KŌDAI MATH. SEM. REP.
23 (1971), 160—167

TENSOR FIELDS LIFTED TO COTANGENT BUNDLES AND DIFFERENTIAL CONCOMITANTS OF TENSOR FIELDS IN THE BASE MANIFOLDS

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Introduction.

Recently, Yano and Akō ([2])[1] defined an operator $\Phi^F(X)$ associated with a given tensor field F of type $(1,1)$ and any vector field X in a differentiable manifold M. By applying the operator $\Phi^F(X)$ to any vector field Y in such a way that $\Phi^F(X)Y=-(\mathcal{L}_Y F)X$, where $\mathcal{L}_Y$ denotes the Lie derivative with respect to Y, they got the differential concomitant (S, T) of tensor fields S of type $(1,2)$ and T of type $(1,t)$, that is a tensor field of type $(1, t+2)$, and the differential concomitant (σ, T) of tensor fields σ of type $(0,2)$ and T of type $(1,t)$, that is a tensor field of type $(0, t+2)$.

In this paper, we investigate the properties of tensor fields lifted to the cotangent bundle of M and try to get systematically the differential concomitant (S, T) of tensor fields S of type $(1,s)$ and T of type $(1,t)$, that is a tensor field of type $(1, s+t)$, and the differential concomitant (σ, T) of tensor fields σ of type $(0,s)$ and T of type $(1,t)$ respectively, that is a tensor field of type $(0, s+t)$ when S, T and σ are skew-symmetric.

§1. Lifts of tensor fields to cotangent bundles.

Let M be a differentiable manifold of class C^∞ and of dimension n. Let ${}^CT(M)$ be the cotangent bundle of M. Then ${}^CT(M)$ is also a differentiable manifold of class C^∞ and of dimension $2n$.

A point $\tilde{P}$ of ${}^CT(M)$ is an ordered pair (P, ω_P) of a point $P \in M$ and a covector ω_P at P. We denote by π the natural projection ${}^CT(M) \rightarrow M$ given by $\tilde{P}=(P, \omega_P) \rightarrow P$.

Suppose that the manifold M is covered by a system of coordinate neighbourhoods $\{U, x^i\}$ where (x^i) is a system of local coordinates in the neighborhood U. Then, in the open set $\pi^{-1}(U)$ of ${}^CT(M)$, we can introduce local coordinates $(x^i, x^{\bar{i}})$ or (x^I)[2] for $\tilde{P}$ where we put $x^{\bar{i}}=p_i$ and p_i are the components of ω_P with respect to the natural coframe dx^i. We call $(x^i, x^{\bar{i}})$ or (x^I) the coordinates in $\pi^{-1}(U)$ induced

Received June 18, 1970.

1) The number between brackets refers to the Bibliography at the end of the paper.

2) For indices, small letters $i, j, k, \cdots$ run over the range $1, 2, \cdots, n$, and $\bar{i}=i+n$ Capital letters $I, J, K, \cdots$ run over the range $1, 2, \cdots, 2n$.

from (x^i) or simly the induced coordinates in $\pi^{-1}(U)$.

We denote by $\mathcal{T}^r_s(M)$ or symply by $\mathcal{T}^r_s$ the set of tensor fields of class C^∞ and of type (r, s) in M and similarly by $\mathcal{T}^r_s({}^CT(M))$ or simply by $\tilde{\mathcal{T}}^r_s$ the corresponding set of tensor fields in ${}^CT(M)$. And we denote by ${}'\mathcal{T}^r_s, {}'\tilde{\mathcal{T}}^r_s$ the sets of elements of $\mathcal{T}^r_s, \tilde{\mathcal{T}}^r_s$ which are skew-symmetric with respect to all covariant indices, respectively.

Suppose that $\tau\in\mathcal{T}^0_{t+1}$ and that τ has components $\tau_{i_{t+1}i_t\cdots i_1}$ in U. We define an element $\tilde{Q}$ of $\tilde{\mathcal{T}}^0_{t+1}$ whose components $\tilde{Q}_{I_{t+1}I_t\cdots I_1}$ in $\pi^{-1}(U)$ are given by

$$\tilde{Q}_{i_{t+1}i_t\cdots i_1}=\tau_{i_{t+1}i_t\cdots i_1},\qquad \tilde{Q}_{i_{t+1}i_t\cdots \bar{i}_h\cdots i_1}=\cdots=\tilde{Q}_{\bar{i}_{t+1}\bar{i}_t\cdots \bar{i}_1}=0. \tag{1.1}$$

It is well-known that the tensor field ε^{-1} with components in $\pi^{-1}(U)$ given by a matrix

$$(\varepsilon^{BA})=\begin{pmatrix}0 & -I\\ I & 0\end{pmatrix}$$

belongs to $\tilde{\mathcal{T}}^2_0$, where I denotes the unit $n\times n$ matrix (cf. [3]).

By putting

$$\tilde{\tau}_{I_tI_{t-1}\cdots I_1}{}^K=\tilde{Q}_{I_t\cdots I_lBI_{l-1}\cdots I_1}\varepsilon^{BK}, \tag{1.2}$$

we can define a tensor field belonging to $\tilde{\mathcal{T}}^1_t$ with components given by (1. 2) in $\pi^{-1}(U)$. We call this tensor field the l-vertical lift of τ and by $\tau^V_{(l)}$. And we call the 1-vertical lift of τ simply the vertical lift of τ and denote by τ^V (cf. [3], for $s=1$).

By (1. 2), the components $\tilde{\tau}_{I_tI_{t-1}\cdots I_1}{}^K$ of $\tau^V_{(l)}$ in $\pi^{-1}(U)$ are given by

$$\tilde{\tau}_{i_ti_{t-1}\cdots i_1}{}^{\bar{k}}=\tau_{i_ti_{t-1}\cdots i_lki_{l-1}\cdots i_1},\qquad \tilde{\tau}_{I_tI_{t-1}\cdots I_1}{}^{k}=\tilde{\tau}_{i_ti_{t-1}\cdots \bar{i}_h\cdots i_1}{}^{\bar{k}}=\cdots=\tilde{\tau}_{\bar{i}_t\bar{i}_{t-1}\cdots \bar{i}_1}{}^{\bar{k}}=0. \tag{1.3}$$

Conversely, we can easily see that if, for $\tilde{\tau}\in\tilde{\mathcal{T}}^1_t$, its components $\tilde{\tau}_{I_tI_{t-1}\cdots I_1}{}^K$ in $\pi^{-1}(U)$ are given by (1. 3) and $\tau_{i_ti_{t-1}\cdots i_lki_{l-1}\cdots i_1}$ are functions in M, then $\tau_{i_ti_{t-1}\cdots i_lki_{l-1}\cdots i_1}$ define a tensor field $\tau\in\mathcal{T}^0_{t+1}$ and are components of τ in U.

Thus we have

LEMMA 1. 1. *Suppose that $\tau_{i_{t+1}i_t\cdots i_1}$ are functions in M, $\tilde{\tau}_{I_tI_{t-1}\cdots I_1}{}^K$ functions in ${}^CT(M)$ and $\tilde{\tau}_{I_tI_{t-1}\cdots I_1}{}^K$ satisfy the condition* (1. 3). *Then $\tau_{i_{t+1}i_t\cdots i_1}$ define a tensor field $\tau\in\mathcal{T}^0_{t+1}$ and are components of τ in U, if and only if $\tilde{\tau}_{I_tI_{t-1}\cdots I_1}{}^K$ define a tensor field $\tilde{\tau}=\tau^V_{(l)}\in\tilde{\mathcal{T}}^1_t$ and are components of $\tilde{\tau}=\tau^V_{(l)}$ in $\pi^{-1}(U)$.*

In exactly the same way as above, for $T\in\mathcal{T}^1_{t+1}$, we can define the l-vertical lift of T denoted by $T^V_{(l)}\in\tilde{\mathcal{T}}^1_t$. In order to get $T^V_{(l)}$, we replace only $\tilde{Q}_{i_{t+1}i_t\cdots i_1}=\tau_{i_{t+1}i_t\cdots i_1}$ in (1. 1) by $\tilde{Q}_{i_{t+1}i_t\cdots i_1}=p_aT_{i_{t+1}i_t\cdots i_1}{}^a$, where $T_{i_{t+1}i_t\cdots i_1}{}^k$ are components of T in U. Thus the components $\tilde{T}_{I_tI_{t-1}\cdots I_1}{}^K$ of $T^V_{(l)}$ in $\pi^{-1}(U)$ are expressible as follows (cf. [3], for $t=1, 2$):

$$\tilde{T}_{i_t i_{t-1}\cdots i_1}{}^{\bar{k}}=p_a T_{i_t i_{t-1}\cdots i_l k i_{l-1}\cdots i_1}{}^{a},$$
(1. 4)
$$\tilde{T}_{I_t I_{t-1}\cdots I_1}{}^{k}=\tilde{T}_{i_t i_{t-1}\cdots \bar{i}_h\cdots i_1}{}^{\bar{k}}=\cdots=\tilde{T}_{\bar{i}_t \bar{i}_{t-1}\cdots \bar{i}_1}{}^{\bar{k}}=0.$$

From (1. 4), we have a lemma corresponding to Lemma 1. 1.

LEMMA 1. 2. *Suppose that* $T_{i_{t+1} i_t \cdots i_1}{}^{K}$ *are functions in* M, $\tilde{T}_{I_t I_{t-1}\cdots I_1}{}^{K}$ *functions in* ${}^{C}T(M)$ *and* $\tilde{T}_{I_t I_{t-1}\cdots I_1}{}^{K}$ *satisfy the condition* (1. 4). *Then* $T_{i_{t+1} i_t \cdots i_1}{}^{k}$ *define a tensor field* T belonging to $\mathfrak{T}^1_{t+1}$ *and are components of* T *in* U, *if and only if* $\tilde{T}_{I_t I_{t-1}\cdots I_1}{}^{K}$ *define a tensor field* $\tilde{T}=T^V_{(l)}\in\tilde{\mathfrak{T}}^1_t$ *and are components of* $\tilde{T}=T^V_{(l)}$ *in* $\pi^{-1}(U)$.

For $\sigma\in{}'\mathfrak{T}^0_s$, we denote by $d\sigma$ the exterior derivative of σ. We can see that the components of $\tilde{\sigma}_{I_s I_{s-1}\cdots I_1}{}^{K}$ of $(d\sigma)^V\in{}'\tilde{\mathfrak{T}}^1_s$ are expressible in $\pi^{-1}(U)$ as follows:

$$\tilde{\sigma}_{i_s i_{s-1}\cdots i_1}{}^{\bar{k}}=(-1)^s\left\{\partial_k\sigma_{i_s\cdots i_1}-\sum_{h=1}^{s}\partial_{i_h}\sigma_{i_s\cdots i_{h-1}k i_{h+1}\cdots i_1}\right\},$$
(1. 5)
$$\tilde{\sigma}_{I_s I_{s-1}\cdots I_1}{}^{k}=\tilde{\sigma}_{i_s i_{s-1}\cdots \bar{i}_h\cdots i_1}{}^{\bar{k}}=\cdots=\tilde{\sigma}_{\bar{i}_s \bar{i}_{s-1}\cdots \bar{i}_1}{}^{\bar{k}}=0.$$

Suppose now that $S\in{}'\mathfrak{T}^1_s$ and S has components $S_{i_s i_{s-1}\cdots i_1}{}^{k}$ in U. Then

$$\tilde{\sigma}=\frac{1}{s!}p_a S_{i_s i_{s-1}\cdots i_1}{}^{a}dx^{i_s}\wedge dx^{i_{s-1}}\wedge\cdots\wedge dx^{i_1}$$

is an s-form in ${}^{C}T(M)$. Consequently, the exterior derivative $d\tilde{\sigma}$ of $\tilde{\sigma}$ in ${}^{C}T(M)$ belongs to ${}'\tilde{\mathfrak{T}}^0_{s+1}$. We now put

$$d\tilde{\sigma}=\frac{1}{(s+1)!}\tilde{S}_{B_{s+1}B_s\cdots B_1}dx^{B_{s+1}}\wedge dx^{B_s}\wedge\cdots\wedge dx^{B_1}.$$

By putting

(1. 6) $$\tilde{S}_{I_s I_{s-1}\cdots I_1}{}^{K}=(-1)^{s+1}\tilde{S}_{I_s I_{s-1} I_1 B}\varepsilon^{BK},$$

we can define a tensor field belonging to ${}'\tilde{\mathfrak{T}}^1_s$ whose components in $\pi^{-1}(U)$ are given by (1. 6). We call this tensor field the complete lift of S and denote by S^C. By (1. 6), the components $\tilde{S}_{I_s I_{s-1}\cdots I_1}{}^{K}$ of S^C in $\pi^{-1}(U)$ are expressible as follows (cf. [3], for $s=1, 2$):

$$\tilde{S}_{i_s i_{s-1}\cdots i_1}{}^{k}=S_{i_s i_{s-1}\cdots i_1}{}^{k},$$
$$\tilde{S}_{i_s\cdots \bar{i}_h\cdots i_1}{}^{k}=\cdots=\tilde{S}_{\bar{i}_s \bar{i}_{s-1}\cdots \bar{i}_1}{}^{k}=0,$$
(1. 7)
$$\tilde{S}_{i_s i_{s-1}\cdots i_1}{}^{\bar{k}}=p_a\left(\sum_{h-1}^{s}\partial_{i_h}S_{i_s\cdots k\cdots i_1}{}^{a}-\partial_k S_{i_s i_{s-1}\cdots i_1}{}^{a}\right),$$
$$\tilde{S}_{i_s\cdots \bar{i}_h\cdots i_1}{}^{\bar{k}}=S_{i_s\cdots k\cdots i_1}{}^{i_h},$$
$$\tilde{S}_{i_s\cdots \bar{i}_l\cdots \bar{i}_h\cdots i_1}{}^{\bar{k}}=\cdots=\tilde{S}_{\bar{i}_s \bar{i}_{s-1}\cdots \bar{i}_1}{}^{\bar{k}}=0.$$

Suppose that M has a symmetric affine connection Γ whose components in U

are Γ_{ji}^h and ∇ denotes the covariant derivative with respect to Γ. For a tensor field S belonging to $'\mathcal{T}_s^1$ whose components in U are $S_{i_s i_{s-1}\cdots i_1}{}^k$, we here put

$$[\nabla S]_{i_s\cdots i_1 k}{}^a=\sum_{h=1}^{s}\nabla_{i_h}S_{i_s\cdots k\cdots i_1}{}^a-\nabla_k S_{i_s\cdots i_1}{}^a. \tag{1.8}$$

Since the tensor field $[\nabla S]$ with components in U given by (1. 8) belongs to $'\mathcal{T}_{s+1}^1$, $[\nabla S]^V$ belongs to $'\tilde{\mathcal{T}}_s^1$. We now put

$$S^H=S^C-[\nabla S]^V(\in{}'\tilde{\mathcal{T}}_s^1) \tag{1.9}$$

and call S^H the horizontal lift of S (cf. [4], for $s=1$).

In the sequel, whenever we say the horizontal lifts or the covariant derivatives, we suppose that M has a symmetric affine connection Γ.

§ **2.** Differential concomitants of tensor fields in base manifolds.

Let S be a tensor field belonging to $'\mathcal{T}_s^1$ with components $S_{i_s\cdots i_1}{}^k$ in U and T a tensor field belonging to $'\mathcal{T}_t^1$ with components $T_{j_t\cdots j_1}{}^k$ in U. Suppose that ST belongs to $'\mathcal{T}_{s+t-1}^1$, where

$$ST(X_1,\cdots,X_{s-1},Y_1,\cdots,Y_t)=S(X_1,\cdots,X_{s-1},T(Y_1,\cdots,Y_t))$$

for any $X_1,\cdots,X_{s-1},Y_1,\cdots,Y_t\in\mathcal{T}_0^1$. For the tensor fields S and T, we define an operator Φ^C which makes a new tensor field belonging to $\tilde{\mathcal{T}}_{s+t-1}^1$ by

$$\Phi^C(S,T)=(ST)^C-S^C T^C. \tag{2.1}$$

When we denote components of $\Phi^C(S,T)$ in $\pi^{-1}(U)$ by $\Phi^C_{I_s\cdots I_2 J_t\cdots J_1}{}^{I_1}$, (2. 1) is expressible as follows:

$$\Phi^C_{i_s\cdots i_2 j_t\cdots j_1}{}^{\bar{i}_1}=p_b\{(S,T)_{i_s\cdots i_1 j_t\cdots j_1}{}^b+[ST]_{i_s\cdots i_1 j_t\cdots j_1}{}^b\}, \tag{2.2}$$

$$\Phi^C_{i_s\cdots i_2 j_t\cdots \bar{j}_l\cdots j_1}{}^{\bar{i}_1}=S_{i_s\cdots i_2 a}{}^{j_l}T_{j_t\cdots i_1\cdots j_1}{}^a-S_{i_s\cdots i_1}{}^a T_{j_t\cdots a\cdots j_1}{}^{j_l} \tag{2.3}$$

and other remaining components of $\Phi^C_{i_s\cdots I_2 J_t\cdots J_1}{}^{I_1}$ are all zero, where

$$\begin{aligned}(S,T)_{i_s\cdots i_1 j_t\cdots j_1}{}^b=&\,S_{i_s\cdots i_1}{}^a\partial_a T_{j_t\cdots j_1}{}^b-T_{j_t\cdots j_1}{}^a\partial_a S_{i_s\cdots i_1}{}^b\\&-\sum_{h=1}^{s}S_{i_s\cdots a\cdots i_1}{}^b\partial_{i_h}T_{j_t\cdots j_1}{}^a+\sum_{l=1}^{t}T_{j_t\cdots a\cdots j_1}{}^b\partial_{j_l}S_{i_s\cdots i_1}{}^a\end{aligned} \tag{2.4}$$

and

$$[ST]_{i_s\cdots i_1 j_t\cdots j_1}{}^b=\sum_{l=1}^{t}\partial_{j_l}(S_{i_s\cdots i_2 a}{}^b T_{j_t\cdots i_1\cdots j_1}{}^a-S_{i_s\cdots i_1}{}^a T_{j_t\cdots a\cdots j_1}{}^b). \tag{2.5}$$

REMARK. The notation $(S,T)_{i_s\cdots i_1 j_t\cdots j_1}{}^k$ is the generalization of what was introduced by Yano and Akō for $s=1.2$ (cf. [2]).

If conditions

$$\Phi^C_{i_s\cdots i_2 j_t\cdots \bar{j}_l\cdots j_1}{}^{i_1}=0 \qquad (l=1, 2, \cdots, t)$$

are satisfied, then $(ST)_{i_s\cdots i_1 j_t\cdots j_1}{}^k=0$ and consequently we can see that $(S, T)_{i_s\cdots i_1 j_t\cdots j_1}{}^k$ are components of a tensor field belonging to $\mathcal{T}^1_{s+t}$ by virtue of Lemma 1. 2. We denote this tensor field by (S, T).

Thus we have

PROPOSITION 2. 1. *Let S and T be tensor fields belonging to $'\mathcal{T}^1_s$ and $'\mathcal{T}^1_t$ respectively. Suppose that ST belongs to $'\mathcal{T}^1_{s+t-1}$. Then*

$$\Phi^C(S, T)=(S, T)^V_{(t+1)}, \qquad (S, T)\in\mathcal{T}^1_{s+t}$$

if and only if $\Phi^C_{i_s\cdots i_2 j_t\cdots \bar{j}_l\cdots j_1}{}^{i_1}$ $(l=1, 2, \cdots, t)$ vanish, that is,

$$S_{i_s\cdots i_2 a}{}^{j_l}T_{j_t\cdots i_1\cdots j_1}{}^a-S_{i_s\cdots i_1}{}^aT_{j_t\cdots a\cdots j_1}{}^{j_l}=0 \qquad (l=1, 2, \cdots, t). \tag{2. 6}$$

If we here use the horizontal lift in stead of the complete lift in (2. 1), then we have

$$\begin{aligned}\Phi^H(S, T)&=(S, T)^H-S^HT^H\\&=\Phi^C(S, T)-\{'(S, T)+'[S, T]\}^V_{(t+1)}\end{aligned} \tag{2. 7}$$

by (1. 4), (1. 7), (1. 8) and (1. 9), where $'(S, T)$ is a tensor field belonging to $\mathcal{T}^1_{s+t}$, whose components are given by

$$\begin{aligned}'(S, T)_{i_s\cdots i_1 j_t\cdots j_1}{}^k=&S_{i_s\cdots i_1}{}^a\nabla_aT_{j_t\cdots j_1}{}^k-T_{j_t\cdots j_1}{}^a\nabla_aS_{i_s\cdots i_1}{}^k\\&-\sum_{h=1}^{s}S_{i_s\cdots a\cdots i_1}{}^k\nabla_{i_k}T_{j_t\cdots j_1}{}^a+\sum_{l=1}^{t}T_{j_t\cdots a\cdots j_1}{}^k\nabla_{j_l}S_{i_s\cdots i_1}{}^a,\end{aligned} \tag{2. 8}$$

and $'[ST]$ is a tensor field belonging to $\mathcal{T}^1_{s+t}$ whose components are given by

$$'[ST]_{i_s\cdots i_1 j_t\cdots j_1}{}^k=\sum_{l=1}^{t}\nabla_{j_l}(S_{i_s\cdots i_2 a}{}^kT_{j_t\cdots i_1\cdots j_1}{}^a-S_{i_s\cdots i_1}{}^aT_{j_t\cdots a\cdots j_1}{}^k). \tag{2. 9}$$

If condition (2. 6) is satisfied, then $[ST]='[ST]=0$ and, by virtue of Proposition 2. 1, $(S, T)='(S, T)$, from which and (2. 7), we have $\Phi^H(S, T)=0$. Conversely, if $\Phi^H(S, T)=0$, then (2. 6) is clearly satisfied.

Thus we have

PROPOSITION 2. 2. *Let S and T be tensor fields belonging to $'\mathcal{T}^1_s$ and $'\mathcal{T}^1_t$ respectively. Suppose that ST belongs to $'\mathcal{T}^1_{s+t-1}$. Then*

$$\Phi^H(S, T)=0$$

if and only if condition (2. 6) *is satisfied.*

Now, for elements S of $'\mathcal{T}^1_s$, T of $'\mathcal{T}^1_t$ and any $Y_1, \cdots, Y_t\in\mathcal{T}^1_0$, we define a tensor field $\Phi^C_{(Y)}(S, T)$ belonging to $\tilde{\mathcal{T}}^1_{s-1}$ by

$$\Phi^C_{(Y)}(S, T)(\tilde{X}_s, \cdots, \tilde{X}_2)=\Phi^C(S, T)(\tilde{X}_s, \cdots, \tilde{X}_2, Y^C_t, \cdots, Y^C_1), \tag{2. 10}$$

where $\tilde{X}_2, \cdots, \tilde{X}_s \in \tilde{\mathcal{T}}^1_0$.

If we denote components of $\Phi^C_{(Y)}(S, T)$ by $\Phi^C_{(Y)I_s \cdots I_2}{}^{I_1}$, then, by (2. 10),

$$\Phi^C_{(Y)I_s \cdots I_2}{}^{I_1} = \Phi^C_{I_s \cdots I_2 B_t \cdots B_1}{}^{I_1} \tilde{Y}^{B_t}_t \cdots \tilde{Y}^{B_1}_1, \tag{2. 11}$$

where $\tilde{Y}^{B_h}_h$ are components of the complete lift Y^C_h of Y_h $(h=1, 2, \cdots, t)$. From (1. 7), (2. 2), (2. 3) and (2. 11), we can see that $\Phi^C_{(Y)I_s \cdots I_2}{}^{I_1}$ are expressible as follows:

$$\Phi^C_{(Y)I_s \cdots I_2}{}^{i_1} = \Phi^C_{(Y)i_s \cdots \bar{i}_h \cdots i_2}{}^{\bar{i}_1} = \cdots = \Phi^C_{(Y)\bar{i}_s \bar{i}_{s-1} \cdots \bar{i}_2}{}^{\bar{i}_1} = 0,$$

$$\Phi^C_{(Y)i_s \cdots i_2}{}^{\bar{i}_1} = p_b(R_{i_s \cdots i_1 j_t \cdots j_1}{}^b Y^{j_t}_t \cdots Y^{j_1}_1) + p_b('[ST]_{i_s \cdots i_1 j_t \cdots j_1}{}^b Y^{j_t}_t \cdots Y^{j_1}_1)$$

$$-p_b \sum_{l=1}^{t} \{(S_{i_s \cdots i_2 a}{}^{j_l} T_{j_t \cdots i_1 \cdots j_1}{}^a - S_{i_s \cdots i_1}{}^a T_{j_t \cdots a \cdots j_1}{}^{j_l}) Y^{j_t}_t \cdots (\nabla_{j_l} Y^b_l) \cdots Y^{j_1}_1\},$$

where we put

$$R_{i_s \cdots i_1 j_t \cdots j_1}{}^k = (S, T)_{i_s \cdots i_1 j_t \cdots j_1}{}^k + A_{i_s \cdots i_1 j_t \cdots j_1}{}^k + B_{i_s \cdots i_1 j_t \cdots j_1}{}^k, \tag{2. 13}$$

$$A_{i_s \cdots i_1 j_t \cdots j_1}{}^k = \sum_{l=1}^{t} \Gamma^C_{j_l i_1}(S_{i_s \cdots i_2 a}{}^k T_{j_t \cdots C \cdots j_1}{}^a - S_{i_s \cdots i_2 C}{}^a T_{j_t \cdots a \cdots j_1}{}^k) \tag{2. 14}$$

and

$$B_{i_s \cdots i_1 j_t \cdots j_1}{}^k = \sum_{l=1}^{t} \sum_{h=2}^{s} \Gamma^C_{j_l i_h}(S_{i_s \cdots i_2 a}{}^k T_{j_t \cdots i_1 \cdots j_1}{}^a - S_{i_s \cdots C \cdots i_1}{}^a T_{j_t \cdots a \cdots j_1}{}^k). \tag{2. 15}$$

Since

$$'[ST]_{i_s \cdots i_1 j_t \cdots j_1}{}^k Y^{j_t}_t \cdots Y^{j_1}_1$$

and

$$\sum_{l=1}^{t} \{(S_{i_s \cdots i_2 a}{}^{j_l} T_{j_t \cdots i_1 \cdots j_1}{}^a - S_{i_s \cdots i_1}{}^a T_{j_t \cdots a \cdots j_1}{}^{j_l}) Y^{j_t}_t \cdots (\nabla_{j_l} Y^k_l) \cdots Y^{j_1}_1\}$$

are components of tensor fields belonging to $\mathcal{T}^1_s$ respectively, by virtue of Lemma 1. 2, we can see that $R_{i_s \cdots i_1 i_t \cdots j_1}{}^k$ are components of a tensor field belonging to $\mathcal{T}^1_{s+t}$.

Thus we have

PROPOSITION 2. 3. *Suppose that M has a symmetric affine connection Γ and ∇ denotes the covariant derivative with respect to Γ. Let S, T and ST be tensor fields belonging to $'\mathcal{T}^1_s$, $'\mathcal{T}^1_t$ and $'\mathcal{T}^1_{s+t-1}$ respectively. Then* (2. 13) *defines a tensor field R belonging to $\mathcal{T}^1_{s+t}$ and $R_{i_s \cdots i_1 j_t \cdots j_1}{}^k$ are components of R.*

Now, in (2. 3) we make the skew-symmetric part with respect to covariant indices $i_s, \cdots, i_1, j_t, \cdots, j_1$. Then we can see easily that

$$A_{[i_s \cdots i_1 j_t \cdots j_1]}{}^k = B_{[i_s \cdots i_1 j_t \cdots j_1]}{}^k = 0,$$

from which and Proposition 2. 3, we see that $(S, T)_{[i_s \cdots i_1 j_t \cdots j_1]}{}^k$ define a tensor field belonging to $'\mathcal{T}^1_{s+t}$. Moreover we have

$$\frac{(s+t)!}{s!t!}(S, T)_{[i_s\cdots i_1 j_t\cdots j_1]}{}^k=[S, T]_{i_s\cdots i_1 j_t\cdots j_1}{}^k,$$

where $[S, T]$ is the notation induced by Frölicher and Nijenhuis (cf. [1]).
Thus we nave

COROLLARY 2. 4. *Under the same suppositions as in Proposition* 2. 3, $(S, T)_{[i_s\cdots i_1 j_t\cdots j_1]}{}^k$ *define a tensor field belonging to* $'\mathcal{T}^1_{s+t}$ *and*

$$\frac{(s+t)!}{s!t!}(S, T)_{[i_s\cdots i_1 j_t\cdots j_1]}{}^k=[S, T]_{i_s\cdots i_1 j_t\cdots j_1}{}^k.$$

REMARK. In the case where $s=1$, we denote $S^k_{i_1}$ by F^k_i. If F is an almost complex structure in M, then we can see that the condition (2. 6) is equivalent to the condition that T is pure. A pure tensor field T is said to be almost analytic, if $(F, T)=0$. Consequently, for $T\in{}'\mathcal{T}^1_t$, $\Phi^C(F, T)=0$ if and only if T is pure and almost analytic. If T is pure, then $\Phi^H(F, T)=0$.

Next, let σ be a tensor field belonging to $'\mathcal{T}^0_s$ with components $\sigma_{i_s\cdots i_1}$ in U and T a tensor field $\in{}'\mathcal{T}^1_t$ with components $T_{j_t\cdots j_1}{}^k$ in U. Suppose that $\sigma\circ T$ belongs to $'\mathcal{T}^0_{s+t-1}$, where

$$\sigma\circ T(X_1, \cdots, X_t, Y_1, \cdots, Y_{s-1})=\sigma(T(X_1, \cdots, X_t), Y_1, \cdots, Y_{s-1})$$

for any $X_1, \cdots, X_t, Y_1, \cdots, Y_{s-1}\in\mathcal{T}^1_0$. For the tensor fields σ and T, we define an operator Ψ which makes a new tensor field belonging to $\mathcal{T}^1_{s+t-1}$ by

$$\Psi(\sigma, T)=(\sigma\circ T)^*-\sigma^*\circ T^C, \tag{2. 16}$$

where

$$\sigma^*=(-1)^{s+1}(d\sigma)^V \qquad \text{and} \qquad (\sigma\circ T)^*=(-1)^{s+t}(\alpha(\sigma\circ T))^V.$$

When we denote by $\Psi_{J_t\cdots J_1 I_{s-1}\cdots I_1}{}^{I_s}$ the components of $\Psi(\sigma, T)$ in $\pi^{-1}(U)$, (2. 16) is expressible as follows:

$$\Psi_{j_t\cdots j_1 i_{s-1}\cdots i_1}{}^{\bar{i}_s}=-(T, \sigma)_{j_t\cdots j_1 i_s\cdots i_1} \tag{2. 17}$$

and other components $\Psi_{J_t\cdots J_1 I_{s-1}\cdots I_1}{}^{I_s}$ are all zero, where

$$(T, \sigma)_{j_t\cdots j_1 i_s\cdots i_1}=T_{j_t\cdots j_1}{}^a\partial_a\sigma_{i_s\cdots i_1}-\sum_{l=1}^{t}\partial_{j_l}(\sigma\circ T)_{j_t\cdots i_s\cdots j_1 i_{s-1}\cdots i_1}+\sum_{h=1}^{s}\sigma_{i_s\cdots a\cdots i_1}\partial_{i_h}T_{j_t\cdots j_1}{}^a. \tag{2. 18}$$

REMARK. The notation $(T, \sigma)_{j_t\cdots j_1 i_s\cdots i_1}$ is the generalization of what was introduced by Yano and Akō for $t=1, 2$ (cf. [2]).

By making use of Lemma 1. 1, we can see that $(T, \sigma)_{j_t\cdots j_1 i_s\cdots i_1}$ are components of a tensor field belonging to $\mathcal{T}^0_{s+t}$.
Thus we have

PROPOSITION 2. 5. *Let* σ *and* T *be tensor fields belonging to* $'\mathcal{T}_s^0$ *and* $'\mathcal{T}_t^1$ *respectively. Suppose that* $\sigma\circ T$ *belongs to* $'\mathcal{T}_{s+t-1}^0$. *Then*

$$\Psi(\sigma, T)=-(T,\sigma)_{(s)}^V, \qquad (T,\sigma)\in\mathcal{T}_{s+t}^0.$$

REMARK. In the case where $t=1$, we denote $T_{j_1}^k$ by F_j^k. If F is an almost complex structure in M, then we can see that the condition that $\sigma\circ F$ belongs to $'\mathcal{T}_s^0$ is equivalent to the condition that σ is pure. A pure tensor field σ is called to be almost analytic, if $(F,\sigma)=0$. Consequently, for a tensor field σ belonging to $'\mathcal{T}_s^0$, $\Psi(\sigma, F)=0$ if and only if σ is pure and almost analytic.

REMARK. We can verify that

$$(T,\sigma)_{[j_t\cdots j_1 i_s\cdots i_1]}=\frac{s!t!}{(s+t)!}[T,\sigma]_{j_t\cdots j_1 i_s\cdots i_1}-(-1)^{t-1}\frac{t}{s+t}[I,\sigma\circ T]_{j_t\cdots j_1 i_s\cdots i_1}$$

where $[T,\sigma]$ and $[I,\sigma\circ T]$ are the notations introduced by Frölicher and Nijenhuis (cf. [1]) and I is the unit tensor with components δ_j^k.

The author wishes to express his hearty thanks to Prof. K. Yano and Prof. S. Ishihara for their kind help during the preparation of this paper and also his thanks deeply to Prof. I. Sato for his kind and valuable advices.

BIBLIOGRAPHY

[1] FRÖLICHER, A., AND A. NIJENHUIS, Some new cohomology invariant for complex manifold I. Proc. Kon. Ned. Wet. Amsterdam **59** (1956), 540-552.

[2] YANO, K., AND M. AKŌ, On certain operators associated with tensor fields. Kōdai Math. Sem. Rep. **20** (1968), 414-436.

[3] YANO, K., AND E. M. PATTERSON, Vertical and complete lifts from a manifold to its cotangent bundle. J. Math. Soc. Japan **19** (1967) 91-113.

[4] YANO, K., AND E. M. PATTERSON, Horizontal lifts from a manifold to its cotangent bundle. J. Math. Soc. Japan **19** (1967), 185-198.

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