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Tensor Force of the Pion-Theoretical Potential and the Doublet Splitting in n-He^4 Scattering

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In order to investigate whether the spin-orbit force in the theory of nuclear shell structure is due to the tensor force of the pion-theoretical potential, the doublet splitting of the p-phase shifts in low energy n-He^4 scattering is analysed. We get the conclusion that the major part of the experimental doublet splitting can be reproduced by the strong tensor force of the pion-theoretical potential. Also it is shown qualitatively what features of the pion-theoretical potential are important to the binding energy of He^4 and the discontinuity of the binding energies between He^4 and the system of He^4 plus one nucleon.

§ 1. Introduction

Recent developments in researches on nuclear forces, made it clear that all the nucleon-nucleon phenomena up to about 150 Mev are well explained by the pion theory of nuclear forces[1]. The pion-theoretical potential has been established quantitatively in the outer region ($r \gtrsim 2\times10^{-13}$ cm) and at least qualitatively in the intermediate region ($r \simeq 1 \sim 2\times10^{-13}$ cm), where r is the inter-nueleon distance. The inner part ($r \lesssim 1\times10^{-13}$ cm) of nucleon-nucleon interaction, to which the present day pion theory can not give any reliable prediction, has been determined by the comparison with experimental data. Thus we know of the phenomenological effective potential corresponding to main features of nucleon-nucleon interaction in this inner ragion. At present, our knowledge about nuclear forces is sufficient to attack problems of nuclei on the basis of two-body interaction.

The characteristic features of the pion theoretical potential differ essentially from those of phenomenological potentials conventionally adopted so far as will be shown in § 2. The most remarkable one of them is the strong tensor force due to one-pion-exchange process. It is thus very interesting to investigate the relations between the strong tensor force and characteristic properties of nuclei. However, there has yet been no attempt taking into account this feature of two-body interaction. One of important problems in connection with the strong tensor force is whether the spin-orbit coupling force in the shell model can be accounted for by this strong tensor force*.

* As will be discussed in § 2, the two-body spin-orbit potential predicted by the pion theory is too weak to produce the spin-orbit potential in the shell model.

At the present stage, there exist many difficulties in general treatment of this strong tensor force, because one has to take account of the following situations: The mixing of states plays an important role in this case, hence the perturbational approach becomes questionable. Furthermore, significant contributions from the tensor force may sppear at the nuclear surface, and it is desirable to treat the nucleus as a finite system not as an infinite medium.

In the case of lightest nuclei, the above mentioned difficulties do not appear and we can treat the problems directly by adopting the pion-theoretical potential.

The investigation of the doublet splitting of He^5 and Li^5 is the crucial test of the problem whether the strong tensor force is the origin of the spin-orbit coupling force in the shell model. The effects of the spin-orbit force in He^5 and Li^5 appear in the most direct fashion as the wide splitting of the doublet p-phase shifts in the low energy nucleon scattering by He^4. The main purpose of this work is to investigate qualitatively the relation between this wide doublet splitting and the strong tensor force of the pion-theoretical potential. In the course of this investigation we also consider the binding energy of He^4 and the reason why there is no bound state in the system of He^4 plus one nucleon, while He^4 is a tightly bound system. This feature is closely related to the binding energy discontinuity at the closed shell.

Many authors investigated the effect of a tensor force on the doublet splitting. Dancoff[2] estimated the doublet splitting of He^5 in the second order perturbation. Feingold[3] also calculated it in the variation-perturbational way. In these works, besides the defect of the perturbational approach, there exists the unsatisfactory point that the values of the splitting and even its sign depend seriously on the parameters of the wave function, for they treat He^5 as the bound system. In another type of approach[4], the doublet splitting is calculated on the basis of the Fermi gas model in the second order Born approximation with respect to a tensor potential or a modified tensor potential (t-matrix in Brueckner's theory). In these works, however, there are also unsatisfactory points in treating the nuclear surface effects or the mixing of states due to a tensor force.

Here, we follow the procedure developed by Sugie, Hodgson and Robertson[5]. This approach seems to be most reasonable for investigating the spin-orbit coupling force resulting from the tensor force in He^5 and Li^5. However, they got only the small splitting of the p-phase shifts (about 30% of the experimental value), because they used the phenomenological potential with a weaker tensor force than a central one. As discussed by Sugie et al., the main part of the interaction term responsible for the splitting of the phase shifts is proportional to the strength of a tensor force and the mixing ratio of the D-state of He^4 due to a tensor force. Therefore, the strong tensor force characteristic of the pion-theoretical potential is expected to account for the wide experimental splitting. Indeed, it is shown that the major part ($\sim$60%) of the experimental value of the splitting can be reproduced by the pion-theoretical potential.

It is to be noted that, although such a strong tensor force to reproduce the experimental splitting is believed to reduce the binding energy of He^4 utterly, the pion-theoretical potential gives its reasonable value mainly due to the strong attractive force of the two-pion-exchange potential in the singlet even state.

The essential difference between the work of Sugie et al. and ours lies in the properties of the two-body potential. In § 2, we shall show the characteristic features of the pion-theoretical potential. In § 3, we shall recapitulate the procedure deriving the spin-orbit coupling term and discuss the approximations used. The determination of the parameters of the He^4 wave function and the calculation of the binding energy of He^4 will be made in § 4. The qualitative features of the spin-orbit coupling term and the numerical results derived from the pion-theoretical potential will be presented in § 5. § 6 will be devoted to discuss various corrections affecting numerical results. In § 7, we shall summarize the main results obtained.

§ 2. Characteristic features of nucleon-nucleon interaction

In this section we summarize the characteristic features of nucleon-nucleon interaction clarified by the analyses on two-nucleon problems[1]. It should be noted that the features of the outer and intermediate parts ($r \gtrsim 1\times10^{-13}$ cm) have been established pion-theoretically, while those of the inner part ($r \lesssim 1\times10^{-13}$ cm) have been determined by the comparison with experimental data. These features are shown in Fig. 1.

(i) In the outer region ($r \gtrsim 2\times10^{-13}$ cm), the tensor potential is very strong compared with the central one. This feature, the most characteristic one of the pion-theoretical potential, results from the one-pion-exchange potential.

$$V^{(1\pi)}=\left(\frac{g_e^2}{4\pi}\right)\mu c^2\,\frac{(\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2)}{3}\left\{(\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2)+S_{12}\left(1+\frac{3}{\kappa r}+\frac{3}{(\kappa r)^2}\right)\right\}\frac{e^{-\kappa r}}{\kappa r}, \qquad (2\cdot1)$$

where

$$S_{12}=3r^{-2}(\boldsymbol{\sigma}_1\cdot\boldsymbol{r})\,(\boldsymbol{\sigma}_2\cdot\boldsymbol{r})-(\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2),$$

$$\kappa^{-1}=\hbar/\mu c=1.415\times10^{-13}\text{ cm}\quad(\mu\text{ is the pion mass})^*$$

and

$$g_e^2/4\pi\simeq 0.08.$$

The potential in this region is completely described by $V^{(1\pi)}$.

(ii) In the intermediate region ($r\simeq 1\sim2\times10^{-13}$ cm), the contributions from the two-pion-exchange potential $V^{(2\pi)}$ become important in addition to $V^{(1\pi)}$. The tensor part of $V^{(1\pi)}$ is important also in this region. The qualitative features of $V^{(2\pi)}$ have been verified, although there remain some quantitative ambiguities due to different choices of methods in the derivation of $V^{(2\pi)}$. The most essential feature of $V^{(2\pi)}$ is noticed in the strong attractive potentials in the central part of

* We adopt the value $\mu c^2=139.4$Mev.

the charge triplet states (^1E and ^3O)*. The central potentials in ^3E and ^1O and the tensor potential of $V^{(2\pi)}$ are not very effective and their effects can be expressed by the suppression of magnitude of $V^{(1\pi)}$ in this region. The comparison with experimental data also supports this feature[1].

(iii) The two-body spin-orbit coupling potential predicted by the pion theory is very small compared with the static potential in the intermediate region[6)7)8)]. This potential is not strong enough to produce the spin-orbit coupling force in the shell model. It is of the wrong sign in the recent calculation using the dispersion relation[8]. Also, the nucleon-nucleon scatterings up to about 150 Mev can be reproduced by the pion-theoretical potentials without spin-orbit potentials as predicted by Signell and Marshak[9] and by Gammel and Thaler[10]. Therefore, the two-body spin-orbit potential cannot play any essential role, and we can neglect its effect in qualitative discussions on problems of nuclei.

(iv) The main features of the exchange character of the pion-theoretical potential can be represented, from the properties (i) and (ii), as follows:

$$V=(\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2)(\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2)V_c^{(1\pi)}+(\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2)S_{12}V_t^{(1\pi)}$$
$$-\frac{3+(\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2)}{4}\left\{\frac{3+(\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2)}{4}V_c^{(2\pi)}(^3O)+\frac{1-(\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2)}{4}V_c^{(2\pi)}(^1E)\right\}. \qquad (2\cdot2)$$

$V_{c,t}^{(1\pi)}$ are the radial parts of $V^{(1\pi)}$. $V_c^{(2\pi)}(^3O)$ and $V_c^{(2\pi)}(^1E)$ are those of the two-pion-exchange-central potentials in ^3O and ^1E respectively. The minus sign in the third term is added to make $V_c^{(2\pi)}(^3O)$ and $V_c^{(2\pi)}(^1E)$ positive for convenience's sake.

(v) In the inmost part ($r\lesssim0.5\times10^{-13}$cm), there exists the hard-core-like repulsive interaction in all the states. Through comparison with experiments, it has been shown that the effective potential just outside this hard-core can be roughly given by the straight cut-off potential of $V^{(1\pi)}+V^{(2\pi)}$ at the region, $r\cong0.5\sim1\times10^{-13}$ cm[1].

In choosing the detailed forms of $V_c^{(1\pi)}$, etc., in the intermediate and inner regions, the following should be taken into account. In ^3E, $V_t^{(2\pi)}$ is very small and $V^{(1\pi)}$ with the hard-core cut-off is the most reasonable potential for reproducing the deuteron data[1a)11)]. As will be discussed in § 3, $V_t(^3E)$ plays an essential role in the phenomena of He^4 and the system of He^4 plus one nucleon, while $V_t(^3O)$ does not. So, in the case of the present paper, it is allowable to take $V_t=V_t^{(1\pi)}$ in the region, $r\gtrsim1\times10^{-13}$ cm**. $V_c^{(2\pi)}(^1E)$ is stronger than $V_c^{(2\pi)}(^3O)$, i. e. $V_c^{(2\pi)}(^1E)/V_c^{(2\pi)}(^3O)\cong3/2$[1a)11)].

Thus on the basis of the features shown in (i)$\sim$(v) and the situations discussed above, we adopt the potential given by (2·2) as the two-body potential outside the core region. For convenience of analyses, we take the following ap-

* We use the following simple notation for the classification of the two-nucleon states; ^3E (the triplet even state), ^1E (the singlet even state), ^3O (the triplet odd state) and ^1O (the singlet odd state).

** Strictly speaking, $V_t^{(1\pi)}(^3O)$ should be modified due to $V_t^{(2\pi)}(^3O)$ so as to be damped in the region, $r\lesssim1.4\times10^{-13}$cm[1f)].

proximate form* to the pion-theoretical potential.

$$V_c^{(1\pi)}=v_c^{(1)}\,r^2\exp[-\mu^{(1)}\,r^2],$$
$$V_t^{(1\pi)}=v_t^{(L)}\,r^2\exp[-\nu^{(L)}\,r^2]+v_t^{(S)}\,r^2\exp[-\nu^{(S)}\,r^2], \qquad (2\cdot3)$$
$$V_c^{(2\pi)}(^1E)=v_c^{(2)}\,r^2\exp[-\mu^{(2)}\,r^2] \text{ and } V_c^{(2\pi)}(^3O)=(2/3)\,V_c^{(2\pi)}(^1E)$$

$v_c^{(1)}=6.86$, $v_c^{(2)}=700$, $v_t^{(L)}=7.45$ and $v_t^{(S)}=456$ (in unit of Mev$\times10^{26}$ cm^{-2}),

$\mu^{(1)}=0.600$, $\mu^{(2)}=1.94$, $\nu^{(L)}=0.388$ and $\nu^{(S)}=1.76$ (in unit of 10^{26} cm^{-2}).

In order to obtain a good approximate form to $V_t^{(1\pi)}$, we use the sum of potentials with different ranges. The long range part corresponds to $(1/3)\,(g_e^2/4\pi)\,\mu c^2\,(e^{-\varkappa r}/\kappa r)$ and the short range part to the remainings. The errors caused by these approximations are very small (several per cent) except at the tail ($r\gtrsim3\times10^{-13}$ cm), where the potentials themselves are vanishingly small, as shown in Fig. 1.**

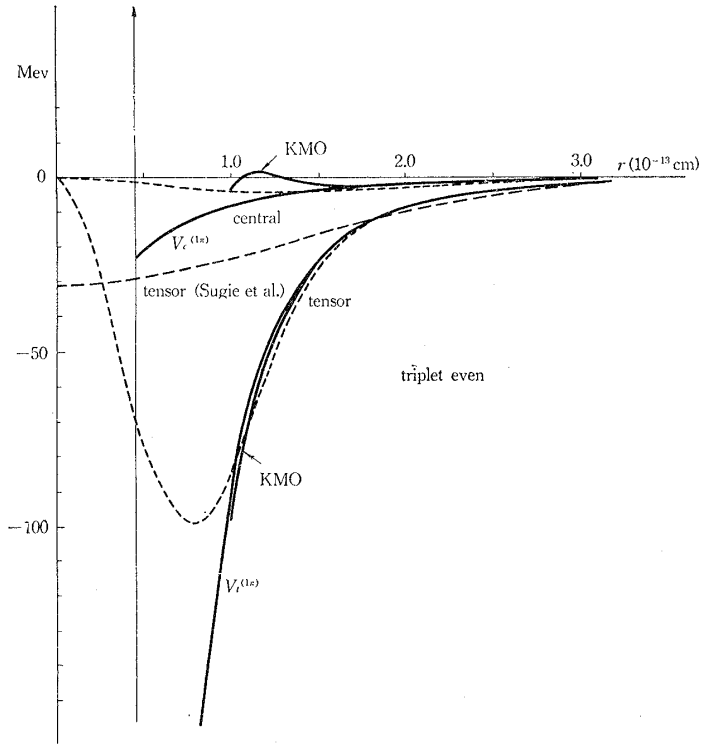

Fig. 1 (a)

* The reason why we use such a particular form comes from the following situation: If we choose this form, we can analytically perform the calculation of the binding energy of He^4 and the derivation of the interaction kernels in n-He^4 scattering. Otherwise cumbersome numerical calculations are needed after eliminating exactly the motion of center of mass. Including the r^2-factor, we can avoid the procedure of Eq. (26) in Sugie et al. The r^2-factor plays the role as damping factor in the inner region, which corresponds to the situation discussed in (v). Also, we can avoid overestimation of the contributions of the potential in the core region ($r\lesssim0{\cdot}5\times10^{-13}$cm) to the binding energy of He^4 and the scattering potential in n-He^4, even in the case where no short-range correlation function is introduced.

** Because the results depend on the overlap integral of the forces and the wave function, the errors in the results that arise from the small incorrectness of the tail may be negligible.

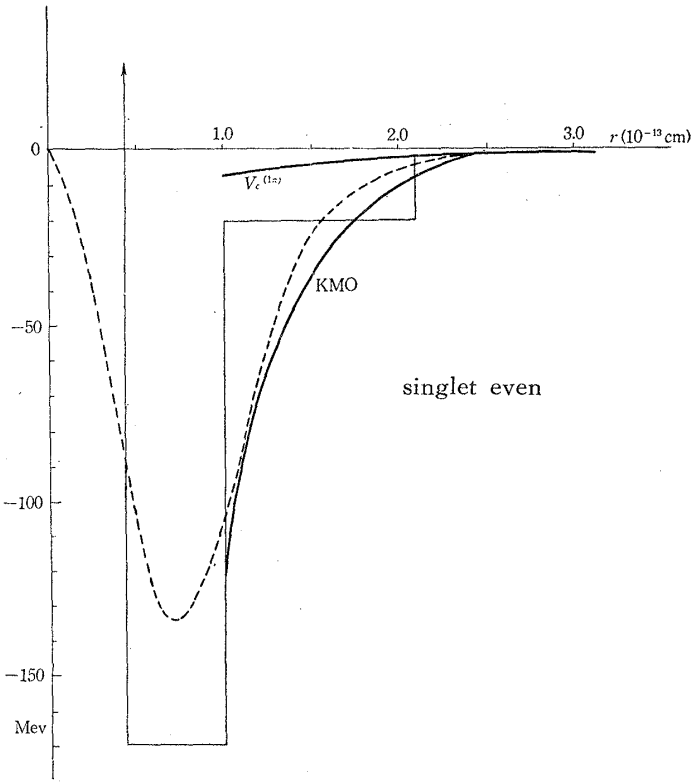

Fig. 1 (b)

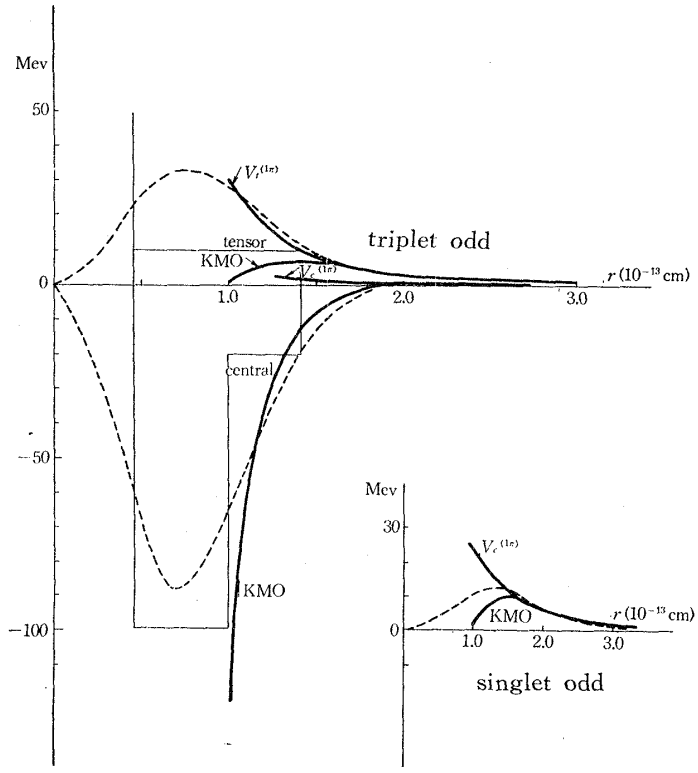

Fig. 1 (c) Fig. 1 (d)

Fig. 1. Two-body potential. $V_c^{(1\pi)}$ and $V_t^{(1\pi)}$ are the central and tensor parts of the one-pion-exchange potential, respectively. The curves denoted by KMO are the one- plus two-pion-exchange potentials of ref. lb). For other pion-theoretical potentials, see ref. la). Also, the effective potentials in the inner region determined by comparison with the experimental data are shown. The dotted curves are the approximate forms given by (2·3) to the pion-theoretical potential.

1(a) Potential in the triplet even state.
1(b) Potential in the singlet even state.*
1(c) Potential in the triplet odd state.*
1(d) Potential in the singlet odd state.

* The square well potentials with the tails of the one-pion-exchange potential are the effective potentials which explain the experimental data.

In the present paper, we perform the calculation using the exchange operators instead of $(\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2)$ and $(\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2)$. If we use the conventional notation for the exchange operators,

$$V=\sum_{i=1,2}(w^{(i)}+m^{(i)}P_M+b^{(i)}P_B+h^{(i)}P_H)\,V_c^{(i\pi)}(r)+S_{12}(w^{(t)}+m^{(t)}P_M)\,V_t^{(1\pi)}(r) \tag{2·4}$$

and put

$$V_c^{(2\pi)}(r)\equiv V_c^{(2\pi)}({}^1E),$$

the exchange character given by (2·2) is rewritten as follows:

$$\begin{aligned} &V_c^{(1\pi)}:\ w^{(1)}=1/3,\quad m^{(1)}=-4/3,\quad b^{(1)}=-2/3\ \text{ and }\ h^{(1)}=2/3,\\ &V_c^{(2\pi)}:\ w^{(2)}=-5/12,\ m^{(2)}=-1/12,\ b^{(2)}=1/12\ \text{ and }\ h^{(2)}=5/12,\\ &V_t^{(1\pi)}:\ w^{(t)}=-1/3\ \text{ and }\ m^{(t)}=-2/3 \end{aligned} \tag{2·5}$$

§ 3. Spin-orbit term

In this section, we outline the method to derive the spin-orbit term*. We discuss what is important to cause the wide splitting of the p-phase shifts in n-He^4 scattering.

First of all, we assume that He^4 remains in the ground state during the scattering process. This will be justified in n-He^4 scattering at low energy ($\lesssim$5 Mev), because the first excited state of He^4 is believed to be very high**. Hereafter, particles 1, 2 and 3 are neutrons, and 4 and 5 are protons. The totally antisymmetric wave function Ψ of the system is then written in the form

$$\Psi=\psi(-1)\phi(1)+\psi(-2)\phi(2)+\psi(-3)\phi(3), \tag{3·1}$$

where $\psi(-i)$ is the antisymmetric wave function of He^4 which does not contain the i-th neutron and $\phi(i)$ describes the i-th neutron in the scattering state. The ground state of He^4 is considered to be principally the ${}^1S_0$-state. Due to the tensor force it has a small admixture of ${}^5D_0$-states. As the first approximation we neglect the ${}^3P_0$-state probability. And, of all the possible spin-angular wave functions, only the principal ${}^1S_0$- and ${}^5D_0$-state wave functions are considered[12]. We take this approximation, because our main aim is to see the qualitative feature which the pion-theoretical potential shows in the lightest nuclei. Then the wave function of He^4 is of the form

* The method of derivation is the same as the one given by Sugie et al.[5], on the whole. Hence our notations follow what they used in most of the cases. Sometimes, we may omit the description of meanings of notations when they seem obvious. The reader who is not familiar with the notations is advised to see the paper by Sugie et al.

** This assumption is justified by the following experimental data. The behavior of the $p_{3/2}$ phase shift is well accounted for by the one level formula. The proton reduced width is above 75% of the sum rule limit. (R. K. Adair, Phys. Rev. **86** (1952), 155; D. C. Dodder and J. L. Gammel, Phys. Rev. **88** (1952), 520.) The $p_{1/2}$ level is much broader. This shows that for E<7 Mev, the present assumption is not so bad.

$$\psi(-1)=\{g_S\chi(\widetilde{23},\ \widetilde{45})+Cg_D w_D\chi(\widetilde{23},\ \widetilde{45})\}/\sqrt{1+C^2}, \tag{3·2}$$

where g_S and g_D represent the normalized spatial parts of the wave functions for the principal $^1S_0$- and $^5D_0$-state, respectively. χ is $\frac{1}{2}\{\alpha(2)\beta(3)-\beta(2)\alpha(3)\}\{\alpha(4)\beta(5)-\beta(4)\alpha(5)\}$ and, if the operand is χ,

$$\begin{aligned} w_D&=\sum_{i>j=2}^{5} r_{ij}^2 S_{ij} \\ &=3(\boldsymbol{\sigma}_2\cdot\boldsymbol{r}_{23})(\boldsymbol{\sigma}_4\cdot\boldsymbol{r}_{45})+3(\boldsymbol{\sigma}_2\cdot\boldsymbol{r}_{45})(\boldsymbol{\sigma}_4\cdot\boldsymbol{r}_{23})-2(\boldsymbol{\sigma}_2\cdot\boldsymbol{\sigma}_4)(\boldsymbol{r}_{23}\cdot\boldsymbol{r}_{45}). \end{aligned} \tag{3·3}$$

The wave function ϕ is obtained from the Schrödinger equation of the n-He^4 system

$$(\sum_i T_i+\sum_{i>j} V_{ij})\Psi=(E_\alpha+E)\Psi. \tag{3·4}$$

Decomposing the wave function ϕ into partial waves

$$\phi(1)=\sum_{Jl}(f_{lJ}(r)/r)\chi_{lJ}^m(\theta\varphi,\ s), \tag{3·5}$$

we get a set of uncoupled differential equations, each of which is specified by two good quantum numbers, J and l, since we neglect the virtual excitation of He^4. The equation* for $f_{lJ}(r)$ is

$$\begin{aligned} &\frac{\hbar^2}{2M'}\left[\frac{d^2}{dr^2}-\frac{l(l+1)}{r^2}+k^2\right]f_{lJ}(r) \\ &\qquad=\int d\tau_{-1}\,d\Omega_1\,\bar{\psi}(-1)\bar{\chi}_{lJ}^m(1)\sum_{l=2}^{5}V_{1j}\psi(-1)\chi_{lJ}^m(1)f_{lJ}(r) \\ &+2\int d\tau_{-1}\,d\Omega_1\,\bar{\psi}(-1)\bar{\chi}_{lJ}^m(1)\sum_{j=2}^{5}V_{1j}\psi(-2)\chi_{lJ}(2)\frac{f_{lJ}(r')}{r'}r \\ &+2\frac{\hbar^2}{2M'}\int d\tau_{-1}\,d\Omega_1\,\bar{\psi}(-1)\bar{\chi}_{lJ}^m(1)[-\nabla_{1,2345}^2-k^2]\psi(-2)\chi_{lJ}^m(2)\frac{f_{lJ}(r')}{r'}r, \end{aligned} \tag{3·6}$$

where

$$\begin{aligned} &k^2=(2M'/\hbar^2)E, \qquad M'=(4/5)M, \\ &\boldsymbol{r}=\boldsymbol{r}_1-(1/4)(\boldsymbol{r}_2+\boldsymbol{r}_3+\boldsymbol{r}_4+\boldsymbol{r}_5) \\ &\boldsymbol{r}'=\boldsymbol{r}_2-(1/4)(\boldsymbol{r}_1+\boldsymbol{r}_3+\boldsymbol{r}_4+\boldsymbol{r}_5). \end{aligned} \tag{3·7}$$

and

The explicit form of $\psi(-1)$ is given in the next section. We calculate the S-S terms and the S-D cross terms, neglecting the effect of the D-D terms.

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* In deriving the equation, we assume $H_\alpha\psi(-1)=E_\alpha\psi(-1)\cdots$(a). Sugie et al. assumed only $(\psi H_\alpha\psi)=E_\alpha\cdots$(b). By doing so, they included the contribution from S_{24}- and S_{34} -terms in their Eq. (34). However, the splitting should be given only by S_{14}- and S_{15} -terms if we could find the exact wave function of He^4, and so these terms are of physical significance. Then we assume the relation (a), though in actual calculations we use the approximate form for $\psi(-1)$.

Equation (3·6) can be expressed in the following abbreviated form*

$$\left\{\frac{d^2}{dr^2}+k^2-\frac{l(l+1)}{r^2}\right\}f_{lJ}(r)=W(r)f_{lJ}(r)+\int k^l(r,\,r')f_{lJ}(r')\,dr'$$
$$+\int k_{Sp}^{lJ}(r,\,r')f_{lJ}(r')\,dr'. \qquad (3\cdot8)$$

The first term in the right-hand side is the so-called potential term. This term is derived from the central potentials without the space exchange operator in the direct terms and those with the space exchange operators in the antisymmetrized terms. The kernel of the second term arises from (1) the remaining parts of the central potentials in the direct and antisymmetrized terms, (2) the J-independent parts from the tensor force in the antisymmetrized terms and (3) the term related to the kinetic energy in the antisymmetrized terms. The J-dependence of the second term in the right-hand side of Eq. (3·8) is caused only by that of the wave function $f_{lJ}(r)$. Hence this term does not give direct contribution to the spin-orbit splitting. The kernel $k_{Sp}^{lJ}(r,\,r')$ itself in the last term depends on J, hence it contributes directly to the splitting. In the following, we discuss this term in detail.

First of all we must mention that the direct tensor terms do not give any contribution. Then the antisymmetrization is essential in the spin-orbit splitting. After integrating the exchange tensor term over all coordinates except $\boldsymbol{r}$ and $\boldsymbol{r}'$, there remain the scalar quantities of the following types:

(I) (scalar function of $\boldsymbol{r}$ and $\boldsymbol{r}'$)

and

(II) $(\boldsymbol{\sigma}\cdot(\boldsymbol{r}\times\boldsymbol{r}'))\cdot$(scalar function of $\boldsymbol{r}$ and $\boldsymbol{r}'$)**.

It is evident that the spin-orbit splitting results only from the terms of the type (II). Expanding the scalar functions of $\boldsymbol{r}$ and $\boldsymbol{r}'$ in terms of the Legendre polynomials $P_l(\cos(\widehat{\boldsymbol{r},\boldsymbol{r}'}))$, we obtain the linear combination of the following expressions*** by integrating the terms belonging to the type (II) over $d\Omega$ and $d\Omega'$;

$$\int\bar{\chi}(-1)\bar{\chi}_{lJ}^{m}(1)\,i(\boldsymbol{\sigma}_3\cdot(\boldsymbol{r}\times\boldsymbol{r}'))\sum_M Y_L^M(\Omega)\,\overline{Y}_L^M(\Omega')\chi_{lJ}^{m}(2)\chi(-2)\,d\Omega\,d\Omega'$$
$$=-\tfrac{1}{2}rr'C_{lJL}, \qquad (3\cdot9)$$

where

$$C_{l,l+1/2,l\mp1}=\pm\frac{l}{2l+1},\quad C_{l,l-1/2,l\mp1}=\mp\frac{l+1}{2l+1},\quad \text{for } l\neq0.$$

* The explicit expressions of each term are given in § 5 and the Appendix.

** This type does not result from the exchange S-D term but from the exchange D-S term in our treatment.

*** This relation is derived by Sugie et al.[5]

The exchange D-S tensor term with Majorana exchange character $\langle D(-1)|S_{14}P_{M,14}|S(-2)\rangle$ is equal to that of Wigner type, since the wave function $g_S(-2)\chi(-2)f_{lJ}(r')/r'$ is invariant under the exchange of the coordinates $\boldsymbol{r}_1 \rightleftarrows \boldsymbol{r}_4$. Thus, the splitting term has the factor $(w^{(t)}+m^{(t)})$, which means that only the tensor force in ${}^3E$ contributes.

We get the following summary as to the splitting kernel $k_{Sp}^{l'J}(r, r')$.

(i) Antisymmetrization is essential to the splitting kernel.

(ii) The splitting kernel is expressed as

$$k_{Sp}^{l'J}(r, r') = (w^{(t)}+m^{(t)})\left(\frac{-C}{1+C^2}\right)v_t({}^3E)\|\boldsymbol{\sigma}\cdot\boldsymbol{l}\|A(\alpha\beta\nu l; r, r'). \qquad (3\cdot10)$$

Here $\|\boldsymbol{\sigma}\cdot\boldsymbol{l}\|$ is the eigen-value of the operator $\boldsymbol{\sigma}\cdot\boldsymbol{l}$, resulting from the numerator of $C_{l,J\mp1}$;

$$\|\boldsymbol{\sigma}\cdot\boldsymbol{l}\| = \begin{cases} l & \text{for } J=l+1/2 \\ -(l+1) & \text{for } J=l-1/2. \end{cases}$$

α and β mean the spreads of the ${}^1S_0$- and ${}^5D_0$- wave functions of He^4, respectively, and ν is the range of the tensor force. From the expression (3·10), we see that the tensor force contributes to the splitting in two ways: (1) proportionally to the strength of the tensor force and (2) through the D-state mixing ratio C of He^4. The strong tensor force characteristic of the pion-theoretical potential is then expected to be favorable to the wide splitting. The remaining part of this paper will show this is the case. As shown in § 4 and § 5, the sign of k_{Sp}^{1J} is the same as what the shell model assumes,, because

$$(w^{(t)}+m^{(t)})v_t({}^3E)<0, \quad C<0 \text{ and } A>0.$$

It will be worth mentioning that also in the case of more general nuclei we expect to get the spin-orbit coupling of the type (3·9) from a tensor force, when we take into account the antisymmetrization and the mixing of the core states due to tensor forces between core nucleons.

§ 4. Wave function and binding energy of He^4

The available data on He^4 are as follows: Its spin is zero, parity even and the experimental binding energy $\sim$28 Mev. Because its spin is zero, we cannot get any information from the E2 and M1 moments. The high energy electron scattering experiments showed that the charge distribution of He^4 can be best fitted by the Gaussian radial distribution with r.m.s. radius 1.61×10^{-13} cm[13].

As in Eq. (3·2), we chose the spin-angular wave functions. Taking account of the results of the high energy electron scattering experiments, Gaussian radial wave functions are chosen, namely

$$g_S=N_S\exp\{-\tfrac{1}{2}\alpha\sum_{i>j=2}^{5}r_{ij}^2\}\ ;\quad g_D=N_D\exp\{-\tfrac{1}{2}\beta\sum_{i>j=2}^{5}r_{ij}^2\}. \qquad (4\cdot1)^*$$

Usually, the parameters of the wave function are determined by the variational calculation. But we do not follow this procedure, because it may be meaningless to determine the values of the parameters, particularly of α, by this method when the wave function has no short-range correlation. Instead of this, we use the result of the high energy e-He^4 scattering experiments in determining α. As the r.m.s. radius $\sqrt{\langle r^2\rangle}$ is mainly determined by the $^1S_0$-state wave function and the contribution from the $^5D_0$-state is estimated to be smaller than 5%, we determine parameter α to fit the experimental value of the r.m.s. radius$\sim1.61\times10^{-13}$ cm.

Regardless of the finite charge distribution of proton, we obtain $\alpha\sim0.11\times10^{26}\,\mathrm{cm}^{-2}$ from the relation $\langle r^2\rangle^{pt}=9/32\alpha$. But through the electron scattering experiments, it has also been shown that the r.m.s. radius of the charge distribution of proton is about $0.7\sim0.8\times10^{-13}$ cm[13]. Taking this fact into account, we obtain the next formula:

$$\langle r^2\rangle^{fs}=\langle r^2\rangle^{pt}+\langle r^2\rangle_p,$$

where $\langle r^2\rangle^{fs}$ is the mean square radius of the charge distribution of He^4 including the effect of the proton finite size, and $\langle r^2\rangle_p$ is that of proton. Substituting the values 1.61×10^{-13} cm and 0.7×10^{-13} cm for $\sqrt{\langle r^2\rangle^{fs}}$ and $\sqrt{\langle r^2\rangle_p}$ respectively, we obtain $\sqrt{\langle r^2\rangle^{pt}}=1.4\times10^{-13}$ cm, from which we determine $\alpha=0.14\times10^{26}\,\mathrm{cm}^{-2}$. This value is considerably smaller than those obtained so far by other authors using the variational calculation without the short-range correlation. Later we shall discuss this point.

Fixing the parameter α to this value, we determine the other parameters β and C by the variational calculation. With nuclear forces of ($r^2\times$Gauss) type radial dependence, the variational expression of the total energy of He^4 becomes

$$\begin{aligned}E_\alpha=\frac{1}{1+C^2}\Bigg[&\frac{\hbar^2}{2M}(18\alpha+26\beta C^2)+\sum_{1\pi,2\pi}v_c\Big\{9(w+m)A^{3/2}\frac{1}{2\alpha+\mu}\\&+C^2\Big(\frac{15}{4}(w+m+b+h)A'^{3/2}+\frac{15}{2}(w-m+b-h)A'^{5/2}\\&+\frac{7}{4}(w+m+b+h)A'^{7/2}\Big)\frac{1}{2\beta+\mu}\Big\}-\sum_{t,S}v_t\Big\{6\sqrt{5}\,C(w^{(t)}+m^{(t)})B'\frac{1}{\alpha+\beta+\nu}\\&+C^2\Big(\frac{21}{2}(w^{(t)}-m^{(t)})B^{5/2}+\frac{7}{2}(w^{(t)}+m^{(t)})B^{7/2}\Big)\frac{1}{2\beta+\nu}\Big\}\Bigg]\end{aligned}$$

where

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* Such functions have no two-body correlation: for example, $\exp\{-\frac{1}{2}\alpha\sum r_{ij}^2\}=\exp\{-2\alpha\sum r_i^2\}$ where $\boldsymbol{r}_i$ is the coordinate of the i-th nucleon relative to the center of mass of He^4.

$$A=\frac{2\alpha}{2\alpha+\mu}\,;\quad B=\frac{2\beta}{2\beta+\nu}\,;$$

$$A'=\frac{2\beta}{2\beta+\mu}\,;\quad B'=\frac{2\alpha}{\alpha+\beta}\left(\frac{2\beta}{\alpha+\beta}\right)^2\left(\frac{2\sqrt{\alpha\beta}}{\alpha+\beta+\nu}\right)^{5/2}. \tag{4·2}$$

This is minimized with respect to β and C. The results are shown in Table 1.

Table 1 Binding energy of He^4 calculated using the pion-theoretical potential without the short-range correlation from Eq. (4·2).

$\beta=0.35\times10^{26}\,\mathrm{cm}^{-2}$; $C=-0.30$ $(\alpha=0.14\times10^{26}\,\mathrm{cm}^{-2})$	
SS ⟨K. E.⟩	46 (Mev)
⟨Central 1π⟩	−11
⟨Central 2π⟩	−55
SD ⟨Tensor 1π⟩	−21
DD ⟨K. E.⟩	16
⟨Central 1π⟩	−0.3
⟨Central 2π⟩	−5.9
⟨Tensor 1π⟩	−1.9
Total	−33 (Mev)

As seen in Table 1, $V_c^{(2\pi)}(^1E)$ plays an important role, and $V_t^{(1\pi)}(^3E)$ also yields large contribution. Without them He^4 will hardly be bound. It is noted that $V_c^{(2\pi)}(^3O)$ and $V_t^{(1\pi)}(^3O)$ do not contribute to the SS and SD terms, respectively. Obviously, the total binding energy is too large*. This is due to the fact that we take no account of the short-range correlation, the main effect of which is to increase the SS⟨K.E.⟩. Although it is difficult to say anything about this effect quantitatively, we estimate this roughly in § 6 using a trial correlation function $\prod_{i>j=2}^{5}(1-\exp(-\gamma r_{ij}^2))$. From this result we may say qualitatively that: 1) we get the reasonable minimum total energy, 2) the value of α minimizing the total energy tends to be much smaller than that determined variationally without correlation ($\alpha=0.50\times10^{26}\,\mathrm{cm}^{-2}$; B.E.=120 Mev. See Fig. 5(d)), 3) the values of C and β are insensitive to the correlation function in our case.

So far, many authors[12),14)] have calculated the binding energy of He^4 to determine the "consistent" phenomenological potentials. From their results it is seen that we could not obtain sufficient binding energy if the tensor force was predominant in ^3E. In our case, the pion-theoretical potential has two central parts with different ranges, i.e. $V_c^{(1\pi)}$ and $V_c^{(2\pi)}$; so, although its tensor force is strong and the contribution of $V_c^{(1\pi)}$ is small, it can reproduce the binding energy of He^4 reasonably as shown above. Moreover, the strong tensor force results in a large value of

* Since, in the intermediate region, $V_c^{(2\pi)}$ is known only qualitatively, so our choice of $V_c^{(2\pi)}$ and the value of the total energy should not be taken seriously. The latter is very sensitive to the choice of the detailed form of the former.

$C^2 \sim 9\%$, which is very advantageous for reproducing the wide splitting of the p-phase shifts in n-He^4 scattering, while the value of C^2 is about 4% according to other authors. Also, by our method of determining the value of α, the bremsstrahlung-weighted cross section[15] in the γ-He^4 reaction is naturally reproduced.

About the corrections due to the correlation and additional $^5D_0$ states, we shall discuss briefly in § 6.

§ 5. Effective potentials and phase shifts

In this section we discuss the p- and s-phase shifts in n-He^4 scattering and show the numerical results.

5·a) *Effective potentials*

The explicit expression of the integro-differential equation is derived from Eq. (3·6) using Eqs. (2·3), (3·2) and (4·1). The result is given in the Appendix.

In order to find out the characteristic features of the interaction terms we rewrite the terms which contain kernels in the form of the effective potential. The abbreviated form for $l=1$ is

$$\frac{\hbar^2}{2M'}\left(\frac{d^2}{dr^2}+k^2-\frac{2}{r^2}\right)f_J(r)=(W(r)+W_J'(r)+W_{Sp}^J(r))f_J(r) \qquad (5\cdot1)$$

where

$$W_J'(r)=\frac{\int k(r,\,r')f_J^0(r')\,dr'}{f_J^0(r)} \qquad (5\cdot2)$$

$$W_{Sp}^J(r)=\frac{\int k_{Sp}^J(r,\,r')f_J^0(r')\,dr'}{f_J^0(r)}. \qquad (5\cdot3)$$

$f_J^0(r)$ is a solution in the square well potential reproducing nearly the experimental splitting. This procedure is allowable, if f_J^0 does not differ much from f_J inside the force range and then the convergence of the iteration in solving the integro-differential equation is good. In our case, we can find such f_J^0 as seen in the following. These potentials and the phase shifts are plotted in Fig. 2.

In $W(r)$, $V_c^{(1\pi)}$ vanishes exactly because of its exchange character. Consequently, $V_c^{(2\pi)}$ is essential in this scattering problem because it composes the whole $W(r)$ which is the main part of the effective potentials. Besides $V_c^{(1\pi)}$, the tensor force, which is important in binding four nucleons, has no effect on extra neutron in the direct term. These circumstances seem to be the reason why five nucleons do not bind, although the $p_{3/2}$-level of this system is just above the zero energy. $W(r)$ is shown in Fig. 3 (a) with the use of the parameters decided in § 4.

In $W_J'(r)$ the main part arises from the $V_c^{(2\pi)}$, V_t and the kinetic energy in the antisymmetrized effect (the last term in Eq. (3·6)). They cancel each other to some extent in $r \lesssim 3.0$. Out of this region, only the kinetic energy term is

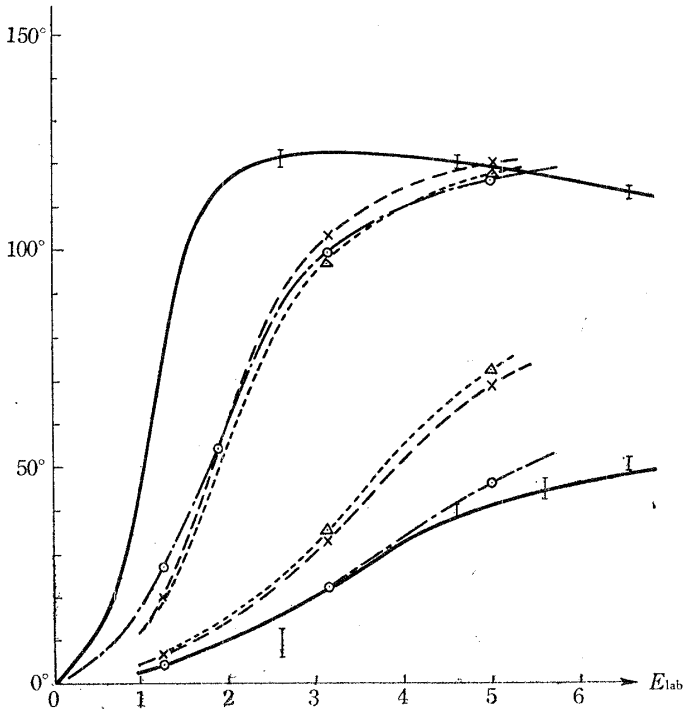

Fig. 2(a). p-phase shift.

I Experiment of n-He^4 (J. D. Seagrave, Phys. Rev. **92** (1953) 1222; Levintov et al., Soviet Phys. JETP **5** (1957) N 2258)

——— Calculated from experiments of p-He^4. (K. W. Brockman, Jr. Phys. Rev. **102** (1956), 391. We wish to thank Dr. Brockman for sending his data prior to the publication.

— — — ⊙ — — — Calculated by the zeroth order wave function.

— — —×— — — Theoretical values at the first step.

- - - - -△- - - - - Theoretical values at the second step. At the third step they scarcely change.

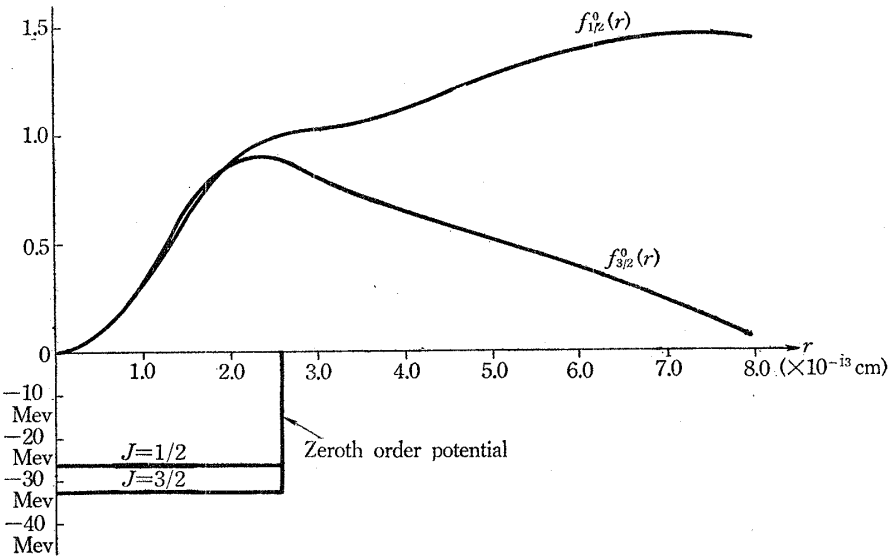

Fig. 2(b). Zeroth order potentials and their wave functions for $l=1$.

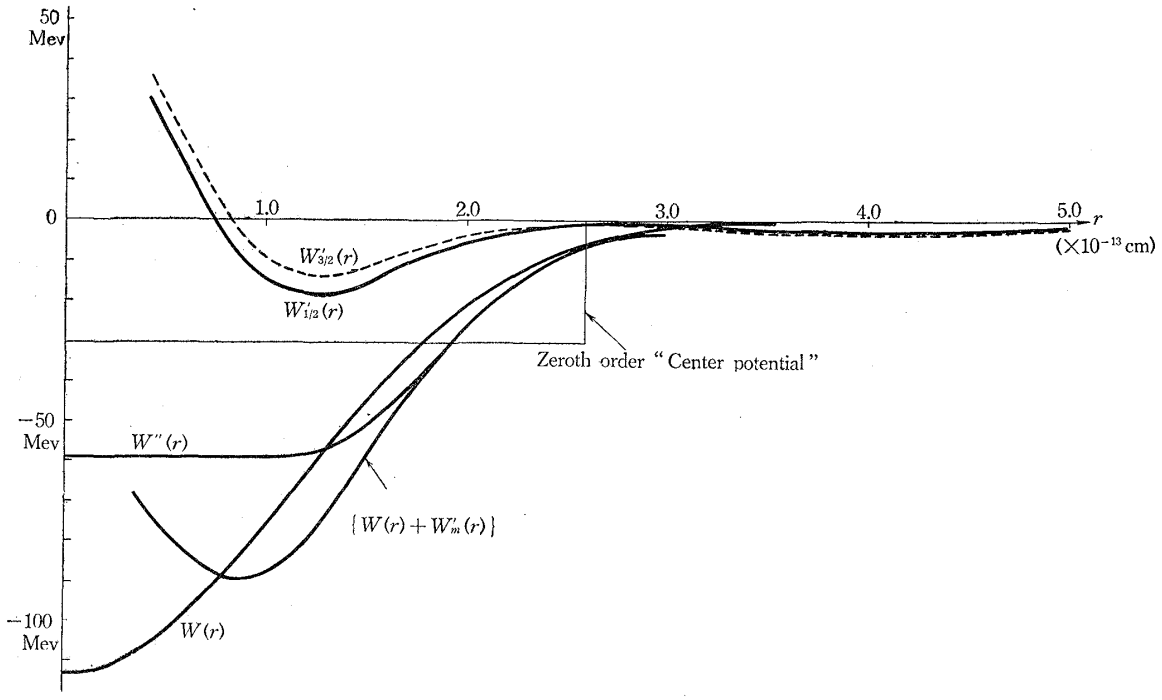

Fig. 3(a). Effective potentials $W(r)$, $W_J'(r)$ and "center potential" $W''(r)$ for $l=1$.

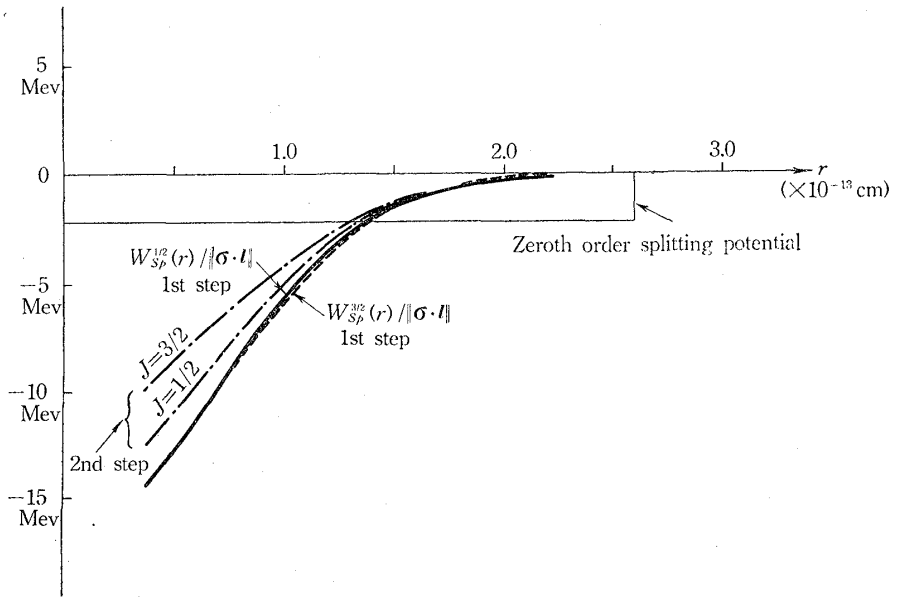

Fig. 3(b). Effective spin-orbit potential $W_{Sp}'(r)$ for $l=1$.

effective but small and attractive. As $k(r, r')$ itself has no J-dependence, the J-dependence of $W_J'(r)$ is small. Its effect to the splitting is reductive in $r\lesssim 3.0$ and constructive in $r\gtrsim 3.0$. This is shown in Fig. 3 (a).

In the last term of Eq. (5·1) the kernels $k_{Sp}^J(r, r')$ themselves split depending

on the total angular momentum J, proportionally to the factor $\|\boldsymbol{\sigma}\cdot\boldsymbol{l}\|$. Now $W^J_{Sp}(r)/\|\boldsymbol{\sigma}\cdot\boldsymbol{l}\|$ is shown in Fig. 3 (b). There we find that the effective spin-orbit potential $W^J_{Sp}(r)$ has only small J-dependence except the kinematical factor $\|\boldsymbol{\sigma}\cdot\boldsymbol{l}\|$.

The J-dependence of the effective potentials through that of $f_J(r)$ is small. This feature comes from the situation that in spite of the wide splitting of the p-phase shifts the wave functions $f_{3/2}(r)$ and $f_{1/2}(r)$ are not very different inside the potential range ($r\lesssim 2.5$) below 5 Mev of the incident energy, (e. g. see Fig. 2. (a)), and outside the range the kernels are small.

5·b) *Numerical calculation*

In the next place, we solve the integro-differential equation (3·8) and calculate the p-phase shifts. As the first step we consider Eq. (5·1) given in terms of effective potentials. Further steps will be discussed in § 6.

The energy dependence of $f_J(r)$ inside the potential range is very small below 5 Mev of the incident energy. Consequently, in this energy range, we use Eq. (5·1) with the effective potentials defined by Eqs. (5·2) and (5·3) at a definite energy.

The wide splitting of the p-phase shifts is related to the "critical" situation that the $p_{3/2}$-level is just above the zero energy, while the $p_{1/2}$-level is not so. Therefore, the p-phase shifts are very sensitive to the details of the potentials, particularly in the $p_{3/2}$-state. Then, we can hardly obtain the reasonable "center potential" to reproduce approximately the weighted mean value of the experimental p-phase shifts, unless we have the very detailed knowledge of two-body interactions and of the treatment of the system. It is readily seen in practice that our effective potentials fail to reproduce the experimental mean phase shift, mainly because $W(r)$ is somewhat too strong. It is, however, noted that we have not taken into consideration the hard-core of the nuclear force and the short-range correlation between two nucleons. Consequently, if we took account of these points, the potential $W(r)$ should be reduced by some amount, particularly for small r. However, we can hardly estimate this effect definitely. So we are obliged to decide this "center potential" phenomenologically by the following procedure.

At first we neglect the outside part ($r\geqq 3.5\times 10^{-13}$ cm) of $W'_J(r)$ because of its smallness, and for $r\leqq 3.5\times 10^{-13}$ cm adopt the mean value $W'_m(r)$ of $W'_{3/2}(r)$ and $W'_{1/2}(r)$, neglecting the small J-dependence of $W'_J(r)$. We then solve the following differential equation

$$\frac{\hbar^2}{2M'}\left(\frac{d^2}{dr^2}+k^2-\frac{2}{r^2}\right)f(r)=W''(r)f(r), \tag{5·4}$$

where $W''(r)$ has the same form as $W(r)+W'_m(r)$ at $r\gtrsim 2\times 10^{-13}$ cm but the inner part of $W''(r)$ is to be decided to give the "center potential". Second, we solve the following differential equation

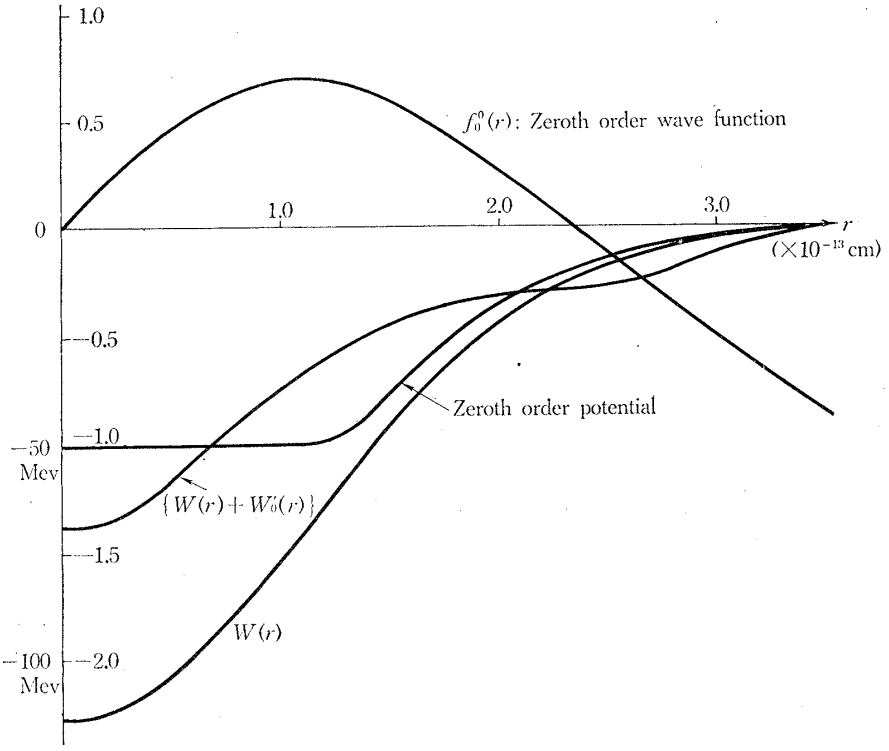

Fig. 4(a). Zeroth order potential, its wave function and effective potentials for $l=0$.

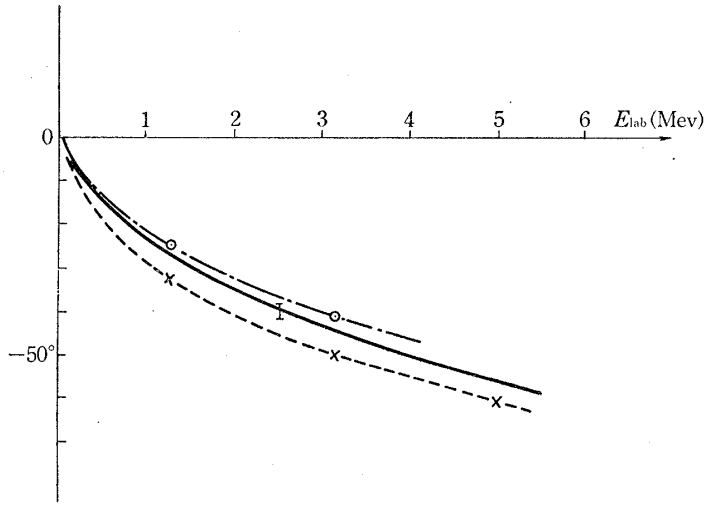

Fig. 4(b). s-phase shift.

I Experiment of n-He^4. See the caption of Fig. 2 (a)

Calculated from experiments of p-He^4. See the caption of Fig. 2 (a)

—— —⊙—— — Calculated by 0-th order wave function.

—— — × —— — Theoretical value.

$$\frac{\hbar^2}{2M'}\left(\frac{d^2}{dr^2}+k^2-\frac{2}{r^2}\right)f_J(r)=(W''(r)+W_{sp}^J(r))f_J(r). \qquad (5\cdot5)$$

In spite of our approximations we can get the essential feature so far as the spin-orbit coupling is concerned. In order to obtain better solutions, we continue the iteration in the original equation, (3·8).

The results are shown in Fig. 3 (a) and Fig. 2 (b). In Fig. 3 (a) the "center potential" $W''(r)$ is shown. It is seen that the tensor force of the pion-theoretical potential gives about 60% of the experimental splitting at the second step iteration. This result is reasonable in comparison with that of Sugie et al. obtaining 30% by the week tensor force in Fig.1 (a). The values of the $W_{sp}^J(r)/\|\boldsymbol{\sigma}\cdot\boldsymbol{l}\|$ at the third step (obtained by using the second step solution) are almost equal to those at the second step. This shows that the convergence of the iteration is good.

Finally, we calculate the s-phase shift. Also in this case the discussion is pushed in the same manner as the p-phase shifts, and the differential equation becomes

$$\frac{\hbar^2}{2M'}\left(\frac{d^2}{dr^2}+k^2\right)f_0(r)=(W(r)+W_0'(r))f_0(r), \qquad (5\cdot6)$$

where

$$W_0'(r)=\frac{\int k(r,r')f_0^0(r')dr'}{f_0^0(r)}. \qquad (5\cdot7)$$

$W(r)$ is the same as that in (5·1) and $k(r, r')$ is abbreviated from the kernels for $l=0$ in (A·1). $f_0^0(r)$ is solved by the similar potential to the "center potential" for $l=1$, reproducing the experimental s-phase shift. In this case, $W_0'(r)$ is mainly repulsive contrary to the p-waves and so the effective potential $W(r)+W_0'(r)$ has a reasonable strength. Although the p-waves are very sensitive to the details of the interaction terms, the s-phase shift is almost determined by the main feature of the effective potential. This is because the s-state is not at the "critical" situation. The potentials and the phase shifts are shown in Fig. 4. It can be concluded that the pion-theoretical potential explains the s-phase shift.

§ 6. Supplementary discussions

6·a) *Short-range correlations*

In the previous calculation the nuclear size parameter α was fixed from the information of the high energy e-He^4 scattering. In the following we show that our value of α is consistent with that determined by the variational method, if the short-range correlation between two nucleons is taken into account.

Let us use the following correlation function so that the calculation can be performed analytically:

$$\prod_{i>j}(1-\exp(-\gamma r_{ij}^2)). \qquad (6\cdot1)$$

Generally, introducing the correlation function, we obtain an additional kinetic energy as the most important effect. Since this additional energy increases with α, the value of α which minimizes the binding energy is to be reduced. However, this type of the correlation function may not be realistic in the core region of nuclear force, because the actual wave function vanishes there. The additional kinetic energy derived from this correlation function has a (negative) contribution in the core region, so we should eliminate this unrealistic contribution by cutting off.

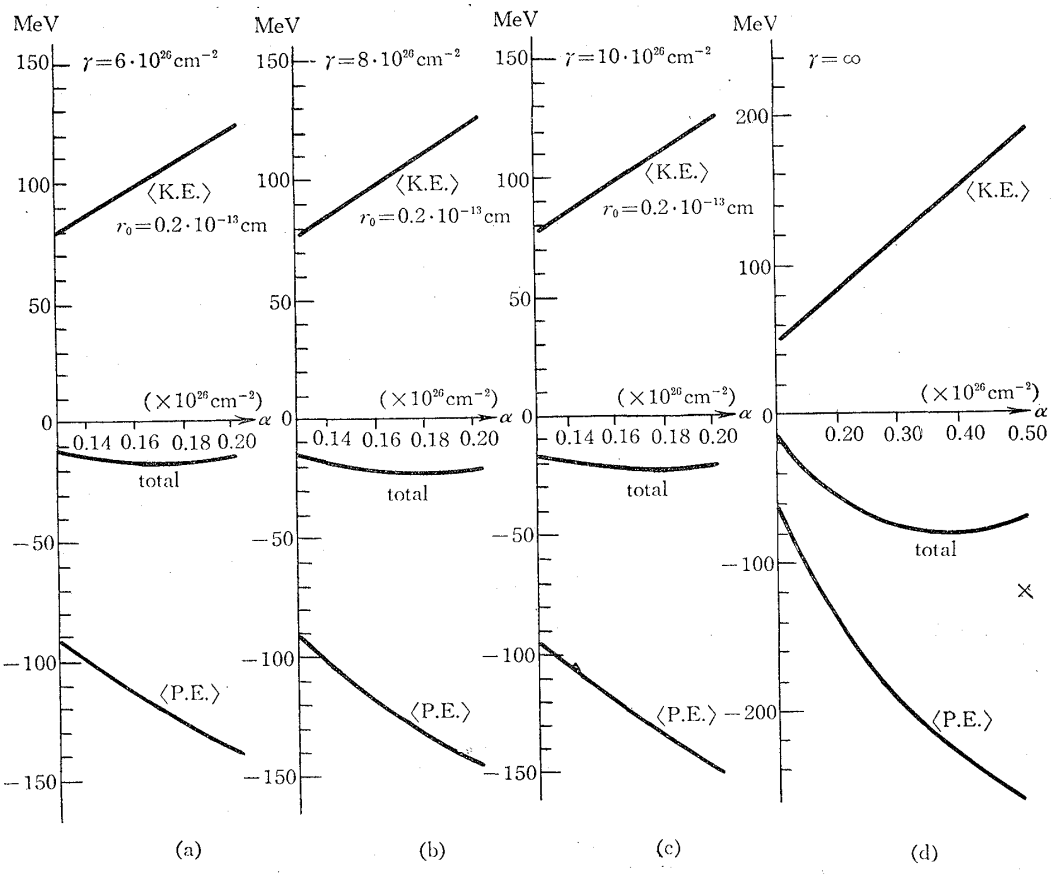

Fig. 5. The α's minimizing the total energy of He^4 are shown for several cases. Solid lines are obtained by fixing β and C to $0.35\times10^{26}\,\mathrm{cm}^{-2}$ and -0.30, respectively (Table 1).

Figs. (a), (b) and (c) show the results with short-range correlation function $\prod_{i>j=2}(1-\exp(-\gamma r_{ij}^2))$. (a), (b) and (c) correspond to $\gamma=6$, 7 and $8\times10^{26}\,\mathrm{cm}^{-2}$, respectively. r_0 is the range of cutting off. In these calculations, we include only two-body clusters, since the effects from the higher clusters are negligible in our cases. The values of C and β are insensitive to the correlation function when $\alpha\lesssim0.2\times10^{26}\,\mathrm{cm}^{-2}$ and $\gamma\gtrsim6\times10^{26}\,\mathrm{cm}^{-2}$.

5 (d) shows the results without short-range correlation ($\gamma=\infty$). The variational calculation with respect to all parameters α, β and C gives the result shown by the cross point (B. E.=120 Mev, $\alpha=0.5\times10^{26}\,\mathrm{cm}^{-2}$, $\beta=0.8\times10^{26}\,\mathrm{cm}^{-2}$ and $C=-0.5$).

Using (6·1), we obtain the following results. Fig. 5 shows the values of α that give the minimum binding energy for fixed β, C, the range of cutting off and several values of γ*. It is seen from Fig. 5 that owing to the introduction of the correlation function the value of α minimizing the binding energy is much reduced to $\alpha<0.2\times10^{26}\,\mathrm{cm}^{-2}$ from $\alpha=0.5\times10^{26}\,\mathrm{cm}^{-2}$ without correlation. In spite of above rough estimation, we can expect to get $\alpha\sim0.14\times10^{26}\,\mathrm{cm}^{-2}$ if we adopt a more realistic correlation function. The situation is same also in the three-body problem, as already studied by Kikuta et al.[16], that is, adopting a correlation function which is consistent with the hard core, the parameter (corresponding to our α) becomes smaller than the value without the correlation.

Moreover, it is to be noted that the short-range correlation does not affect our determination of the value of α through the r.m.s. radius, because in the expectation value of r^2 the contribution from the region, where the correlation is important, is negligibly small.

From the above discussions we conclude that the value $0.14\times10^{26}\,\mathrm{cm}^{-2}$ of α is consistent with the whole variational treatment with the hard core and it is rather appropriate to n-He^4 scattering problem.

Speaking of n-He^4 scattering in relation to α, the magnitude of the splitting of the scattering potential and so the p-phase shifts are rather insensitive to the change of α near $\alpha\sim0.14\times10^{26}\,\mathrm{cm}^{-2}$ as seen from Eq. (A·6)**.

6·b) *Corrections to the numerical value of the splitting*

(1) *The additional* $^5D_0$ *states in* He^4. According to Abraham et al.[14b)], the additional $^5D_0$ states have some effect on the binding energy of He^4. So, introducing these states in our problem, they would make the binding energy and the splitting of the p-phase shifts larger, since their effect seems to be at least additive to the principal $^5D_0$ state.

(2) *So-called* S_{24} *and* S_{34} *terms discussed in* § 3. There remain two independent terms, S_{24} and S_{34}, with respect to the tensor terms, if the wave function of He^4 is not exact. While S_{34} term is independent of the splitting, S_{24} becomes the same as S_{14} after the interchange $1\rightleftarrows2$, hence in the kernel from S_{24} the coordinates $\boldsymbol{r}$ and $\boldsymbol{r}'$ exchange each other as compared with that from S_{14}. Then, speaking about the splitting, we now get $k^J_{Sp}(rr')+k^J_{Sp}(r'\,r)$ instead of $k^J_{Sp}(r\,r')$ in (3·8). This additional kernel may increase the splitting of the p-phase shifts.

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* In the next step, we estimated the effect of short-range correlation to β and C fixing α to $\alpha=0.14\times10^{26}\mathrm{cm}^{-2}$. We found that C and β are insensitive to the correlation function in our case.

** It is because, firstly, the integrand of the splitting potential, in which α appears in the form of $\alpha+\beta$, is not so affected by the change of α on account of $\alpha<\beta$ in our case, and secondly, the fractional variation with respect to α of the other factor, which depends on r, is nearly equal to $\Delta\alpha/\alpha$ over the region where the contribution to the splitting is most important.

As for the effect to the result by the change of β, the effect may be estimated to be small as seen in Sugie et al., provided that our β should be replaced for their 2α because $\alpha<\beta$ in our case.

However, the contribution of all kernels resulting from inaccuracy of the wave function of He^4, including this additional splitting kernel, tends to vanish as the wave function of He^4 becomes exact. Therefore, we cannot say anything definite about this effect.

(3) *J-dependence of W_J' and iteration.* Finally we speak about the approximations in the calculation of phase shifts. In § 5, we neglected the small J-dependence of $W_J'(r)$. It is reductive in $r \lesssim 3.0\times10^{-13}$ cm and constructive in $r \gtrsim 3.0\times10^{-13}$ cm to the splitting. Its net effect on the splitting of the p-phase shifts is estimated to be reductive, 15% at most. Our result in the previous section was obtained at the second step of the iteration method. At the third step it is found that the effective potentials $W_{Sp}'(r)/\|\boldsymbol{\sigma}\cdot\boldsymbol{l}\|$ reconstructed by using the second step solution scarcely change. There is a reason* to consider that the successive effective splitting potentials fall inside the bounded region between the second step ones for $J=1/2$ and $J=3/2$. Thus we estimate the error of our result which arises by successive iteration to be negligible.

After all, as for the splitting, we conclude that these various corrections will not change essentially the results that the tensor force of the pion-theoretical potential explains about 60% of the splitting of the p-phase shifts.

§ 7. Concluding remarks

Applying the pion-theoretical potential to He^4 and n-He^4, we obtained valuable information on the relations between the characteristic features of nucleon-nucleon interaction and important properties of nuclei. We summarize the main results in the following.

(i) The wide splitting of the p-phase shifts in the low energy n-He^4 scattering can be explained by the strong tensor force of the one-pion-exchange potential in the triplet even state, if we take into accout the Pauli principle and the mixing of of ${}^5D_0$-states of He^4 due to this tensor force. Basing upon this results, therefore, we can expect that the spin-orbit coupling in the shell model is originated from the

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* At the second step, the reduction of $W_{Sp}{}^{1/2}(r)$ is mainly due to the reduction in the overlap integral $\int k_{Sp}{}^{1/2} f_{1/2}{}^{(1)} dr'$ arising from the change of the form of the effective potential. Although this reduction of $W_{Sp}{}^{1/2}(r)$ makes the total scattering potential a few Mev deeper for $r<1.2\times10^{-13}$cm and so the wave function $f_{1/2}{}^{(2)}$ somewhat larger at this region, the change of $W_{Sp}{}^{1/2}(r)$ constructed by this $f_{1/2}{}^{(2)}$ is much smaller than that of the first step to the second step. The trend is to make $W_{Sp}{}^{1/2}(r)$ smaller because of the division of the larger value of $f_{1/2}{}^{(2)}$, but, of course, $W_{Sp}{}^{1/2}$ does not become smaller than $W_{Sp}{}^{3/2}(r)$ at the second step. On the other hand, the reduction of $W_{Sp}{}^{3/2}(r)$ is partly due to that in the overlap integral and partly, in $r<1.2\times10^{-13}$cm, due to the division by $f_{3/2}{}^{(1)}$, which is larger than $f_{3/2}{}^{(0)}$ because of the large discrepancy between the effective scattering potentials there. The smaller $W_{Sp}{}^{3/2}(r)$ becomes, the smaller $f_{3/2}$ becomes and then $W_{Sp}{}^{3/2}(r)$ at the next step becomes larger, but the absolute value of the change is very small. Thus the effective potential $W_{Sp}{}^{3/2}(r)$ will converge at a slightly larger absolute value than that of the second step.

strong tensor forces between an extra nucleon and core particles, through the exchange effects among these particles and the mixing of the core states due to the tensor forces acting between core particles.*,**

(ii) The binding energy of He^4 can be explained by the following main features of nucleon-nucleon interaction: the strong short-range attractive force of the two-pion-exchange potential in the singlet even state, the strong tensor force of the one-pion-exchange potential in the triplet even state and the short-range repulsion. The parameter representing the spread of the wave function is consistent with the experimental data of e-He^4 scattering, if we properly take into account the short-range correlation effects due to the hard-core-like repulsive interaction. In three-dody system, essential contributions to the energy seem to come from the above mentioned three parts of nucleon-nucleon interaction. Therefore, we expect that the pion-theoretical potential may explain all data of the nuclei with $A \leqq 4$.

(iii) The strong tensor force gives an important contribution to the binding energy of He^4. On the other hand, in the system of He^4 plus one nucleon, the direct (non-exchange) contribution from the tensor force to the interaction acting on the extra nucleon vanishes exactly, even though we include the $^5D_0$-state. The main contribution from the one-pion-exchange central potential also vanishes. These situations help us to explain the discontinuity in the binding energy at the closed shell. Also in this case we see the importance of the attractive force of the two-pion-exchange potential in the two-nucleon charge triplet states.

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Appendix

Eq. (3·6) is rewritten with the use of Eqs. (2·3), (3·2) and (4·1) as follows:

$$\frac{\hbar^2}{2M'}\left\{\frac{d^2}{dr^2}+k^2-\frac{l(l+1)}{r^2}\right\}f_{lJ}(r)=W(r)f_{lJ}(r)+\int k^l_{K.E.}(r,\,r')f_{lJ}(r')\,dr'$$
$$+\int k_c{}^l(r,\,r')f_{lJ}(r')\,dr'+\int k_t{}^l(r,\,r')f_{lJ}(r')\,dr'+\int k^{lJ}_{Sp}(r,\,r')f_{lJ}(r')\,dr', \tag{A·1}$$

where $k^l_{K.E.}(r, r')$ is derived from the kinetic energy in the antisymmetrized term, $k^l_c(r, r')$ is from the central potentials and $k^l_t(r, r')$ is from the tensor potential. These have no J-dependence.

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* This program is now being pushed in the case of general nuclei by Takagi, Watari and Yasuno.

** T. Terasawa and T. Terasawa and A. Arima obtained about half the doublet splittings of the energy levels of He^5, N^{15} and O^{17} in the second order perturbation, by taking into account the same effect.

$k_{Sp}^{lJ}(r, r')$ is the splitting kernel derived from the tensor potential. The expressions of $W(r)$ and these kernels are given in the following.

$$W(r)=\sum_{i=1,2}(4w^{(i)}+2b^{(i)}-m^{(i)}-2h^{(i)})\frac{v_c^{(i)}}{1+C^2}\,p_{(i)}^{5/2}\left(p_{(i)}r^2+\frac{9}{32\alpha}\right)\exp(-p_{(i)}r^2),$$

$$p_{(i)}\equiv\frac{16\alpha}{16\alpha+3\mu^{(i)}}. \tag{A·2}$$

$$k_{K.E.}^{l}(r, r')=\frac{1}{1+C^2}\frac{32}{15}\left(\frac{3\alpha}{\pi}\right)^{1/2}\zeta^2\left\{\left(16\alpha^2+r'^2-\frac{8(l+3)}{\zeta}+\frac{4}{\zeta^2}k^2\right)\mathcal{J}_{l+1/2}(\zeta rr')\right.$$
$$\left.+8rr'\,\mathcal{J}_{l+3/2}(\zeta rr')\right\}\exp\left\{-\frac{17}{16}\zeta(r^2+r'^2)\right\},$$

$$\zeta\equiv 6\left(\frac{8}{15}\right)^2\alpha. \tag{A·3}$$

$\mathcal{J}_{l+1/2}(x)\equiv xi^l j_l(ix)$, where j_l is the spherical Bessel function of order l.

$$k_c^{l}(r, r')=k_c^{l\xi}(r, r')+k_c^{l\eta}(r, r'),$$

$$k_c^{l\xi}(r, r')=\sum_{i=1,2}(4m^{(i)}+2h^{(i)}-w^{(i)}-2b^{(i)})\frac{v_c^{(i)}}{1+C^2}\left(\frac{16}{15}\right)^3\left(\frac{3\alpha}{\pi}\right)^{1/2}\frac{18\alpha}{4\alpha-3\mu^{(i)}}$$
$$\times\left\{\left(r^2+r'^2+\frac{2l}{\xi^{(i)}}\right)\mathcal{J}_{l+1/2}(\xi^{(i)}rr')-2rr'\,\mathcal{J}_{l+3/2}(\xi^{(i)}rr')\right\}$$
$$\times\exp\left\{-\frac{8}{75}(17\alpha+6\mu^{(i)})(r^2+r'^2)\right\},$$

$$k_c^{l\eta}(r, r')=-\sum_{i=1,2}(w^{(i)}+m^{(i)})\frac{v_c^{(i)}}{1+C^2}\frac{128}{5}\left(\frac{3\alpha}{\pi}\right)^{1/2}\frac{2\alpha}{2\alpha+\mu^{(i)}}q_{(i)}^{3/2}\left[\left\{\left(\frac{4}{15}\right)^2 q_{(i)}\right.\right.$$
$$\left.\times\left(16r^2+r'^2-\frac{8l}{\eta^{(i)}}\right)+\frac{1}{4\alpha}\right\}\mathcal{J}_{l+1/2}(\eta^{(i)}rr')+2\left(\frac{8}{15}\right)^2 q_{(i)}\,rr'\,\mathcal{J}_{l+3/2}(\eta^{(i)}rr')\left.\right]$$
$$\times\exp\left[-\frac{4}{75}q_{(i)}\{(34\alpha+27\mu^{(i)})r^2+(34\alpha+7\mu^{(i)})r'^2\}\right],$$

$$\xi^{(i)}\equiv\frac{32}{75}(4\alpha-3\mu^{(i)}),\qquad \eta^{(i)}\equiv\frac{128}{25}\frac{2\alpha+\mu^{(i)}}{6\alpha+\mu^{(i)}}\alpha,$$

$$q_{(i)}=\frac{6\alpha}{6\alpha+\mu^{(i)}}. \tag{A·4}$$

$$k_t^{l}(r, r')=\sum_{j=1,2}k_t^{l(j)}(r, r'),$$

$$k_t^{l(j)}(r, r')=-\sum_{L,S}(w^{(t)}+m^{(t)})\frac{C}{1+C^2}v_t\frac{2^6\alpha^{9/4}\beta^{13/4}}{(\alpha+\beta)^6}b^{1/2}d\left[\left\{S_l^{(j)}-3\left(\frac{16}{15}\right)^2\right.\right.$$
$$\times(\alpha+\beta)(T_l^{(j)}r^2+T_l'^{(j)}r'^2)+9\left(\frac{16}{15}\right)^4(\alpha+\beta)^2(U^{(j)}r^4+2U'^{(j)}r^2r'^2+U''^{(j)}r'^4)\left.\right\}$$
$$\times\mathcal{J}_{l+1/2}(\lambda rr')+3\left(\frac{16}{15}\right)^2(\alpha+\beta)rr'\left\{-V^{(j)}+3\left(\frac{16}{15}\right)^2(\alpha+\beta)(W^{(j)}r^2\right.$$

$$+W'^{(j)}r'^2)\Big\}\mathcal{J}_{l+3/2}(\lambda rr')\Big]\exp\{-\gamma'^{(j)}r^2-\delta^{(j)}r'^2\},$$

$$\lambda\equiv 3\left(\frac{8}{15}\right)^2(1+2a)(\alpha+\beta)$$

$$a\equiv\frac{\nu}{3(\alpha+\beta)+\nu}\qquad b\equiv\frac{3(\alpha+\beta)}{3(\alpha+\beta)+\nu}\qquad d\equiv\frac{\alpha+\beta}{\alpha+\beta+\nu}$$

$$\gamma^{(1)}\equiv\frac{24}{225}(16\alpha+\beta)+3\left(\frac{16}{15}\right)^2(\alpha+\beta)a$$

$$\gamma^{(2)}\equiv\frac{24}{225}(\alpha+16\beta)+3\left(\frac{16}{15}\right)^2(\alpha+\beta)a$$

$$\delta^{(1)}\equiv\frac{24}{225}(\alpha+16\beta)+\frac{3}{16}\left(\frac{16}{15}\right)^2(\alpha+\beta)a$$

$$\delta^{(2)}\equiv\frac{24}{225}(16\alpha+\beta)+\frac{3}{16}\left(\frac{16}{15}\right)^2(\alpha+\beta)a$$

$$S_l^{(1)}\equiv\frac{5}{2}b^2+\frac{5}{6}lbd(25a+17b)+2l(l-1)ad^2\left(7a+\frac{17}{3}b\right)$$

$$S_l^{(2)}\equiv\frac{5}{2}b^2+\frac{20}{3}lbd(2a+b)+8l(l-1)ad^2\left(a+\frac{2}{3}b\right)$$

$$T_l^{(1)}\equiv\frac{5}{6}b^2(5a+b)+labd\left(9a+\frac{19}{3}b\right)$$

$$T_l^{(2)}\equiv\frac{10}{3}b^2(2a+b)+8labd\left(a+\frac{2}{3}b\right)$$

$$T_l'^{(1)}\equiv\frac{5}{24}b^2(5a+4b)+\frac{1}{16}labd\left(19a+\frac{49}{3}b\right),$$

$$T_l'^{(2)}\equiv\frac{5}{24}b^2(2a+b)+\frac{1}{2}labd\left(a+\frac{2}{3}b\right),$$

$$U^{(1)}\equiv ab^2\left(a+\frac{1}{3}b\right),\quad U^{(2)}\equiv 2ab^2\left(a+\frac{2}{3}b\right),$$

$$U'^{(1)}\equiv\frac{1}{32}ab^2(21a+17b),\quad U'^{(2)}=\frac{3}{16}U^{(2)},$$

$$U''^{(1)}\equiv\frac{1}{128}ab^2\left(3a+\frac{8}{3}b\right),\quad U''^{(2)}=\frac{1}{256}U^{(2)},$$

$$V^{(1)}\equiv\frac{5}{24}b^2(25a+17b)-abd\left(7a+\frac{17}{3}b\right),$$

$$V^{(2)}\equiv\frac{5}{3}b^2(2a+b)-4abd\left(a+\frac{2}{3}b\right),$$

$$W^{(1)}\equiv\frac{1}{4}ab^2\left(9a+\frac{19}{3}b\right),\quad W^{(2)}=U^{(2)},$$

$$W'^{(1)} \equiv \frac{1}{64} ab^2 \left(19a + \frac{49}{3} b\right), \quad W'^{(2)} = \frac{1}{16} U^{(2)}. \tag{A·5}$$

In $k_t'(a, r')$, (1) and (2) do not denote the number of exchanged pion in potentials.

$$\begin{aligned} k_{Sp}^{lJ}(r, r') = & -\sum_{L,S} (w^{(t)} + m^{(t)}) \frac{C}{1+C^2} v_t \frac{64}{15} \left(\frac{3}{5\pi}\right)^{1/2} \frac{2^6 \alpha^{9/4} \beta^{13/4}}{(\alpha+\beta)^5} 3b^{3/2} dr r' \\ & \times \exp\{-\gamma^{(1)} r^2 - \delta^{(1)} r'^2\} \sum_{L=l\pm 1} C_{lJK} \Big[\left\{ X_L - 3\left(\frac{16}{15}\right)^2 (\alpha+\beta)(Yr^2 + Y' r'^2) \right\} \\ & \times \mathcal{J}_{L+1/2}(\lambda r r') - 3\left(\frac{16}{15}\right)^2 (\alpha+\beta) r r' Z \mathcal{J}_{L+3/2}(\lambda r r') \Big], \\ & X_L \equiv \frac{8}{3} b + L \frac{32}{15} ad, \quad Y \equiv \frac{16}{15} ab, \quad Y' = \frac{1}{16} Y, \quad Z = \frac{1}{2} Y. \end{aligned} \tag{A·6}$$

C_{lJL} is defined in Eq. (3·9)

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