Tôhoku Math. Journ.
Vol. 19, No. 1, 1967.

# TENSOR FIELDS AND CONNECTIONS ON CROSS-SECTIONS IN THE COTANGENT BUNDLE

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(Received August 13, 1966)

**Introduction.** E. M. Patterson and the present author [6] recently studied vertical and complete lifts of tensor fields and connections from a manifold $M$ to its cotangent bundle ${}^cT(M)$. When a 1-form is given in an $n$-dimensional manifold $M$, the 1-form defines a cross-section in the cotangent bundle ${}^cT(M)$, which is an $n$-dimensional submanifold in the $2n$-dimensional cotangent bundle ${}^cT(M)$.

The main purpose of the present paper is to study the behaviour on the cross-section of the lifts of tensor fields and connections in a manifold $M$ to its cotangent bundle ${}^cT(M)$.

In §1 and §2, we review the results obtained in [6] on vertical and complete lifts of tensor fields and connections from a manifold to its cotangent bundle ${}^cT(M)$. In §3, we study the behaviour of the lifts of tensor fields and of Riemann extension of connections [1] on the cross-sections. We examine, in §4, the behaviour of the lifts of almost complex structures on the cross-sections. We show in §5 that the tensor discovered by Slebodzinski [3] appears in our present theory. Finally we study in §6 the behaviour of the complete lift of a connection on the cross-sections.

The manifold, functions, vector fields, 1-forms, tensor fields and connections appearing in the discussion will be supposed to be of the differentiability class $C^\infty$.

The indices $A, B, C, D, \cdots$ run from 1 to $2n$, the indices $a, b, c, \cdots, h, i, j, \cdots$ from 1 to $n$ and the indices $\bar{a}, \bar{b}, \bar{c}, \cdots, \bar{h}, \bar{i}, \bar{j}, \cdots$ from $n+1$ to $2n$. We use the notations $x^A = (x^h, x^{\bar{h}})$ and $x^{\bar{h}} = p_h$.

**1. Vertical lifts of tensor fields.** Let $M$ be an $n$-dimensional differentiable manifold of class $C^\infty$, ${}^cT(M)$ its cotangent bundle, and $\pi$ the projection ${}^cT(M) \to M$. Let the manifold $M$ be covered by a system of coordinate neighbourhoods $\{U; x^h\}$, where $(x^h)$ is a local coordinate system defined in the neighbourhood $U$. Let $(p_i)$ be the cartesian coordinate system in each cotangent space ${}^cT_P(M)$ at $P$ of $M$ with respect to the natural coframe $dx^i$

in $M$, $P$ being an arbitrary point in $U$ whose coordinates are $(x^h)$. Then we can introduce local coordinates $(x^h, p_i)$ in the open set $\pi^{-1}(U)$ of ${}^cT(U)$. We call them coordinates induced in $\pi^{-1}(U)$ from $\{U; x^h\}$ or simply *induced coordinates* in $\pi^{-1}(U)$. The projection $\pi$ is represented by $(x^h, p_i) \to (x^h)$.

In ${}^cT(M)$, there exists a 1-form

$$p = p_i dx^i, \tag{1.1}$$

which we call the *basic 1-form* in ${}^cT(M)$. The exterior derivative of $p$ is

$$dp = dp_i \wedge dx^i. \tag{1.2}$$

We call this the *basic 2-form* in ${}^cT(M)$. If we put

$$dp = \frac{1}{2} \varepsilon_{CB} dx^C \wedge dx^B, \tag{1.3}$$

we see that $\varepsilon_{CB}$ given by

$$\varepsilon_{CB} = \begin{pmatrix} 0 & \delta_i^j \\ -\delta_j^i & 0 \end{pmatrix} \tag{1.4}$$

are components of a tensor field of type (0, 2) in ${}^cT(M)$. Consequently we can define a tensor field $\varepsilon^{BA}$ of type (2, 0) by

$$\varepsilon_{BC} \varepsilon^{AB} = \delta_C^A \tag{1.5}$$

and find that $\varepsilon^{BA}$ has components

$$\varepsilon^{BA} = \begin{pmatrix} 0 & -\delta_i^h \\ \delta_h^i & 0 \end{pmatrix}. \tag{1.6}$$

We now take a function $f$ in $M$. The function $f \circ \pi$ in ${}^cT(M)$ induced from $f$ in $M$ is called the *vertical lift of $f$* and is denoted by

$$f^V = f \circ \pi. \tag{1.7}$$

A vector field $X$ in $M$ is, in a natural way, regarded as a function in ${}^cT(M)$. This function is called the *vertical lift of the vector field $X$* to ${}^cT(M)$ and is denoted by $X^V$. When $X$ in $M$ has local components $X^h$ with respect to the natural frame $\partial_h$ in $M$, $X^V$ in ${}^cT(M)$ has local expression

$$X^V = p_i X^i. \tag{1.8}$$

When a 1-form $\omega = \omega_i dx^i$ is given in $M$, it is also regarded as a 1-form in ${}^cT(M)$. If we write $\tilde{\omega} = \tilde{\omega}_B dx^B$, then $\tilde{\omega}$ has components

$$\tilde{\omega}_B = (\omega_i, 0)$$

in ${}^cT(M)$. Thus we can define a vector field $\tilde{\omega}_B \varepsilon^{BA}$ in ${}^cT(M)$. We call this vector field in ${}^cT(M)$ the *vertical lift of a 1-form* in $M$ to ${}^cT(M)$ and denote it by $\omega^V$. The $\omega^V$ has the components

$$\omega^V = \begin{pmatrix} 0 \\ \omega_i \end{pmatrix}. \tag{1.9}$$

The vertical lift $\omega^V$ of a 1-form $\omega$ in $M$ to ${}^cT(M)$ satisfies

$$\begin{cases} \omega^V f^V = 0, & \text{for any } f \in \mathfrak{T}_0^0(M), \\ \omega^V X^V = (\omega(X))^V, & \text{for any } X \in \mathfrak{T}_0^1(M), \end{cases} \tag{1.10}$$

which characterize $\omega^V$, where $\mathfrak{T}_s^r(M)$ denotes the set of tensor fields of type $(r, s)$ in $M$.

When we are given a tensor field $F$ of type (1, 1) in $M$ with local components $F_i^h$, we can easily see that $\tilde{F}_B dx^B = p_a F_i^a dx^i$ is a 1-form in ${}^cT(M)$. Thus we can define a vector field $\tilde{F}_B \varepsilon^{BA}$ in ${}^cT(M)$. We call this the *vertical lift of the tensor field* $F$ of type (1, 1) in $M$ to ${}^cT(M)$ and denote it by $F^V$. The $F^V$ has the components

$$F^V = \begin{pmatrix} 0 \\ p_a F_i^a \end{pmatrix}. \tag{1.11}$$

The $F^V$ satisfies

$$\begin{cases} F^V f^V = 0, & \text{for any } f \in \mathfrak{T}_0^0(M), \\ F^V X^V = (FX)^V, & \text{for any } X \in \mathfrak{T}_0^1(M), \end{cases} \tag{1.12}$$

which characterize $F^V$.

We also have

$$[F^V, G^V] = (FG - GF)^V, \quad \text{for any } F, G \in \mathfrak{T}_1^1(M). \tag{1.13}$$

Suppose that there is given a vector-valued 2-form $N$ in $M$ with local components $N_{ji}{}^h$. We can easily see that

$$\widetilde{N}_{CB}\, dx^C \wedge dx^B = p_a N_{ji}{}^a dx^j \wedge dx^i$$

is a 2-form in ${}^cT(M)$ and consequently that $\widetilde{N}_{CB}\varepsilon^{BA}$ is a tensor field of type (1, 1) in ${}^cT(M)$. We call this the *vertical lift of the vector-valued 2-form* $N$ in $M$ to ${}^cT(M)$, and denote it by $N^V$. The $N^V$ has components

$$N^V = \begin{pmatrix} 0 & 0 \\ p_a N_{ji}{}^a & 0 \end{pmatrix}. \tag{1.14}$$

The $N^V$ satisfies

$$\begin{cases} N^V \omega^V = 0, & \text{for any } \omega \in \mathfrak{T}_1^0(M), \\ N^V F^V = 0, & \text{for any } F \in \mathfrak{T}_1^1(M). \end{cases} \tag{1.15}$$

We can repeat the same argument and define the *vertical lift of a vector-valued r-form* in $M$ to ${}^cT(M)$.

**2. Complete lifts of tensor fields.** Suppose that there is given a vector field $X$ in $M$. From $X$ we can construct a function $X^V = p_a X^a$ in $M$. The gradient $\widetilde{X}_B$ of $X^V$ has components

$$\widetilde{X}_B = (p_a \partial_i X^a, X^i)$$

in ${}^cT(M)$. We can define a vector field $-\widetilde{X}_B\varepsilon^{BA}$ corresponding to this gradient in ${}^cT(M)$. We call this vector field the *complete lift* of $X$ in $M$ to ${}^cT(M)$ and denote it by $X^C$. The $X^C$ has components

$$X^C = \begin{pmatrix} X^h \\ -p_a \partial_i X^a \end{pmatrix}. \tag{2.1}$$

The complete lift of $X$ in $M$ to ${}^cT(M)$ has properties

$$\begin{cases} X^C f^V = (Xf)^V, & \text{for any } f \in \mathfrak{T}_0^0(M), \\ X^C Y^V = [X, Y]^V, & \text{for any } Y \in \mathfrak{T}_0^1(M), \end{cases} \tag{2.2}$$

which characterize the complete lift $X^C$. The complete lift $X^C$ of $X$ in $M$ to ${}^cT(M)$ has further properties:

(2.3) $$[X^C, \omega^V] = (\pounds_X \omega)^V, \quad \text{for any} \quad \omega \in \mathfrak{T}_1^0(M),$$

where $\pounds_X$ denotes the Lie derivative with respect to $X$,

(2.4) $$[X^C, F^V] = (\pounds_X F)^V, \quad \text{for any} \quad F \in \mathfrak{T}_1^1(M),$$

(2.5) $$N^V X^C = (N_X)^V,$$

for any vector-valued 2-form in $M$, where $N_X$ is a tensor field of type (1, 1) such that $N_X Y = N(X, Y)$ for any $Y \in \mathfrak{T}_0^1(M)$, and

(2.6) $$[X^C, Y^C] = [X, Y]^C, \quad \text{for any } X, Y \in \mathfrak{T}_0^1(M).$$

We note here that (1.15) and (2.5) characterize the vertical lift $N^V$.

Now take a tensor field $F$ of type (1, 1) in $M$ with local components $F_i^h$. Then $p_a F_i^a dx^i$ is a 1-form in ${}^cT(M)$ and its exterior differential

$$d(p_a F_i^a dx^i) = p_a \partial_j F_i^a dx^j \wedge dx^i + F_i^a dp_a \wedge dx^i$$

gives, when it is written as $\frac{1}{2}\widetilde{F}_{CB} dx^C \wedge dx^B$, a tensor field of type (0, 2) whose components are

$$\widetilde{F}_{CB} = \begin{pmatrix} p_a(\partial_j F_i^a - \partial_i F_j^a) & F_i^j \\ -F_j^i & 0 \end{pmatrix}.$$

We define a tensor field of type (1, 1) by $\widetilde{F}_{CB}\varepsilon^{BA}$ and call this tensor field the *complete lift of* $F$ in $M$ to ${}^cT(M)$ and denote it by $F^C$. The $F^C$ has components

(2.7) $$F^C = \begin{pmatrix} F_i^h & 0 \\ p_a(\partial_i F_h^a - \partial_h F_i^a) & F_h^i \end{pmatrix}.$$

The complete lift $F^C$ has the properties

(2.8)
$$F^C \omega^V = (\omega F)^V, \qquad \text{for any } \omega \in \mathfrak{T}_1^0(M),$$
$$F^C G^V = (GF)^V, \qquad \text{for any } G \in \mathfrak{T}_1^1(M),$$
$$F^C X^C = (FX)^C + (\pounds_X F)^V, \qquad \text{for any } X \in \mathfrak{T}_0^1(M)$$

which characterize $F^C$, where $\omega F$ denotes a 1-form defined by

$$(\omega F)(X) = \omega(FX), \quad \text{for any } X \in \mathfrak{T}_0^1(M).$$

The complete lift $F^C$ of $F \in \mathfrak{T}_1^1(M)$ has further properties

$$(2.9) \qquad \begin{aligned} F^C N^V &= (NF)^V, \quad \text{for any } N \in \mathfrak{T}_2^1(M), \\ N^V F^C &= (NF)^V, \quad \text{for any } N \in \mathfrak{T}_2^1(M), \end{aligned}$$

where $NF$ is a tensor field of type (1, 2) defined by

$$(NF)(X, Y) = N_X(FY), \quad \text{for any } X, Y \in \mathfrak{T}_0^1(M), \text{ and}$$

$$(2.10) \qquad F^C G^C + G^C F^C = (FG+GF)^C + (N_{F,G})^V,$$

for any $F, G \in \mathfrak{T}_1^1(M)$, where $N_{F,G}$ is the Nijenhuis tensor formed with $F$ and $G$:

$$(2.11) \qquad \begin{aligned} 2N_{F,G}(X, Y) = {} & [FX, GY] + [GX, FY] \\ & - F[GX, Y] - G[FX, Y] - F[X, GY] - G[X, FY] \\ & + (FG+GF)[X, Y] \end{aligned}$$

for any $X, Y \in \mathfrak{T}_0^1(M)$. From equation (2.10) we have, on putting $F=G$,

$$(2.12) \qquad (F^C)^2 = (F^2)^C + N^V,$$

where $N$ is the Nijenhuis tensor formed from $F$:

$$(2.13) \qquad N(X, Y) = [FX, FY] - F[FX, Y] - F[X, FY] + F^2[X, Y].$$

From (2.12), we see that, when $F$ defines a complex structure, that is, $F^2 = -1$ and $N = 0$, its complete lift $F^C$ to ${}^cT(M)$ defines an almost complex structure in ${}^cT(M)$.

We can moreover prove that, $F$ being an almost complex structure,

$$(2.14) \qquad F^C + \frac{1}{2}(NF)^V$$

is also an almost complex structure. (See, Satô, [2]).

Now take a vector-valued 2-form $N$ in $M$ with local components $N_{ji}{}^h$. Then $p_a N_{ji}{}^a dx^j \wedge dx^i$ is a 2-form in ${}^cT(M)$ and consequently its exterior differential

$$d(p_a N_{ji}{}^a dx^j \wedge dx^i) = p_a(\partial_k N_{ji}{}^a)\, dx^k \wedge dx^j \wedge dx^i + N_{ji}{}^a dp_a \wedge dx^j \wedge dx^i,$$

gives, when it is written as $\frac{1}{3}\tilde{N}_{DCB}dx^D\wedge dx^C\wedge dx^B$, a tensor field $\tilde{N}_{DCB}$ of type (0, 3) in ${}^cT(M)$, where

$$N_{kji} = p_a(\partial_k N_{ji}{}^a + \partial_j N_{ik}{}^a + \partial_i N_{kj}{}^a),$$
$$N_{ji\bar{h}} = N_{i\bar{h}j} = N_{\bar{h}ji} = N_{ji}{}^h,$$

all the other components being zero, from which we can define a tensor field of type (1, 2) $\tilde{N}_{CBE}\varepsilon^{EA}$ in ${}^cT(M)$. We call this *the complete lift of* $N$ in $M$ to ${}^cT(M)$ and denote it by $N^C$. The $N^C$ has components

$$\text{(2.15)}\qquad \begin{aligned} \tilde{N}_{ji}{}^h &= N_{ji}{}^h, \\ \tilde{N}_{ji}{}^{\bar{h}} &= -p_a(\partial_j N_{ih}{}^a + \partial_i N_{hj}{}^a + \partial_h N_{ji}{}^a), \\ \tilde{N}_{j\bar{i}}{}^{\bar{h}} &= N_{jh}{}^i, \\ \tilde{N}_{\bar{j}i}{}^{\bar{h}} &= N_{hi}{}^j, \end{aligned}$$

all the others being zero. The $N^C$ satisfies

$$\text{(2.16)}\qquad N^C(X^C, Y^C) = (N(X, Y))^C - ((\pounds_X N)_Y - (\pounds_Y N)_X + N_{[X,Y]})^V,$$

which characterizes $N^C$.

We can prove that the Nijenhuis tensor of the complete lift $F^C$ of $F$ is the complete lift $N^C$ of the Nijenhuis tensor $N$ formed with $F$.

We know that if $F$ defines a complex structure in $M$, then $F^C$ defines an almost complex structure in ${}^cT(M)$. Following (2.16) and the fact above, $F^C$ actually defines a complex structure in ${}^cT(M)$.

Suppose now that $F$ defines an almost complex structure in $M$. We know that $\bar{F} = F^C + \frac{1}{2}(NF)^V$ defines an almost complex structure in ${}^cT(M)$. We thus consider the Nijenhuis tensor $\bar{N}$ of $\bar{F}$. The Nijenhuis tensor $\bar{N}$ of $\bar{F}$ has components

$$\text{(2.17)}\qquad \begin{aligned} \bar{N}_{ji}{}^h &= N_{ji}{}^h, \\ \bar{N}_{ji}{}^{\bar{h}} &= -p_a(\partial_j N_{ih}{}^a + \partial_i N_{hj}{}^a + \partial_h N_{ji}{}^a) \\ &\quad + \frac{1}{2}p_a\{F_j{}^t\partial_t(N_{is}{}^a F_h{}^s) - F_i{}^t\partial_t(N_{js}{}^a F_h{}^s) \\ &\quad - (\partial_j(N_{is}{}^a F_t{}^s) - \partial_i(N_{js}{}^a F_t{}^s))F_h{}^t \\ &\quad + (\partial_j F_t{}^a - \partial_t F_j{}^a)N_{is}{}^t F_h{}^s - (\partial_i F_t{}^a - \partial_t F_i{}^a)N_{js}{}^t F_h{}^s \end{aligned}$$

$$- (\partial_j F_h{}^t - \partial_h F_j{}^t) N_{is}{}^a F_t{}^s + (\partial_i F_h{}^t - \partial_h F_i{}^t) N_{js}{}^a F_t{}^s$$

$$- (\partial_j F_i{}^t - \partial_i F_j{}^t) N_{ts}{}^a F_h{}^s + \frac{1}{2}(N_{js}{}^a N_{ih}{}^s - N_{is}{}^a N_{jh}{}^s)\} ,$$

all the others being zero.

**3. Lifts of vector fields on the cross-sections.** Suppose that there is given a global 1-form $W$ in $M$ whose local expression is $W = W_i(x)\, dx^i$. Then the 1-form $W$ defines a cross-section in ${}^cT(M)$, whose parametric representation is

$$x^h = x^h , \quad p_h = W_h(x) . \tag{3.1}$$

Thus the tangent vectors $B_i{}^A = \partial_i x^A$ to the cross-section have components

$$B_i{}^A = \begin{pmatrix} \delta_i^h \\ \partial_i W_h \end{pmatrix} . \tag{3.2}$$

On the other hand, the fibre being represented by

$$x^h = \text{const.}, \quad p_h = p_h , \tag{3.3}$$

the tangent vectors $C_{\bar{i}}{}^A = \partial_{\bar{i}} x^A$ to the fibre have components

$$C_{\bar{i}}{}^A = C^{iA} = \begin{pmatrix} 0 \\ \delta_h^i \end{pmatrix} . \tag{3.4}$$

The vectors $B_i{}^A$ and $C_{\bar{i}}{}^A$, being linearly independent, form a frame along the cross-section. We call this the *frame* $(B, C)$ along the cross-section. The coframe $(B^h{}_A, C^{\bar{h}}{}_A)$ corresponding to this frame is given by

$$\begin{aligned} B^h{}_A &= (\delta_i^h, 0) \\ C^{\bar{h}}{}_A = C_{hA} &= (-\partial_i W_h, \delta_h^i) . \end{aligned} \tag{3.5}$$

We call this coframe the *coframe* $(B, C)$ along the cross-section.

The basic 1-form $p = p_i\, dx^i$ has the expression $p = W_i\, dx^i$ and the basic 2-form the expression $dp = \frac{1}{2}(\partial_j W_i - \partial_i W_j)\, dx^j \wedge dx^i$ on the cross-section. The vertical lift $\omega^V$ of a 1-form $\omega = \omega_i\, dx^i$ has the expression

$$C_{\bar{i}}{}^{A}\omega^{\bar{i}} = C^{iA}\omega_i = \begin{pmatrix} 0 \\ \omega_i \end{pmatrix} \tag{3.6}$$

on the cross-section.

The complete lift $X^C$ of a vector field $X$ in $M$ to ${}^cT(M)$, having components (2. 1) with respect to the natural frame, has components

$$\begin{pmatrix} X^h \\ -\pounds_X W_h \end{pmatrix}$$

with respect to the frame $(B, C)$ along the cross-section. Thus We have

$$X^C:\ B_i{}^A X^i - C^{iA}(\pounds_X W_i), \tag{3.7}$$

from which

PROPOSITION 3.1. *The complete lift $X^C$ of a vector field $X$ in $M$ to ${}^cT(M)$ is tangent to the cross-section determined by a 1-form $W$ in $M$ if and only if the Lie derivative of $W$ with respect to $X$ vanishes in $M$.*

Suppose now that an affine connection $\nabla$ without torsion is given in $M$ and denote by $\Gamma_{ji}^h$ the components of the connection. Then

$$ds^2 = 2\,\delta p_i\, dx^i, \tag{3.8}$$

where

$$\delta p_i = dp_i - \Gamma_{ji}^h dx^j p_h, \tag{3.9}$$

defines a Riemannian metric in ${}^cT(M)$. We call this metric in ${}^cT(M)$ the Riemann extension of $\nabla$ and denote it by $\nabla^R$ [1]. With respect to the Riemann extension $\nabla^R$, the fibre given by $dx^h = 0$ is null and the horizontal distribution given by $\delta p_i = 0$ is also null.

The Riemann extension $\nabla^R$ has components

$$\nabla^R: \begin{pmatrix} -2\Gamma_{ji}^h p_h & \delta_i^j \\ \delta_j^i & 0 \end{pmatrix} \tag{3.10}$$

with respect to the natural frame and components

$$\nabla^R : \begin{pmatrix} \nabla_j W_i + \nabla_i W_j & \delta_i^j \\ \delta_j^i & 0 \end{pmatrix} \tag{3.11}$$

with respect to the frame $(B, C)$ along the cross-section, from which we have

PROPOSITION 3.2. *If $M$ has an affine connection $\nabla$ without torsion and ${}^cT(M)$ has the Riemann extension $\nabla^R$ as its metric, then the cross-section determined by a 1-form $W$ in $M$ is null with respect to $\nabla^R$ if and only if*

$$\nabla_j W_i + \nabla_i W_j = 0 . \tag{3.12}$$

PROPOSITION 3.3. *When $M$ has Riemannian metric $g$ and the Levi-Civita connection $\nabla$ of $g$ and ${}^cT(M)$ has the Riemann extension $\nabla^R$ as its metric, the cross-section determined by a 1-form $W$ in $M$ is null with respect to $\nabla^R$ if and only if $W$ is a Killing vector field in $M$.*

**4. Lifts of almost complex structures on cross-sections.** Suppose that the manifold $M$ has a complex structure $F$. Then the cotangent bundle ${}^cT(M)$ has the complex structure $F^C$.

Now the $F^C$ has the components (2.7) with respect to the natural frame and consequently has components

$$\begin{pmatrix} F_i{}^h & 0 \\ (\partial_i F_h{}^a - \partial_h F_i{}^a) W_a - F_i{}^t \partial_t W_h + F_h{}^t \partial_i W_t, & F_h{}^i \end{pmatrix} \tag{4.1}$$

with respect to the frame $(B, C)$ along the cross-section determined by $W$. Thus we have

$$\begin{aligned} \widetilde{F}_B^A B_i{}^B &= F_i{}^h B_h{}^A + \{(\partial_i F_h{}^a - \partial_h F_i{}^a) W_a - F_i{}^t \partial_t W_h + F_h{}^t \partial_i W_t\} C^{hA} , \\ \widetilde{F}_B^A C_{\bar{i}}{}^B &= F_t{}^i C^{tA} . \end{aligned} \tag{4.2}$$

Thus the cross-section is analytic if and only if

$$P_{ih} = (\partial_i F_h{}^a - \partial_h F_i{}^a) W_a - F_i{}^t \partial_t W_h + F_h{}^t \partial_i W_t = 0 . \tag{4.3}$$

We can easily verify that $P_{ih}$ are components of a tensor field of type (0, 2) in $M$. On the other hand, equation (4.3) is the condition for $W_i$ to be covariant analytic. [5]. Thus we have

PROPOSITION 4.1. *Suppose that $M$ has a complex structure $F$. Then the cross-section determined by a 1-form $W$ in ${}^cT(M)$ with complex structure $F^C$ is analytic if and only if $W$ is covariant analytic in $M$.*

Now suppose that $M$ has an almost complex structure $F$. Then the Nijenhuis tensor $\tilde{N}$ of the complete lift $F^C$ of $F$ has components (2.15) with respect to the natural frame in ${}^cT(M)$. Thus we have

$$\tilde{N}_{CB}{}^A B_j{}^C B_i{}^B = N_{ji}{}^h B_h{}^A - Q_{jih} C^{hA}, \tag{4.4}$$

where

$$\begin{aligned} Q_{jih} = & (\partial_j N_{ih}{}^a + \partial_i N_{hj}{}^a + \partial_h N_{ji}{}^a) W_a \\ & + N_{ji}{}^t \partial_t W_h - N_{ih}{}^t \partial_j W_t - N_{hj}{}^t \partial_i W_t . \end{aligned} \tag{4.5}$$

We can easily verify that $Q_{jih}$ are components of a tensor field of type (0, 3) in $M$. From (4.4) we have

PROPOSITION 4.2. *In order that $\tilde{N}_{CB}{}^A B_j{}^C B_i{}^B$ be tangent to the cross-section determined by a 1-form $W$, it is necessary and sufficient that $Q_{jih}=0$ in $M$.*

We know that when $M$ has an almost complex structure $F$, the cotangent bundle ${}^cT(M)$ has also an almost complex structure

$$\bar{F} = F^C + \frac{1}{2}(NF)^V .$$

The almost complex structure $\bar{F}$ has components

$$\bar{F}: \begin{pmatrix} F_i{}^h & 0 \\ (\partial_i F_h{}^a - \partial_h F_i{}^a + \frac{1}{2} N_{it}{}^a F_h{}^t) W_a & F_h{}^i \end{pmatrix}$$

with respect to the natural frame in ${}^cT(M)$ and components

$$\bar{F}: \begin{pmatrix} F_i{}^h & 0 \\ (\partial_i F_h{}^a - \partial_h F_i{}^a) W_a - F_i{}^t \partial_t W_h + F_h{}^t \partial_i W_t + \frac{1}{2} N_{it}{}^a F_h{}^t W_a) & F_h{}^i \end{pmatrix}$$

with respect to the frame $(B, C)$ along the cross-section determined by the 1-form $W$. Thus we have

$$\begin{aligned}
&\overline{F}_B^A B_i^B = F_i^h B_h^A \\
(4.6)\qquad &\quad + \{(\partial_i F_h^a - \partial_h F_i^a) W_a - F_i^t \partial_t W_h + F_h^t \partial_i W_t + \frac{1}{2} N_{it}^a F_h^t W_a\} C^{hA}, \\
&\overline{F}_B^A C^{iB} = F_h^i C^{hA}.
\end{aligned}$$

Thus the cross-section is almost analytic if and only if

$$(\partial_i F_h^a - \partial_h F_i^a) W_a - F_i^t \partial_t W_h + F_h^t \partial_i W_t + \frac{1}{2} N_{it}^a F_h^t W_a = 0 .$$

But the last equation means that $W_i$ is almost covariant analytic [2], [5]. Thus we have

PROPOSITION 4.3. *Suppose that $M$ has an almost complex structure $F$. Then the cross-section determined by $W$ in ${}^cT(M)$ with almost complex structure $F^C + \frac{1}{2}(NF)^V$ is almost analytic if and only if $W$ is almost covariant analytic in $M$.*

We now consider the Nijenhuis tensor $\overline{N}$ of $\overline{F} = F^C + \frac{1}{2}(NF)^V$. The $\overline{N}$ has components (2.17), or equivalently, by virtue of the relation $N_{is}^a F_h^s = -N_{ih}^s F_s^a$,

$$\begin{aligned}
&\overline{N}_{ji}^h = N_{ji}^h , \\
(4.8)\qquad &\overline{N}_{ji}^{\bar h} = -p_a(\partial_j N_{ih}^a + \partial_i N_{hj}^a + \partial_h N_{ji}^a) \\
&\quad - \frac{1}{2} p_a [F_j^t \partial_t (N_{ih}^s F_s^a) - F_i^t \partial_t (N_{jh}^s F_s^a) \\
&\quad + \{\partial_j (N_{is}^a F_t^s) - \partial_i (N_{js}^a F_t^s)\} F_h^t \\
&\quad + (\partial_j F_t^a - \partial_t F_j^a) N_{ih}^s F_s^t - (\partial_i F_t^a - \partial_t F_i^a) N_{jh}^s F_s^t \\
&\quad - (\partial_j F_h^t - \partial_h F_j^t) N_{it}^s F_s^a + (\partial_i F_h^t - \partial_h F_i^t) N_{jt}^s F_s^a \\
&\quad - (\partial_j F_i^t - \partial_i F_j^t) N_{th}^s F_s^a \\
&\quad - \frac{1}{2} (N_{js}^a N_{ih}^s - N_{is}^a N_{jh}^s)] ,
\end{aligned}$$

all the others being zero. Thus we have

$$\overline{N}_{CB}^A B_j^C B_i^B = N_{ji}^h B_h^A + R_{jih} C^{hA} , \qquad (4.9)$$

where

$$
(4.10)\qquad \begin{aligned} R_{jih} = &- N_{ji}{}^{t}\partial_t W_h - (\partial_j N_{ih}{}^{a} + \partial_i N_{hj}{}^{a} + \partial_h N_{ji}{}^{a}) W_a \\ &- \frac{1}{2}[F_j{}^{t}\partial_t(N_{ih}{}^{s}F_s{}^{a}) - F_i{}^{t}\partial_t(N_{jh}{}^{s}F_s{}^{a}) \\ &+ \{\partial_j(N_{is}{}^{a}F_t{}^{s}) - \partial_i(N_{js}{}^{a}F_t{}^{s})\}F_h{}^{t} \\ &+ (\partial_j F_t{}^{a} - \partial_t F_j{}^{a}) N_{ih}{}^{s}F_s{}^{t} - (\partial_i F_t{}^{a} - \partial_t F_i{}^{a}) N_{jh}{}^{s}F_s{}^{t} \\ &- (\partial_j F_h{}^{t} - \partial_h F_j{}^{t}) N_{it}{}^{s}F_s{}^{a} + (\partial_i F_h{}^{t} - \partial_h F_i{}^{t}) N_{jt}{}^{s}F_s{}^{a} \\ &- (\partial_j F_i{}^{t} - \partial_i F_j{}^{t}) N_{th}{}^{s}F_s{}^{a} - \frac{1}{2}(N_{js}{}^{a}N_{ih}{}^{s} - N_{is}{}^{a}N_{jh}{}^{s})] W_a . \end{aligned}
$$

We can easily verify that $R_{jih}$, or rather $R_{jih}$ minus last term containing $N_{ij}{}^{e}N_{hu}{}^{i}$'s are components of a tensor field of type (0, 2) in $M$. From (4.9) we have

PROPOSITION 4.4. *The vector $\overline{N}_{CB}{}^{A}B_j{}^{C}B_i{}^{B}$ is tangent to the cross-section determined by $W_i$ if and only if $R_{jih} = 0$.*

**5. The Slebodzinski tensor.** From (4. 10), we have

$$
(5.1)\qquad \begin{aligned} R_{jih} + R_{ihj} + R_{hji} = &- N_{ji}{}^{t}\partial_t W_h - N_{ih}{}^{t}\partial_t W_j - N_{hj}{}^{t}\partial_t W_i \\ &- 3(\partial_j N_{ih}{}^{a} + \partial_i N_{hj}{}^{a} + \partial_h N_{ji}{}^{a}) W_a \\ &- [(F_j{}^{t}\partial_t N_{ih}{}^{s} + F_i{}^{t}\partial_t N_{hj}{}^{s} + F_h{}^{t}\partial_t N_{ji}{}^{s}) F_s{}^{a}W_a \\ &- \{N_{ji}{}^{s}(\partial_s F_h{}^{t}) + N_{ih}{}^{s}(\partial_s F_j{}^{t}) + N_{hj}{}^{s}(\partial_s F_i{}^{t})\} F_t{}^{a}W_a \\ &- (\partial_j N_{ih}{}^{a} + \partial_i N_{hj}{}^{a} + \partial_h N_{ji}{}^{a}) W_a \\ &- \{(\partial_j F_i{}^{t} - \partial_i F_j{}^{t}) N_{th}{}^{s} \\ &+ (\partial_i F_h{}^{t} - \partial_h F_i{}^{t}) N_{tj}{}^{s} \\ &+ (\partial_h F_j{}^{t} - \partial_h F_j{}^{t}) N_{ti}{}^{s}\} F_s{}^{a}W_a \\ &+ \frac{1}{2}(N_{jt}{}^{a}N_{ih}{}^{t} + N_{it}{}^{a}N_{hj}{}^{t} + N_{ht}{}^{a}N_{ji}{}^{t}) W_a] \end{aligned}
$$

But, we have on the other hand

$$
(5.2)\qquad \begin{aligned} &\partial_j N_{ih}{}^{a} + \partial_i N_{hj}{}^{a} + \partial_h N_{ji}{}^{a} \\ &\quad = -\frac{1}{2}\{(\partial_j N_{ih}{}^{t} - \partial_i N_{jh}{}^{t}) + (\partial_i N_{hj}{}^{t} - \partial_h N_{ij}{}^{t}) \end{aligned}
$$

$$
\begin{aligned}
&+ (\partial_h N_{ji}{}^t - \partial_j N_{hi}{}^t)\} F_t{}^s F_s{}^a \\
&= -\frac{1}{2}\{\partial_j(N_{ih}{}^t F_t{}^s) - N_{ih}{}^t \partial_j F_t{}^s - \partial_i(N_{jh}{}^t F_t{}^s) + N_{jh}{}^t \partial_i F_t{}^s \\
&+ \partial_i(N_{hj}{}^t F_t{}^s) - N_{hj}{}^t \partial_i F_t{}^s - \partial_h(N_{ij}{}^t F_t{}^s) + N_{ij}{}^t \partial_h F_t{}^s \\
&+ \partial_h(N_{ji}{}^t F_t{}^s) - N_{ji}{}^t \partial_h F_t{}^s - \partial_j(N_{hi}{}^t F_t{}^s) + N_{hi}{}^t \partial_j F_t{}^s\} F_s{}^a \\
&= \frac{1}{2}\{\partial_j(N_{it}{}^s F_h{}^t) + N_{ih}{}^t \partial_j F_t{}^s - \partial_i(N_{jt}{}^s F_h{}^t) - N_{jh}{}^t \partial_i F_t{}^s \\
&+ \partial_i(N_{ht}{}^s F_j{}^t) + N_{hj}{}^t \partial_i F_t{}^s - \partial_h(N_{it}{}^s F_j{}^t) - N_{ij}{}^t \partial_h F_t{}^s \\
&+ \partial_h(N_{jt}{}^s F_i{}^t) + N_{ji}{}^t \partial_h F_t{}^s - \partial_j(N_{ht}{}^s F_i{}^t) - N_{hi}{}^t \partial_j F_t{}^s\} F_s{}^a \\
&= \frac{1}{2}\{F_j{}^t(\partial_i N_{ht}{}^s - \partial_h N_{it}{}^s) + F_i{}^t(\partial_h N_{jt}{}^s - \partial_j N_{ht}{}^s) \\
&+ F_h{}^t(\partial_j N_{it}{}^s - \partial_i N_{jt}{}^s) \\
&+ (\partial_j F_i{}^t - \partial_i F_j{}^t) N_{th}{}^s + (\partial_i F_h{}^t - \partial_h F_i{}^t) N_{tj}{}^s \\
&+ (\partial_h F_j{}^t - \partial_j F_h{}^t) N_{ti}{}^s \\
&+ 2(N_{ji}{}^t \partial_h F_t{}^s + N_{ih}{}^t \partial_j F_t{}^s + N_{hj}{}^t \partial_i F_t{}^s)\} F_s{}^a .
\end{aligned}
$$

Thus, we have from (5.1)

$$
\begin{aligned}
(5.3)\qquad & R_{jih} + R_{ihj} + R_{hji} \\
&= -(N_{ji}{}^t \partial_t W_h + N_{ih}{}^t \partial_t W_j + N_{hj}{}^t \partial_t W_i) - (\partial_j N_{ih}{}^a + \partial_i N_{hj}{}^a + \partial_h N_{ji}{}^a) W_a \\
&- [F_j{}^t \partial_t N_{ih}{}^s + F_i{}^t \partial_t N_{hj}{}^s + F_h{}^t \partial_t N_{ji}{}^s \\
&+ \frac{1}{2}\{F_j{}^t(\partial_i N_{ht}{}^s - \partial_h N_{it}{}^s) + F_i{}^t(\partial_h N_{jt}{}^s - \partial_j N_{ht}{}^s) + F_h{}^t(\partial_j N_{it}{}^s - \partial_i N_{jt}{}^s)\} \\
&+ N_{ji}{}^t(\partial_h F_t{}^s - \partial_t F_h{}^s) + N_{ih}{}^t(\partial_j F_t{}^s - \partial_t F_j{}^s) + N_{hj}{}^t(\partial_i F_t{}^s - \partial_t F_i{}^s) \\
&- \frac{1}{2}\{(\partial_j F_i{}^t - \partial_i F_j{}^t) N_{th}{}^s + (\partial_i F_h{}^t - \partial_h F_i{}^t) N_{tj}{}^s \\
&+ (\partial_h F_j{}^t - \partial_j F_h{}^t) N_{ti}{}^s\}] F_s{}^a W_a \\
&- \frac{1}{2}[N_{jt}{}^a N_{ih}{}^t + N_{it}{}^a N_{hj}{}^t + N_{ht}{}^a N_{ji}{}^t] W_a ,
\end{aligned}
$$

that is

$$
\begin{aligned}
(5.4)\qquad & R_{jih} + R_{ihj} + R_{hji} + Q_{jih} + N_{ih}{}^t(\partial_j W_t - \partial_t W_j) \\
&\qquad + N_{hj}{}^t(\partial_i W_t - \partial_t W_i) + S_{jih}{}^s F_s{}^a W_a
\end{aligned}
$$

$$+\frac{1}{2}(N_{jt}{}^{a}N_{ih}{}^{t}+N_{it}{}^{a}N_{hj}{}^{t}+N_{ht}{}^{a}N_{ji}{}^{t})W_{a}=0\,,$$

where

$$\begin{aligned}
S_{jih}{}^{s} &= F_j{}^t\partial_t N_{ih}{}^s + F_i{}^t\partial_t N_{hj}{}^s + F_h{}^t\partial_t N_{ji}{}^s \\
&\quad + \frac{1}{2}\{F_j{}^t(\partial_i N_{ht}{}^s - \partial_h N_{it}{}^s) + F_i{}^t(\partial_h N_{jt}{}^s - \partial_j N_{ht}{}^s) \\
&\quad + F_h{}^t(\partial_j N_{it}{}^s - \partial_i N_{jt}{}^s)\} \\
&\quad + N_{ji}{}^t(\partial_h F_t{}^s - \partial_t F_h{}^s) + N_{ih}{}^t(\partial_j F_t{}^s - \partial_t F_j{}^s) + N_{hj}{}^t(\partial_i F_t{}^s - \partial_t F_i{}^s) \\
&\quad - \frac{1}{2}\{(\partial_j F_i{}^t - \partial_i F_j{}^t)N_{th}{}^s + (\partial_i F_h{}^t - \partial_h F_i{}^t)N_{tj}{}^s \\
&\quad + (\partial_h F_j{}^t - \partial_j F_h{}^t)N_{ti}{}^s\}\,. 
\end{aligned} \tag{5.5}$$

Equation (5.4) shows the tensor character of $S_{jih}{}^{s}$. This is the tensor first introduced by Slebodzinski. [3]. (The expression of Slebodzinski tensor in Math. Rev. 30 (1965), p. 652, 3438, should be read as 2[ ]+[ ]+2[ ]−[ ].) T. J. Willmore [4] showed that this tensor is identically zero.

**6. Complete lift of a connection on cross-sections.** Suppose that there is given a symmetric affine connection $\nabla$ in $M$ whose components are $\Gamma^k_{ji}$. Then (3.10) defines a Riemannian metric in ${}^cT(M)$ which is called the Riemann extension of $\nabla$.

We construct the Levi-Civita connection $\nabla^C$ from this Riemann extension and call it *complete lift of the symmetric affine connection* $\nabla$ to the cotangent bundle ${}^cT(M)$. The complete lift $\nabla^C$ has components $\widetilde{\Gamma}^A_{CB}$ given by

$$\begin{aligned}
&\widetilde{\Gamma}^h_{ji}=\Gamma^h_{ji}\,,\quad \widetilde{\Gamma}^h_{j\bar{i}}=0\,,\quad \widetilde{\Gamma}^h_{\bar{j}i}=0\,,\quad \widetilde{\Gamma}^h_{\bar{j}\bar{i}}=0\,.\\
&\widetilde{\Gamma}^{\bar{h}}_{ji}=p_a(\partial_h\Gamma^a_{ji}-\partial_j\Gamma^a_{ih}-\partial_i\Gamma^a_{jh}+2\Gamma^a_{ht}\Gamma^t_{ji})\,,\\
&\widetilde{\Gamma}^{\bar{h}}_{j\bar{i}}=-\Gamma^i_{jh}\,,\quad \widetilde{\Gamma}^{\bar{h}}_{\bar{j}i}=-\Gamma^j_{hi}\,,\quad \widetilde{\Gamma}^{\bar{h}}_{\bar{j}\bar{i}}=0\,,
\end{aligned} \tag{6.1}$$

and the curvature tensor of the complete lift $\nabla^C$ components $\widetilde{R}_{DCB}{}^{A}$ given by

$$\begin{aligned}
&\widetilde{R}_{kji}{}^{h}=R_{kji}{}^{h}\,,\\
&\widetilde{R}_{kji}{}^{\bar{h}}=(\nabla_h R_{kji}{}^{a}-\nabla_i R_{kjh}{}^{a}\\
&\qquad +\Gamma^a_{ht}R_{kji}{}^{t}+\Gamma^a_{kt}R_{ihj}{}^{t}+\Gamma^a_{jt}R_{hik}{}^{t}+\Gamma^a_{it}R_{kjh}{}^{t})\,p_a\,,\\
&\widetilde{R}_{kj\bar{i}}{}^{\bar{h}}=-R_{kjh}{}^{i}\,,\quad \widetilde{R}_{k\bar{j}i}{}^{\bar{h}}=-R_{hik}{}^{j}\,,\quad \widetilde{R}_{\bar{k}ji}{}^{\bar{h}}=-R_{hij}{}^{k}\,,
\end{aligned} \tag{6.2}$$

all the others being zero, where $R_{kji}{}^{h}$ are components of the curvature tensor of $\nabla$.

Suppose now that there is given a global 1-form $W$ in $M$. Then the $W$ defines a cross-section in ${}^cT(M)$. The vectors (3. 2) are tangent to the cross-section and (3. 4) are $n$ linearly independent vectors which are not tangent to the cross-section. We take the vectors $C^{iA}$ as normals to the cross-section and define an affine connection induced on the cross-section. The components of the induced affine connection are given by

$$(\partial_j B_i{}^A + \widetilde{\Gamma}^A_{CB} B_j{}^C B_i{}^B) B^h{}_A = \Gamma^h_{ji} . \tag{6. 3}$$

From this equation we see that the quantity

$$\partial_j B_i{}^A + \widetilde{\Gamma}^A_{CB} B_j{}^C B_i{}^B - \Gamma^h_{ji} B_h{}^A \tag{6. 4}$$

is a linear combination of the vectors $C_{\bar{i}}{}^A$. To find the coefficients, we put $A=\bar{h}$ in (6. 4) and find

$$\begin{aligned} &\partial_j \partial_i W_h + W_a(\partial_h \Gamma^a_{ji} - \partial_j \Gamma^a_{ih} - \partial_i \Gamma^a_{jh} + 2\Gamma^a_{ht} \Gamma^t_{ji}) \\ &\quad - \Gamma^a_{jh} \partial_i W_a - \Gamma^a_{hi} \partial_j W_a - \Gamma^a_{ji} \partial_a W_h \\ &= \nabla_j \nabla_i W_h + R_{hij}{}^a W_a . \end{aligned}$$

Thus representing (6. 4) by ${}'\nabla_j B_i^A$, we have

$${}'\nabla_j B_i^A = (\nabla_j \nabla_i W_h + R_{hij}{}^a W_a) C^{hA}, \tag{6. 5}$$

which is the equation of Gauss for the cross-section determined by $W_i$. Thus we have

PROPOSITION 6.1. *In order that the cross-section in ${}^cT(M)$ determined by a 1-form $W$ in $M$ with symmetric affine connection $\nabla$ be totally geodesic, it is necessary and sufficient that $W$ satisfies*

$$\nabla_j \nabla_i W_h + R_{hij}{}^a W_a = 0 . \tag{6. 6}$$

On the other hand, since the components $\widetilde{\Gamma}^A_{CB}$ are given by (6. 1) we can easily verify that

$$\partial_j C_{\bar{i}}{}^A + \widetilde{\Gamma}^A_{CB} B_j{}^C C_{\bar{i}}{}^B - \Gamma^i_{jh} C_{\bar{h}}{}^A = 0 ,$$

that is

$$\partial_j C^{iA} + \widetilde{\Gamma}^A_{CB} B_j{}^C C^{iB} - \Gamma^i_{jh} C^{hA} = 0 .$$

Thus denoting by $'\nabla_j C^{iA}$ the left hand member of this equation, we get

$$'\nabla_j C^{iA} = 0. \tag{6.7}$$

This is the equation of Weingarten for the cross-section.

Applying the operator $'\nabla_k$ to (6.5), we find

$$'\nabla_k{}'\nabla_j B_i^A = \nabla_k(\nabla_j \nabla_i W_h + R_{hij}{}^a W_a) C^{hA},$$

from which, remembering that

$$'\nabla_k{}'\nabla_j B_i^A - {}'\nabla_j{}' \nabla_k B_i^A = \widetilde{R}_{DCB}{}^A B_k{}^D B_j{}^C B_i{}^B - R_{kji}{}^h B_h{}^A,$$

we find

$$\begin{aligned} \widetilde{R}_{DCB}{}^A B_k{}^D B_j{}^C B_i{}^B - R_{kji}{}^h B_h{}^A = [(\nabla_k R_{hij}{}^a - \nabla_j R_{hik}{}^a) W_a \\ - R_{kji}{}^a \nabla_a W_h - R_{kjh}{}^a \nabla_i W_a + R_{hij}{}^a \nabla_k W_a - R_{hik}{}^a \nabla_j W_a] C^{hA}. \end{aligned} \tag{6.8}$$

Thus we have

PROPOSITION 6.2. *In order that $\widetilde{R}_{DCB}{}^A B_k{}^D B_j{}^C B_i{}^B$ is tangent to the cross-section, it is necessary and sufficient that*

$$\begin{aligned} &(\nabla_k R_{hij}{}^a - \nabla_j R_{hik}{}^a) W_a \\ &= R_{kji}{}^a \nabla_a W_h + R_{kjh}{}^a \nabla_i W_a - R_{hij}{}^a \nabla_k W_a + R_{hik}{}^a \nabla_j W_a. \end{aligned} \tag{6.9}$$

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