h = x^h, \quad y^h = V^h(x^i), \quad z^h = V^k(x^i) \partial_k V^h(x^i)$$

with respect to the induced coordinate system (ξ^A) , where $V = V^h(x^i) \partial_h$ is the local expression of V in U . We denote the equations (2.1) by

$$(2.2) \quad \xi^A = \xi^A(x^i),$$

i.e., $\xi^h = x^h$, $\xi^{n+h} = V^h$, $\xi^{2n+h} = V^h \partial_k V^h$.

Taking account of (1.3) and (2.1), we have along $\gamma_V(M)$ the equations

$$(2.3) \quad f^{\Pi} = (\mathcal{L}_V^2 f)^0, \quad f^I = (\mathcal{L}_V f)^0, \quad f^0 = f^0$$

for $f \in \mathcal{Q}^0(M)$, where \mathcal{L}_V denotes the Lie derivation with respect to V and $\mathcal{L}_V^2 = \mathcal{L}_V \mathcal{L}_V$.

If we put $B_i^A = \partial_i \xi^A$, we get along $\gamma_V(M)$ n local vector fields B_i tangent to the cross-section which have the components of the form

$$(2.4) \quad (B_i^A) = \begin{pmatrix} \partial_i^h \\ \partial_i V^h \\ (\partial_i V^k)(\partial_k V^h) + V^k \partial_i \partial_k V^h \end{pmatrix}$$

with respect to the induced coordinate system (ξ^A) . For an element X of $\mathcal{Q}^1(M)$ with local expression $X = X^i \partial / \partial x^i$, we denote by BX the vector field with components $B_i^A X^i$, which is defined globally along $\gamma_V(M)$ by virtue of (1.2). The mapping $B_p: T_p(M) \rightarrow T_\sigma(T_2(M))$ ($\sigma = \gamma_V(p)$) defined by the correspondence $X_p \rightarrow (BX)_\sigma$, is the differential mapping γ_V' of the cross-section mapping $\gamma_V: M \rightarrow T_2(M)$. Thus $B_p: T_p(M) \rightarrow T_\sigma(T_2(M))$ is an isomorphism and $B_p(T_p(M))$ is the tangent space of the cross-section $\gamma_V(M)$ at the point $\sigma = \gamma_V(p)$.

We consider along the cross-section $\gamma_V(M)$ n local vector fields C_i and n local

vector fields $D_{\bar{i}}$ along $\gamma_V(M)$, which have respectively components of the form

$$(2.5) \quad (C_{\bar{i}}^A) = \begin{pmatrix} 0 \\ \frac{1}{2} \delta_i^h \\ \partial_i V^h \end{pmatrix}, \quad (D_{\bar{i}}^A) = \begin{pmatrix} 0 \\ 0 \\ \delta_i^h \end{pmatrix}$$

in the induced coordinate system (ξ^A) . For an element X of $\mathcal{F}_0^1(M)$ with local expression $X = X^i \partial_i$, we denote by CX and DX the vector fields with components $C_{\bar{i}}^A X^i$ and $D_{\bar{i}}^A X^i$ respectively. Then according to (1.2), CX and DX are defined along $\gamma_V(M)$. We now defined two mappings $C_p: T_p(M) \rightarrow T_o(T_2(M))$ and $D_p: T_p(M) \rightarrow T_o(T_2(M))$ ($\sigma = \gamma_V(p)$) respectively by

$$(2.6) \quad C_p X_p = (CX)_\sigma, \quad D_p X_p = (DX)_\sigma$$

X being an arbitrary element of $\mathcal{F}_0^1(M)$. It is easily verified by virtue of (2.5) that the two mappings C_p and D_p defined by (2.6) are isomorphisms of $T_p(M)$ into $T_o(T_2(M))$ ($\sigma = \gamma_V(p)$).

Putting

$$N_\sigma^{(1)} = C_p T_p(M), \quad N_\sigma^{(2)} = D_p T_p(M) \quad (\sigma = \gamma_V(p)),$$

we have the following direct sum representation of $T_o(T_2(M))$:

$$T_o(T_2(M)) = T_o(\gamma_V(M)) + N_\sigma^{(1)} + N_\sigma^{(2)}.$$

The $3n$ local vector fields $B_i, C_{\bar{i}}$ and $D_{\bar{i}}$ along $\gamma_V(M)$ are expressed respectively by

$$(2.7) \quad B_i = B \partial_i, \quad C_{\bar{i}} = C \partial_i, \quad D_{\bar{i}} = D \partial_i$$

and form a local family of frames $\{B_i, C_{\bar{i}}, D_{\bar{i}}\}$ along $\gamma_V(M)$, which are called the *adapted frames* of $\gamma_V(M)$. The n local vector fields B_i span $T_o(\gamma_V(M))$, $C_{\bar{i}}$ span $N_\sigma^{(1)}$ and $D_{\bar{i}}$ span $N_\sigma^{(2)}$, all at $\sigma \in \gamma_V(M)$.

Taking account of (2.4), (2.5) and (2.7), we have along $\gamma_V(M)$

$$(2.9) \quad \begin{aligned} X^{II} &= BX + 2C(\mathcal{L}_V X) + D(\mathcal{L}_V^2 X), \\ X^I &= \quad CX \quad + D(\mathcal{L}_V X), \\ X^0 &= \quad \quad \quad DX, \end{aligned}$$

or equivalently

$$\begin{aligned}
 X^{\text{II}} &= (X^i)^0 B_i + 2(\mathcal{L}_V X^i)^0 C_i + (\mathcal{L}_V^2 X^i)^0 D_i, \\
 (2.10) \quad X^{\text{I}} &= (X^i)^0 C_i + (\mathcal{L}_V X^i)^0 D_i, \\
 X^0 &= (X^i)^0 D_i
 \end{aligned}$$

for any element X of $\mathcal{F}_1^0(M)$ with local expression $X = X^i \partial_i$.

§ 3. Prolongations of tensor fields in the cross-sections.

Let there be given a vector field \tilde{X} along $\gamma_V(M)$. Putting

$$\tilde{X} = \tilde{X}^i B_i + \tilde{X}^i C_i + \tilde{X}^i D_i$$

we call $(\tilde{X}^\alpha) = (\tilde{X}^i, \tilde{X}^i, \tilde{X}^i)$ ³⁾ the components of \tilde{X} in the adapted frame. Similarly, for any tensor field \tilde{T} of type (1, 2) along $\gamma_V(M)$, we denote by

$$(\tilde{T}^\alpha_{\beta\gamma}) = (\tilde{T}^{ji^h}, \tilde{T}^{ji^h}, \tilde{T}^{ji^h}, \dots, \tilde{T}^{ji^h})$$

its components in the adapted frame. Thus by means of (2.10), the lifts X^0 , X^{I} and X^{II} have along $\gamma_V(M)$ components of the form

$$(3.1) \quad (X^{0\alpha}) = \begin{pmatrix} 0 \\ 0 \\ X^h \end{pmatrix}, \quad (X^{\text{I}\alpha}) = \begin{pmatrix} 0 \\ X^h \\ \mathcal{L}_V X^h \end{pmatrix}, \quad (X^{\text{II}\alpha}) = \begin{pmatrix} X^h \\ 2\mathcal{L}_V X^h \\ \mathcal{L}_V^2 X^h \end{pmatrix}$$

in the adapted frame, where X is a vector field in M with local expression $X = X^i \partial_i$. In (3.1) we have identified the 0-th lift $(X^h)^0$, $(\mathcal{L}_V X^h)^0$ and $(\mathcal{L}_V^2 X^h)^0$ respectively with functions X^h , $\mathcal{L}_V X^h$ and $\mathcal{L}_V^2 X^h$. In the sequel we sometimes use such identification.

Let there be given an element ω of $\mathcal{F}_1^0(M)$ with local expressions $\omega = \omega_i dx^i$. Then its lifts ω^0 , ω^{I} and ω^{II} have respectively components of the form

$$\begin{aligned}
 (\omega^0_\beta) &= (\omega_i, 0, 0), \\
 (3.2) \quad (\omega^{\text{I}}_\beta) &= \left(\mathcal{L}_V \omega_i, \frac{1}{2} \omega_i, 0 \right), \\
 (\omega^{\text{II}}_\beta) &= (\mathcal{L}_V^2 \omega_i, \mathcal{L}_V \omega_i, \omega_i)
 \end{aligned}$$

in the adapted frame. In fact by virtue of (2.3), (3.1) and (1.5), we have along $\gamma_V(M)$, for example,

3) We use Greek indices α, β, \dots to represent the components in the adapted frame.

$$\begin{aligned} & \omega^{ii}X^i + 2\omega^{ii}(\mathcal{L}_V X^i) + \omega^{ii}(\mathcal{L}_V^2 X^i) \\ &= \mathcal{L}_V^2(\omega_i X^i) \\ &= (\mathcal{L}_V^2 \omega_i)X^i + 2(\mathcal{L}_V \omega_i)(\mathcal{L}_V X^i) + \omega_i(\mathcal{L}_V^2 X^i) \end{aligned}$$

for arbitrary element X of $\mathcal{T}_0^1(M)$ with local expression $X=X^i\partial_i$, and there exists an element X of $\mathcal{T}_0^1(M)$ such that at a given point, for any given values (a^h, b^h, c^h) ,

$$X^h = a^h, \quad \mathcal{L}_V X^h = b^h \quad \text{and} \quad \mathcal{L}_V^2 X^h = c^h$$

hold. The other relations stated in (3.2) are obtained similarly.

Taking account of (1.6), (3.1) and (3.2) we find components of 0-th, 1-st and 2-nd lifts of any tensor field in M with respect to the adapted frame. For example, for an element h of $\mathcal{T}_0^2(M)$ we have

$$(h^0_{\beta\alpha}) = \begin{pmatrix} h_{ji} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad (h^1_{\beta\alpha}) = \begin{pmatrix} \mathcal{L}_V h_{ji} & \frac{1}{2} h_{ji} & 0 \\ \frac{1}{2} h_{ji} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

(3.3)

$$(h^{II}_{\beta\alpha}) = \begin{pmatrix} \mathcal{L}_V^2 h_{ji} & \mathcal{L}_V h_{ji} & h_{ji} \\ \mathcal{L}_V h_{ji} & \frac{1}{2} h_{ji} & 0 \\ h_{ji} & 0 & 0 \end{pmatrix},$$

h_{ji} being the components of h . For an element F of $\mathcal{T}_0^1(M)$,

$$(F^0_{\beta^\alpha}) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ F_i^h & 0 & 0 \end{pmatrix}, \quad (F^I_{\beta^\alpha}) = \begin{pmatrix} 0 & 0 & 0 \\ F_i^h & 0 & 0 \\ \mathcal{L}_V F_i^h & \frac{1}{2} F_i^h & 0 \end{pmatrix},$$

(3.4)

$$(F^{II}_{\beta^\alpha}) = \begin{pmatrix} F_i^h & 0 & 0 \\ 2\mathcal{L}_V F_i^h & F_i^h & 0 \\ \mathcal{L}_V^2 F_i^h & \mathcal{L}_V F_i^h & F_i^h \end{pmatrix},$$

F_i^h being the components of F . For an element S of $\mathcal{T}_2^1(M)$,

$$(3.5) \quad \begin{aligned} S^0_{ji^h} &= 0, & S^0_{ji^{\bar{h}}} &= 0, & S^0_{ji^{\bar{h}}} &= S_{ji^h}, \\ S^I_{ji^h} &= 0, & S^I_{ji^{\bar{h}}} &= S_{ji^h}, & S^I_{ji^{\bar{h}}} &= \mathcal{L}_V S_{ji^h}, \\ S^{II}_{ji^h} &= S_{ji^h}, & S^{II}_{ji^{\bar{h}}} &= 2\mathcal{L}_V S_{ji^h}, & S^{II}_{ji^{\bar{h}}} &= \mathcal{L}_V^2 S_{ji^h}, \end{aligned}$$

S_{ji^h} being the components of S .

The linear isomorphism B defined in § 2 is the differential mapping γ_V' of the cross-section mapping $\gamma_V: M \rightarrow \gamma_V(M)$. Then we denote sometimes by $\gamma_V'X$ the vector field BX , X being an arbitrary element of $\mathcal{T}_0^1(M)$. Given an element ω of $\mathcal{T}_0^1(M)$, we denote by $\gamma_V'\omega$ the image of ω by the dual mapping of B^{-1} (=the restriction of π_2 to $\gamma_V(M)$). The mapping γ_V' is extended as a linear mapping $\gamma_V': \mathcal{T}(M) \rightarrow \mathcal{T}(\gamma_V(M))$ by

$$\gamma_V'(P \otimes Q) = (\gamma_V'P) \otimes (\gamma_V'Q),$$

P and Q being arbitrary tensor fields in M .

Now we will define the operation $\#$ in $\mathcal{T}(T_2(M))$ as follows. For an element \tilde{X} of $\mathcal{T}_0^1(T_2(M))$, we put

$$\tilde{X}^\# = \tilde{X}^i B_i.$$

Let $\tilde{\omega}$ be a tensor field of type (0, 1) defined along $\gamma_V(M)$. Then putting along $\gamma_V(M)$

$$\tilde{\omega}^\#(BX) = \tilde{\omega}(BX)$$

for $X \in \mathcal{T}_0^1(M)$, we can define an element $\tilde{\omega}^\#$ of $\mathcal{T}_0^1(\gamma_V(M))$ which is called the 1-form induced in $\gamma_V(M)$ from $\tilde{\omega}$. Let \tilde{h} be a tensor field of type (0, 2) defined along $\gamma_V(M)$. Then putting along $\gamma_V(M)$

$$\tilde{h}^\#(BX, BY) = \tilde{h}(BX, BY)$$

for $X, Y \in \mathcal{T}_0^1(M)$, we can define an element $\tilde{h}^\#$ of $\mathcal{T}_0^2(\gamma_V(M))$ which is called the tensor field induced in $\gamma_V(M)$ from \tilde{h} . Let \tilde{F} be a tensor field of type (1, 1) defined along $\gamma_V(M)$ such that, for any vector field \tilde{A} tangent to $\gamma_V(M)$, $\tilde{F}\tilde{A}$ is also tangent to $\gamma_V(M)$. Then putting

$$\tilde{F}^\#(BX) = \tilde{F}(BX)$$

for $X \in \mathcal{T}_0^1(M)$, we can define an element $\tilde{F}^\#$ of $\mathcal{T}_0^1(\gamma_V(M))$ which is called the tensor field induced in $\gamma_V(M)$ from \tilde{F} . Let \tilde{S} be a tensor field of type (1, 2) defined along $\gamma_V(M)$ such that for any vector field \tilde{A}, \tilde{B} tangent to $\gamma_V(M)$, $\tilde{S}(\tilde{A}, B)$ is tangent to

$\gamma_V(M)$. Then putting

$$\tilde{S}^*(BX, BY) = \tilde{S}(BX, BY)$$

for $X, Y \in \mathcal{T}_0^1(M)$, we can define an element \tilde{S}^* of $\mathcal{T}_2^1(\gamma_V(M))$, which is called the tensor field induced in $\gamma_V(M)$ from \tilde{S} .

We have from (3. 1),

PROPOSITION 3. 1. *Let X be an element of $\mathcal{T}_0^1(M)$. Then X^{11} is tangent to $\gamma_V(M)$ if and only if $\mathcal{L}_V X = 0$. In this case $X^{11*} = \gamma_V' X$ holds. For any element X of $\mathcal{T}_0^1(M)$, $X^{0*} = 0$ and $X^{1*} = 0$ hold.*

We have from (3. 2),

PROPOSITION 3. 2. *For any element ω of $\mathcal{T}^1(M)$,*

$$\omega^{11*} = \gamma_V'(\mathcal{L}_V \omega), \quad \omega^{1*} = \gamma_V'(\mathcal{L}_V \omega) \quad \text{and} \quad \omega^{0*} = \gamma_V' \omega$$

hold.

We have from (3. 3)

PROPOSITION 3. 3. *For any element h of $\mathcal{T}_2^0(M)$,*

$$h^{11*} = \gamma_V'(\mathcal{L}_V h), \quad h^{1*} = \gamma_V'(\mathcal{L}_V h) \quad \text{and} \quad h^{0*} = \gamma_V' h$$

hold, and hence $h^{0*}(BX, BY) = h(X, Y)^0$.

PROPOSITION 3. 4. *Let g be a Riemannian metric in M . Then g^{0*} is a Riemannian metric in $\gamma_V(M)$ and γ_V is isometry, i.e. $g^{0*} = \gamma_V' g$.*

Suppose that the vector field V in M satisfies the condition $\mathcal{L}_V g = cg$, g being a Riemannian metric in M and c a constant, that is, V is an infinitesimal homothetic transformation with respect to g . Then we have from Proposition 3. 3 the relation $g^{11*} = cg^{1*} = c^2 g^{0*}$.

If for each point σ of $\gamma_V(M)$ the tangent space $T_\sigma(\gamma_V(M))$ is invariant by the action of a tensor field \tilde{F} defined along $\gamma_V(M)$, then the cross-section $\gamma_V(M)$ is said to be invariant by \tilde{F} . For any $F \in \mathcal{T}^1(M)$, we have from (3. 4)

$$F^0(BX) = DF X, \quad F^1(BX) = C(FX) + D((\mathcal{L}_V F)X),$$

$$F^{11}(BX) = B(FX) + 2C((\mathcal{L}_V F)X) + D((\mathcal{L}_V^2 F)X)$$

for $X \in \mathcal{T}_0^1(M)$. Thus we have

PROPOSITION 3. 5. *Let F be an element of $\mathcal{F}_1^1(M)$. The cross-section $\gamma_V(M)$ is invariant by F^{11} if and only if $\mathcal{L}_V F=0$. In this case, $F^{11*}=\gamma_V' F$ holds. The lifts F^0 and F^1 do not leave $\gamma_V(M)$ invariant, unless $F=0$.*

PROPOSITION 3. 6. *If F is an almost complex structure such that $\mathcal{L}_V F=0$, then F^{11*} is an almost complex structure in $\gamma_V(M)$ and $F^{11*}=\gamma_V' F$ holds.*

If a Riemannian metric g in M satisfies the condition

$$g(FX, FY)=g(X, Y) \quad \text{for any } X, Y \in \mathcal{F}_0^1(M),$$

then (g, F) is called an almost Hermitian structure in M . If $\mathcal{L}_V F=0$ holds, then we get along $\gamma_V(M)$

$$\begin{aligned} g^{0*}(F^{11*}BX, F^{11*}BY) &= (\gamma_V' g)((\gamma_V' F)X, (\gamma_V' F)Y) \\ &= (g(FX, FY))^0 \end{aligned}$$

because of Proposition 3. 3 and 3. 5. Thus we have

PROPOSITION 3. 7. *Suppose that there is given an almost Hermitian structure (g, F) in M . If $\mathcal{L}_V F=0$, then (g^{0*}, F^{11*}) is an almost Hermitian structure in $\gamma_V(M)$.*

For any $S \in \mathcal{F}_2^1(M)$, we have from (3. 5)

$$\begin{aligned} S^0(BX, BY) &= D(S(X, Y)), \\ (3. 7) \quad S^1(BX, BY) &= C(S(X, Y)) + D((\mathcal{L}_V S)(X, Y)), \\ S^1(BX, BY) &= B(S(X, Y)) + 2C((\mathcal{L}_V S)(X, Y)) + D((\mathcal{L}_V^2 S)(X, Y)) \end{aligned}$$

for any $X, Y \in \mathcal{F}_0^1(M)$. Thus we get

PROPOSITION 3. 8. *Let S be an element of $\mathcal{F}_2^1(M)$. The vector fields $S^{11}(BX, BY)$ is tangent to $\gamma_V(M)$ for arbitrary elements X, Y of $\mathcal{F}_0^1(M)$, if and only if $\mathcal{L}_V S=0$, and in this case $S^{11*}=\gamma_V' S$ holds. The vector fields $S^0(BX, BY)$ and $S^1(BX, BY)$ are not tangent to $\gamma_V(M)$, unless $S=0$.*

If an element F of $\mathcal{F}_1^1(M)$ satisfies $\mathcal{L}_V F=0$, then its Nijenhuis tensor satisfies $\mathcal{L}_V N_F=0$. By virtue of (1. 8), Proposition 3. 5 and 3. 8, we have

$$N_{F^{11*}}=N_{F^{11*}}=\gamma_V' N_F$$

in the case that $\mathcal{L}_V F=0$. Thus we have

PROPOSITION 3.9. *Let F be an element of $\mathcal{F}_0^1(M)$ such that $\mathcal{L}_V F=0$. Then the vector field $N_{F^{11}}(BX, BY)$ is tangent to $\gamma_V(M)$ for arbitrary elements X, Y of $\mathcal{F}_0^1(M)$, and $N_{F^{11}} = N_{F^{11}} = \gamma_V' N_F$ hold. Especially $N_{F^{11}}$ vanishes identically in $\gamma_V(M)$ if and only if $N_F=0$.*

Consequently taking account of Proposition in §1 and Proposition 3.5, we get

PROPOSITION 3.10. *If a complex structure F satisfies the condition $\mathcal{L}_V F=0$, then F^{11} is a complex structure in $\gamma_V(M)$.*

§ 4. Prolongations of affine connections in cross-sections.

First of all, we recall some formulas on Lie derivations (cf. [4]). Let there be given an affine connection ∇ with coefficients Γ_{ji}^h . For vector fields X with local expression $X=X^i\partial_i$ and V , we have formulas as

$$(4.1) \quad \mathcal{L}_V(\nabla_j X^h) - \nabla_j(\mathcal{L}_V X^h) = (\mathcal{L}_V \Gamma_{ji}^h) X^i,$$

$$(4.2) \quad \nabla_k(\mathcal{L}_V \Gamma_{ji}^h) - \nabla_j(\mathcal{L}_V \Gamma_{ki}^h) = \mathcal{L}_V R_{kji}^h,$$

where R_{kji}^h denotes the components of the curvature tensor R of ∇ . Hence we have

$$(4.3) \quad \mathcal{L}_V^2(\nabla_j X^h) - \nabla_j(\mathcal{L}_V^2 X^h) = (\mathcal{L}_V^2 \Gamma_{ji}^h) X^i + 2(\mathcal{L}_V \Gamma_{ji}^h)(\mathcal{L}_V X^i),$$

$$(4.4) \quad \nabla_k(\mathcal{L}_V^2 \Gamma_{ji}^h) - \nabla_j(\mathcal{L}_V^2 \Gamma_{ki}^h) + 2(\mathcal{L}_V \Gamma_{ki}^h)(\mathcal{L}_V \Gamma_{ji}^h) - 2(\mathcal{L}_V \Gamma_{ji}^h)(\mathcal{L}_V \Gamma_{ki}^h) = \mathcal{L}_V^2 R_{kji}^h.$$

Taking account of (1.9) and (2.9), we have along $\gamma_V(M)$

$$(4.5) \quad \begin{aligned} \nabla^{11}_{Y^{11}} X^{11} &= B(\nabla_Y X) + 2C(\mathcal{L}_V(\nabla_Y X)) + D(\mathcal{L}_V^2(\nabla_Y X)) \\ &= (\nabla_Y X^h)^0 B_h + 2(\mathcal{L}_V \nabla_Y X^h)^0 C_{\bar{h}} + (\mathcal{L}_V^2 \nabla_Y X^h)^0 D_{\bar{h}} \end{aligned}$$

for $X, Y \in \mathcal{F}_0^1(M)$, where $X=X^i\partial_i$ is the local expression of X . On the other hand, taking account of (1.4) and (2.10), we have along $\gamma_V(M)$

$$\nabla^{11}_{Y^{11}} X^{11} = \nabla^{11}_{Y^{11}}((X^h)^0 B_h + 2(\mathcal{L}_V X^h)^0 C_{\bar{h}} + (\mathcal{L}_V^2 X^h)^0 D_{\bar{h}})$$

i.e.

$$\begin{aligned} \mathcal{V}^{\text{II}}_{Y^{\text{II}}}X^{\text{II}} &= (X^h)^0\mathcal{V}^{\text{II}}_{Y^{\text{II}}}B_h + 2(\mathcal{L}_V X^h)^0\mathcal{V}^{\text{II}}_{Y^{\text{II}}}C_{\bar{h}} + (\mathcal{L}_{V^2} X^h)^0\mathcal{V}^{\text{II}}_{Y^{\text{II}}}D_{\bar{h}} \\ &\quad + (Y^i\partial_i X^h)^0B_h + 2(Y^i\partial_i(\mathcal{L}_V X^h))^0C_{\bar{h}} + (Y^i\partial_i(\mathcal{L}_{V^2} X^h))^0D_{\bar{h}}, \end{aligned}$$

where $Y = Y^i\partial_i$ is the local representation of Y . For an arbitrary point σ of $\gamma_V(M)$, there exists a vector field Y in M with initial conditions $Y = \partial_j$, $\mathcal{L}_V Y = 0$, $\mathcal{L}_{V^2} Y = 0$ at $p = \pi_2(\sigma)$. Then at σ , $Y^{\text{II}} = BY = B\partial_j = B_j$, and the value of $\mathcal{V}^{\text{II}}_{Y^{\text{II}}}X^{\text{II}}$ at σ is $\mathcal{V}^{\text{II}}_{B_j}X^{\text{II}}$. Comparing the two equations (4. 5) and (4. 6), we have at $\sigma \in \gamma_V(M)$,

$$\begin{aligned} &(X^h)^0\mathcal{V}^{\text{II}}_{B_j}B_h + 2(\mathcal{L}_V X^h)^0\mathcal{V}^{\text{II}}_{B_j}C_{\bar{h}} + (\mathcal{L}_{V^2} X^h)^0\mathcal{V}^{\text{II}}_{B_j}D_{\bar{h}} \\ &= (\mathcal{V}_j X^h - \partial_j X^h)^0B_h + 2\{\mathcal{L}_V(\mathcal{V}_j X^h) - \mathcal{V}_j(\mathcal{L}_V X^h)\}^0C_{\bar{h}} \\ &\quad + \{\mathcal{L}_{V^2}(\mathcal{V}_j X^h) - \mathcal{V}_j(\mathcal{L}_{V^2} X^h)\}^0D_{\bar{h}} \\ &= (\Gamma_{j_i}^h X^i)^0B_h + 2\{\mathcal{L}_V(\mathcal{V}_j X^h) - \mathcal{V}_j(\mathcal{L}_V X^h) + \Gamma_{j_i}^h \mathcal{L}_V X^i\}^0C_{\bar{h}} \\ &\quad + \{\mathcal{L}_{V^2}(\mathcal{V}_j X^h) - \mathcal{V}_j(\mathcal{L}_{V^2} X^h) + \Gamma_{j_i}^h \mathcal{L}_{V^2} X^i\}^0D_{\bar{h}} \end{aligned}$$

which implies by virtue of (4. 1) and (4. 3),

$$\begin{aligned} &(X^h)^0\mathcal{V}^{\text{II}}_{B_j}B_h + 2(\mathcal{L}_V X^h)^0\mathcal{V}^{\text{II}}_{B_j}C_{\bar{h}} + (\mathcal{L}_{V^2} X^h)^0\mathcal{V}^{\text{II}}_{B_j}D_{\bar{h}} \\ (4. 6) \quad &= (\Gamma_{j_i}^h X^i)^0B_h + 2\{(\mathcal{L}_V \Gamma_{j_i}^h)X^i + \Gamma_{j_i}^h(\mathcal{L}_V X^i)\}^0C_{\bar{h}} \\ &\quad + \{(\mathcal{L}_{V^2} \Gamma_{j_i}^h)X^i + 2(\mathcal{L}_V \Gamma_{j_i}^h)(\mathcal{L}_V X^i) + \Gamma_{j_i}^h(\mathcal{L}_{V^2} X^i)\}^0D_{\bar{h}}. \end{aligned}$$

Let a^h , $b^{\bar{h}}$ and $c^{\bar{h}}$ be arbitrary real numbers. For any point σ of $\gamma_V(M)$, there exists a vector field X in M with local expression $X = X^i\partial_i$ such that it satisfies $X^h = a^h$, $\mathcal{L}_V X^h = b^{\bar{h}}$, $\mathcal{L}_{V^2} X^h = c^{\bar{h}}$ at $p = \pi_2(\sigma)$. Thus (4. 6) gives

$$\begin{aligned} \mathcal{V}^{\text{II}}_{B_j}B_i &= (\Gamma_{j_i}^h)^0B_h + 2(\mathcal{L}_V \Gamma_{j_i}^h)^0C_{\bar{h}} + (\mathcal{L}_{V^2} \Gamma_{j_i}^h)^0D_{\bar{h}}, \\ (4. 7) \quad \mathcal{V}^{\text{II}}_{B_j}C_i &= (\Gamma_{j_i}^h)^0C_{\bar{h}} + (\mathcal{L}_V \Gamma_{j_i}^h)^0D_{\bar{h}}, \\ \mathcal{V}^{\text{II}}_{B_j}D_i &= (\Gamma_{j_i}^h)^0D_{\bar{h}}. \end{aligned}$$

Putting

$$\begin{aligned} \mathcal{V}^{\text{II}}_{B_j}B_i &= \mathcal{V}^{\text{II}}_{B_j}B_i - (\Gamma_{j_i}^h)^0B_h, \\ (4. 8) \quad \mathcal{V}^{\text{II}}_{B_j}C_i &= \mathcal{V}^{\text{II}}_{B_j}C_i - (\Gamma_{j_i}^h)^0C_{\bar{h}}, \\ \mathcal{V}^{\text{II}}_{B_j}D_i &= \mathcal{V}^{\text{II}}_{B_j}D_i - (\Gamma_{j_i}^h)^0D_{\bar{h}}, \end{aligned}$$

we have

$$(4.9) \quad \nabla^{\text{II}}_j B_i = 2(\mathcal{L}_V \Gamma^h_{ji})^0 C_h + (\mathcal{L}_V^2 \Gamma^h_{ji})^0 D_h,$$

$$(4.10) \quad \nabla^{\text{II}}_j C_i = (\mathcal{L}_V \Gamma^h_{ji})^0 D_h,$$

$$(4.11) \quad \nabla^{\text{II}}_j D_i = 0.$$

These are nothing but the structure equations for the cross-section $\gamma_V(M)$, which imply

PROPOSITION 4.1. *The cross-section $\gamma_V(M)$ is totally geodesic in $T_2(M)$ with respect to the connection ∇^{II} , if and only if the vector field V is infinitesimal affine transformation in M with respect to ∇ (i.e. $\mathcal{L}_V \Gamma^h_{ji} = 0$).*

Taking account of (4.8) we have

$$(4.12) \quad \nabla^{\text{II}}_{B_j} B X = (\nabla_j X^h)^0 B_h + (X^h)^0 \nabla_j^{\text{II}} B_i$$

for $X \in \mathcal{F}^1_0(M)$. For $Y = Y^i \partial_i$, putting $\nabla_{B Y^i} B_h = Y^j \nabla_j^{\text{II}} B_h$, we have

$$\nabla^{\text{II}}_{B Y} B X = B(\nabla_Y X) + (X^h)^0 \nabla_{B Y}^{\text{II}} B_h.$$

We can now defined an affine connection ∇^* in $\gamma_V(M)$ by the equation

$$(4.13) \quad \nabla^*_{B Y} B X = B(\nabla_Y X)$$

for $X, Y \in \mathcal{F}^1_0(M)$ and call ∇^* the affine connection induced in $\gamma_V(M)$ from ∇ , or the induced affine connection of $\gamma_V(M)$. Now (4.12) is written as

$$(4.14) \quad \nabla^{\text{II}}_{B Y} B X = \nabla^*_{B Y} B X + (X^h)^0 \nabla_{B Y}^{\text{II}} B_h$$

for $X = X^i \partial_i, Y \in \mathcal{F}^1_0(M)$.

Let there be given an element h of $\mathcal{F}^0_2(M)$. By virtue of Proposition 3.3, we have

$$BZ(h^{0*}(B X, B Y)) = (\nabla^*_{B Z} h^{0*})(B X, B Y) + h^{0*}(\nabla^*_{B Z} B X, B Y) + h^{0*}(B X, \nabla^*_{B Z} B Y)$$

for $X, Y, Z \in \mathcal{F}^1_0(M)$. On the other hand, taking account of (2.11), we have

$$\begin{aligned} (BZ)(h^{0*}(BX, BY)) &= (BZ)(h(X, Y))^0 = (Z(h(X, Y)))^0 \\ &= ((\nabla_Z h)(X, Y))^0 + (h(\nabla_Z X, Y))^0 + (h(X, \nabla_Z Y))^0 \\ &= (\nabla_Z h)^{0*}(BX, BY) + h^{0*}(B(\nabla_Z X), BY) + h^{0*}(BX, B(\nabla_Z Y)) \end{aligned}$$

for $X, Y, Z \in \mathcal{F}_1^1(M)$. If we compare the two equations obtained above, taking account of (4. 13), we have

$$(4. 15) \quad \nabla^*_{BZ} h^{0*} = (\nabla_Z h)^{0*}$$

for $Z \in \mathcal{F}_1^1(M)$.

When an affine connection ∇ in M is torsionfree, ∇^{II} is torsionfree in $T_2(M)$ too (cf. (1. 10)). Hence the induced connection ∇^* of $\gamma_\nabla(M)$ is also torsionfree. Thus we obtain from (4. 15)

PROPOSITION 4. 2. *Let g be a Riemannian metric in M and ∇ the Riemannian connection determined by g in M . Then the connection ∇^* induced in $\gamma_\nabla(M)$ from ∇ is the Riemannian connection determined by the induced metric g^{0*} of $\gamma_\nabla(M)$.*

Let there be given an element F of $\mathcal{F}_1^1(M)$ satisfying the condition $\mathcal{L}_\nabla F = 0$, then by virtue of (4. 13) and Proposition 3. 5, we have

$$(4. 16) \quad \nabla_{BZ}^* F^{II*} = (\nabla_Z F)^{II}$$

for $Z \in \mathcal{F}_1^1(M)$. Thus we have

PROPOSITION 4. 3. *Let F be an element of $\mathcal{F}_1^1(M)$ satisfying the condition $\mathcal{L}_\nabla F = 0$. If $\nabla F = 0$ in M , then $\nabla^* F^{II*} = 0$ in $\gamma_\nabla(M)$.*

An almost Hermitian structure (g, F) in M is called Kählerian if $\nabla F = 0$, ∇ being the Riemannian connection determined by g .

PROPOSITION 4. 4. *If (g, F) is a Kählerian structure satisfying the condition $\mathcal{L}_\nabla F = 0$, so is (g^{0*}, F^{II*}) in $\gamma_\nabla(M)$.*

Operating $\nabla^{II}_{B_k}$ to the first equation (4. 7) and taking the skew symmetric part of the equation obtained with respect to the indices j and k , by virtue of (2. 11) and (4. 7) we have

$$\begin{aligned} &\nabla^{II}_{B_k} \nabla^{II}_{B_j} B_i - \nabla^{II}_{B_j} \nabla^{II}_{B_k} B_i \\ &= (R_{kji}{}^h)^0 B_h + 2\{\nabla_k(\mathcal{L}_\nabla \Gamma_{ji}^h) - \nabla_j(\mathcal{L}_\nabla \Gamma_{ki}^h)\}^0 C_{\bar{h}} \\ &\quad + \{(\nabla_k \mathcal{L}_\nabla^2 \Gamma_{ji}^h - \nabla_j \mathcal{L}_\nabla^2 \Gamma_{ki}^h) + 2(\mathcal{L}_\nabla \Gamma_{ki}^h)(\mathcal{L}_\nabla \Gamma_{ji}^i) - 2(\mathcal{L}_\nabla \Gamma_{ji}^h)(\mathcal{L}_\nabla \Gamma_{ki}^i)\}^0 D_{\bar{h}} \end{aligned}$$

which reduces to

$$(4. 17) \quad R^{II}(B_k, B_j)B_i = (R_{kji}{}^h)^0 B_h + 2(\mathcal{L}_\nabla R_{kji}{}^h)^0 C_{\bar{h}} + (\mathcal{L}_\nabla^2 R_{kji}{}^h)^0 D_{\bar{h}}$$

because of (1.10), (4.2) and (4.4), where $R_{kji}{}^h$ denote the components of the curvature tensor R of the given affine conection ∇ and R^{II} the 2-nd lift of R to $T_2(M)$. As a direct consequence of (4.17), we have

PROPOSITION 4.5. *Let R and R^{II} be the curvature tensors of affine connections ∇ given in M and ∇^{II} , respectively. Then the curvature transformation $R^{II}(BX, BY)$ X and Y being arbitrary elements of $\mathcal{L}_0^1(M)$, leaves invariant the tangent space of the cross-section $\gamma_V(M)$ at each point of $\gamma_V(M)$, if and only if $\mathcal{L}_V R = 0$. In this case $R^{III} = \gamma_V' R$ holds, where R^{III} denotes the tensor field induced in $\gamma_V(M)$ from R .*

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