Tensor Products and Correlation Estimates with Applications to Nonlinear Schrödinger Equations

J. COLLIANDER
University of Toronto

M. GRILLAKIS
University of Maryland

AND

N. TZIRAKIS
University of Illinois at Urbana-Champaign

Abstract

We prove new interaction Morawetz-type (correlation) estimates in one and two dimensions. In dimension 2 the estimate corresponds to the nonlinear diagonal analogue of Bourgain's bilinear refinement of Strichartz. For the two-dimensional case we provide a proof in two different ways. First, we follow the original approach of Lin and Strauss but applied to tensor products of solutions. We then demonstrate the proof using commutator vector operators acting on the conservation laws of the equation. This method can be generalized to obtain correlation estimates in all dimensions. In one dimension we use the Gauss-Weierstrass summability method acting on the conservation laws. We then apply the two-dimensional estimate to nonlinear Schrödinger equations and derive a direct proof of Nakanishi's H^1 scattering result for every L^2-supercritical nonlinearity. We also prove scattering below the energy space for a certain class of L^2-supercritical equations. © 2009 Wiley Periodicals, Inc.

1 Introduction

In this paper we obtain new a priori estimates for solutions of the nonlinear Schrödinger equation in one and two dimensions.[1] We also provide a systematic way to obtain the known interaction a priori estimates for dimensions higher than 3. These estimates are monotonicity formulae that take advantage of the conservation of the momentum of the equation. Due to the pioneering work [19], estimates of this type are referred to as *Morawetz estimates* in the literature. We then apply these estimates to study the global behavior of solutions to the nonlinear Schrödinger equation. To be more precise, we want to study the global-in-time behavior of

[1] The same estimates have been independently and simultaneously (see [9, 22]) obtained by F. Planchon and L. Vega [21] with different proofs.

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solutions to the following initial value problem

$$\begin{cases} iu_t + \Delta u - |u|^{p-1}u = 0, & x \in \mathbb{R}^n,\ t \in \mathbb{R}, \\ u(x,0) = u_0(x) \in H^s(\mathbb{R}^n), \end{cases} \tag{1.1}$$

with $p > 1$. Here we investigate the L^2-supercritical equation in two dimensions under the natural scaling of the equation, and thus we restrict p to $p > 3$. Scaling refers to the fact that if $u(x,t)$ is a solution to (1.1), then

$$u^\lambda(x,t) = \lambda^{-\frac{2}{p-1}} u\left(\frac{x}{\lambda}, \frac{t}{\lambda^2}\right)$$

is also a solution. The problem is then called H^s-critical if the scaling leaves the homogeneous $\dot{H}^s$ norm invariant. This happens exactly when $s = \frac{n}{2} - \frac{2}{p-1}$. We denote the critical index by s_c and thus

$$s_c = \frac{n}{2} - \frac{2}{p-1}. \tag{1.2}$$

The problem of the existence of local-in-time solutions for (1.1) is well studied by many authors, and a summary of the results can been found in [3, 4, 24]. Thus depending on the strength of the nonlinearity and the dimension, the local solutions are well understood. In this paper we will consider problems that are locally well posed and refer the reader to [4, 24] for the proofs.

The local well-posedness definition that we use here reads as follows: for any choice of initial data $u_0 \in H^s$, there exists a positive time $T = T(\|u_0\|_{H^s})$ depending only on the norm of the initial data such that a solution to the initial value problem exists on the time interval $[0,T]$, it is unique in a certain Banach space of functions $X \subset C([0,T], H^s_x)$, and the solution map from H^s_x to $C([0,T], H^s_x)$ depends continuously on the initial data on the time interval $[0,T]$. If the time T can be proved to be arbitrarily large, we say that the Cauchy problem is globally well-posed.

To extend a local solution to a global one, we need some a priori information about the norms of the solution. This usually comes from conservation laws. For example, solutions of equation (1.1) satisfy mass conservation

$$\|u(t)\|_{L^2} = \|u_0\|_{L^2} \tag{1.3}$$

and smooth solutions also satisfy energy conservation

$$E(u)(t) = \frac{1}{2}\int |\nabla u(t)|^2\,dx + \frac{1}{p+1}\int |u(t)|^{p+1}\,dx = E(u_0). \tag{1.4}$$

These two conservation laws identify H^1 and L^2 as important spaces concerning the initial value problem (1.1). We can use them to extend the local solutions for all times. For example, based on energy conservation we immediately get that for initial data $u(t_0) = u_0 \in H^1$ we have that $\|u(t)\|_{H^1} \leq C(u_0, t_0)$ for all times.

In order to use this information to iterate the local solutions, the time of local resolution T has to be estimated from below in terms of the norms of the initial data in H^1, $T \geq M(\|u_0\|_{H^1})$, for some strictly positive and nonincreasing function M. This is not the case for the L^2 norm of the L^2-critical problem that corresponds to the case of $p = 1 + \frac{4}{n}$, since the local time depends not only on the norm of the initial data but also on the profile. On the other hand, since the equation (1.1) is energy subcritical in dimensions 1 and 2 for any p, we have that $T \geq M(\|u_0\|_{H^1})$. Thus one can iterate the local resolution and solve the Cauchy problem at time t_{k-1} $(1 \leq k < \infty)$ with initial data $u(t_{k-1})$ up to time $t_k = t_{k-1} + T_k$ with local time $T_k \geq M(\|u(t_{k-1})\|_{H^1})$. Now if the series $\sum T_k$ converges, then on one hand T_k tends to 0, but on the other hand $T_k \geq M(C(u_0, t_0, I))$ where $I = [t_0, t_0 + \sum T_k]$, which is a contradiction. Thus the series $\sum T_k$ diverges and u can be continued for all times in H^1.

In situations where the Cauchy problem is globally well-posed, we can address the question of describing and classifying the asymptotic behavior in time for global solutions. A possible method to attack the question is to compare the given dynamics with suitably chosen simpler asymptotic dynamics. The method applies to a wide variety of dynamical systems and in particular to some systems defined by nonlinear PDEs and give rise to scattering theory. For the semilinear problem (1.1), the first obvious candidate is the free dynamics generated by the group $S(t) = e^{it\Delta}$. The comparison between the two dynamics gives rise to the following two questions:

(1) Let $v_+(t) = S(t)u_+$ be the solution of the free equation. Does there exist a solution u of equation (1.1) that behaves asymptotically as v_+ as $t \to \infty$, typically in the sense that for a Banach space X

$$\|u(t) - v_+\|_X \to 0 \quad \text{when } t \to \infty. \tag{1.5}$$

If this is true, then one can define the map $\Omega_+ : u_+ \to u(0)$. The map is called the wave operator and the problem of existence of u for given u_+ is referred to as the problem of the *existence of the wave operator*. The analogous problem arises as $t \to -\infty$.

(2) Conversely, given a solution u of (1.1), does there exist an asymptotic state u_+ such that $v_+(t) = S(t)u_+$ behaves asymptotically as $u(t)$, typically in the sense of (1.5)? If that is the case for any u with initial data in X for some $u_+ \in X$, one says that *asymptotic completeness* holds in X.

Asymptotic completeness is a much harder problem than the existence of the wave operators except in the case of small-data theory, which follows pretty much from the iteration method proof of the local well-posedness. Asymptotic completeness requires a repulsive nonlinearity and usually proceeds through the derivation of a priori estimates for general solutions. As we have already mentioned, these estimates take advantage of the momentum conservation law

$$\vec{p}(t) = \Im \int_{\mathbb{R}^n} \bar{u} \nabla u \, dx = \vec{p}(0). \tag{1.6}$$

We can establish, for example, the generalized virial inequality [18],[2]

$$\int_0^T \int_{\mathbb{R}^n} (-\Delta\Delta a(x))|u(x,t)|^2\, dx\, dt + \frac{2(p-1)}{p+1}\int_0^T \int_{\mathbb{R}^n} 2\Delta a|u(x,t)|^{p+1} dx\, dt \lesssim \sup_{[0,T]} |M_a(t)| \tag{1.7}$$

where $a(x)$ is a convex function, u is a solution to (1.1), and $M_a(t)$ is the Morawetz action defined by

$$M_a(t) = 2\int_{\mathbb{R}^n} \nabla a \cdot \Im(\bar{u}(x)\nabla u(x)) dx \tag{1.8}$$

One can use this identity as a starting point and derive a priori interaction Morawetz inequalities. These estimates can be achieved by translating the origin in the integrands of (1.7) to an arbitrary point y and then averaging [11] against the L^1 mass density $|u(y)|^2\, dy$, or by considering the tensor product of two solutions of (1.1) and use the fact that the operation of tensoring the two solutions results again in a defocusing nonlinearity.[3]

Both of these methods depend on the fact that for dimension $n \geq 3$ the distribution $-\Delta\Delta|x|$ is positive. The estimate one can obtain for $n \geq 3$ is

$$\|D^{-\frac{n-3}{2}}(|u|^2)\|_{L_t^2 L_x^2} \lesssim \|u\|_{L_t^\infty \dot{H}_x^{1/2}} \|u\|_{L_t^\infty L_x^2}. \tag{1.9}$$

For $n = 3$ this estimate reduces to

$$\|u\|^2_{L_t^4 L_x^4} \lesssim \|u\|_{L_t^\infty \dot{H}_x^{1/2}} \|u\|_{L_t^\infty L_x^2}. \tag{1.10}$$

This estimate is historically the first interaction Morawetz estimate and was obtained in [11]. For $n \geq 4$ it was derived in [25, 26]. The estimate in three dimensions has important consequences. It can be used to prove scattering in the energy space for the three-dimensional problem for any $p - 1 > \frac{4}{3}$. This result was obtained in [16], but the estimate (1.10) gives a very short and elegant proof. One can also combine this estimate with the "I-method" to show global well-posedness and scattering to the three-dimensional cubic nonlinear Schrödinger equation below the energy space [11].

For solutions below the energy threshold, the first result of global well-posedness was established in [2] by decomposing the initial data into low frequencies and high frequencies and estimating separately the evolution of low and high frequencies. The key observation was that the high frequencies behave "essentially unitarily." The method was applied to the cubic equation in two dimensions and established that the solution is globally well-posed with initial data in $H^s(\mathbb{R}^2)$

[2] In fact, one can write an identity.

[3] This idea emerged in a conversation between Andrew Hassell and Terry Tao.

for any $s > \frac{3}{5}$. Moreover, if we denote with S_t the nonlinear flow and with $S(t) = e^{it\Delta}u_0$ the linear group, the high/low frequency method shows in addition that $(S_t - S(t))\,u_0 \in H^1(\mathbb{R}^2)$ for all times provided $u_0 \in H^s, s > \frac{3}{5}$.

Inspired by [2], the I-method (see [11] and references therein) is based on the almost conservation of a certain modified energy functional. The idea is to replace the conserved quantity $E(u)$ that is no longer available for $s < 1$, with an "almost conserved" variant $E(Iu)$, where I is a smoothing operator of order $1 - s$ that behaves like the identity for low frequencies and like a fractional integral operator for high frequencies. Thus, the operator I maps H^s_x to H^1_x. Notice that Iu is not a solution to (1.1), and hence we expect an energy increment. This increment is in fact quantifying $E(Iu)$ as an "almost conserved" energy. The key is to prove that on intervals of fixed length, where local well-posedness is satisfied, the increment of the modified energy $E(Iu)$ decays with respect to a large parameter N. (For the precise definition of I and N, we refer the reader to Section 2.) This requires delicate estimates on the commutator between I and the nonlinearity.

In addition to the H^1 scattering problem, a frequency-localized version of (1.10) is a main ingredient in the proof that the $\dot{H}^1$-critical NLS is globally well-posed and scatters in three dimensions [12]. Note that if (1.9) were true for $n = 2$, we would have

$$\|D^{\frac{1}{2}}(|u|^2)\|_{L^2_t L^2_x} \lesssim \|u\|_{L^\infty_t \dot{H}^{1/2}_x} \|u\|_{L^\infty_t L^2_x}. \tag{1.11}$$

This estimate can be considered as the diagonal, nonlinear analogue of the bilinear refinement of Strichartz in [2] and has many interesting applications. A weaker local-in-time estimate was recently obtained [14]:

$$\|u\|^2_{L^4_T L^4_x} \lesssim T^{\frac{1}{4}} \|u_0\|_{L^2_x} \|u\|_{L^\infty_T \dot{H}^{1/2}}. \tag{1.12}$$

This estimate is very useful since the $L^4_t L^4_x$ norm is a Strichartz norm and can help one to get a global solution assuming control on the local norms. Note the restriction that u has to be at least as regular as an $H^{1/2}$ solution. This estimate was recently improved [8] to

$$\|u\|^2_{L^4_T L^4_x} \lesssim T^{\frac{1}{6}} \|u_0\|^{4/3}_{L^2_x} \|u\|^{2/3}_{L^\infty_T \dot{H}^{1/2}}. \tag{1.13}$$

This a priori estimate along with the I-method was used to establish global well-posedness for the cubic nonlinear Schrödinger equation in two dimensions for any $s > \frac{2}{5}$. Note that these refinements suggest the global Strichartz estimate that would immediately imply for $\theta = 0$, global well-posedness and scattering for the L^2-critical problem

$$\|u\|^2_{L^4_T L^4_x} \lesssim T^{\frac{\theta}{2}} \|u_0\|^{2(1-\theta)}_{L^2_x} \|u\|^{2\theta}_{L^\infty_T \dot{H}^{1/2}}. \tag{1.14}$$

Unfortunately, an argument in [14] shows that by using the above methods, estimate (1.13) is the best possible.

A byproduct of our analysis in [8] provides a new estimate in one dimension, which reads

$$\|u\|_{L_T^6 L_x^6} \lesssim T^{\frac{1}{6}} \|u_0\|_{L_x^2} \|u\|^{1/3}_{L_T^\infty \dot{H}^{1/2}}. \tag{1.15}$$

This estimate was used to prove global well-posedness for the one-dimensional L^2-critical problem for any $s > \frac{1}{3}$ [13]. Note that for all the above problems, the solution is below the $H^{1/2}$ threshold and the a priori estimates are not applicable. One has to introduce a smooth cutoff of the initial data and control certain error terms using multilinear harmonic analysis techniques.

In this paper we prove that (1.11) is indeed true. It is proved by refining the tensor product approach that we mentioned above. Using Sobolev embedding, an immediate consequence of (1.11) is the following:

$$\|u\|^2_{L_t^4 L_x^8} \lesssim \|u_0\|_{L_x^2} \|u\|_{L_T^\infty \dot{H}^{1/2}}. \tag{1.16}$$

One can use this estimate to obtain a simplified proof of the H^1 scattering result in [20] in two dimensions for any $p > 3$. Such a proof avoids the induction on energy argument and produces a better bound on the space-time size of the solution. For completeness we present the proof in Section 4.

We now state the main theorems of this paper. The estimates contained in Theorems 1.1 and 1.2 below were simultaneously and independently obtained [21, 22] by Planchon and Vega.

THEOREM 1.1 (Correlation Estimate in Two Dimensions) *Let u be an $H^{1/2}$ solution to* (1.1) *on the space-time slab $I \times \mathbb{R}^2$. Then*

$$\|D^{\frac{1}{2}}(|u|^2)\|_{L_t^2 L_x^2} \lesssim \|u\|_{L_t^\infty \dot{H}_x^{1/2}} \|u\|_{L_t^\infty L_x^2}. \tag{1.17}$$

THEOREM 1.2 (Correlation Estimates in One Dimension) *Let u be an H^1 solution to* (1.1) *on the space-time slab $I \times \mathbb{R}$. Then*

$$\|\partial_x(|u|^2)\|_{L_t^2 L_x^2} \lesssim \|u\|^{1/2}_{L_t^\infty \dot{H}_x^1} \|u\|^{3/2}_{L_t^\infty L_x^2} \tag{1.18}$$

and

$$\|u\|^{p+3}_{L_t^{p+3} L_x^{p+3}} \lesssim \|u\|^3_{L_t^\infty L_x^2} \|u\|_{L_t^\infty \dot{H}_x^1}. \tag{1.19}$$

THEOREM 1.3 (Asymptotic Completeness in $H^1(\mathbb{R}^2)$) *Let $u_0 \in H^1(\mathbb{R}^2)$. Then there exists a unique global solution u to the initial value problem*

$$\begin{cases} iu_t + \Delta u = |u|^{p-1}u, & p > 1, \\ u(0,x) = u_0(x). \end{cases} \tag{1.20}$$

Moreover, if $p > 3$ there exist $u_\pm \in H^1(\mathbb{R}^2)$ such that

$$\|u(t) - e^{it\Delta}u_\pm\|_{H^1(\mathbb{R}^2)} \to 0 \quad \text{as } t \to \pm\infty.$$

THEOREM 1.4 (Asymptotic Completeness below $H^1(\mathbb{R}^2)$) *Let $u_0 \in H^s(\mathbb{R}^2)$. Then for each positive integer $k \geq 2$, there exists a regularity threshold $s_k = 1 - \frac{1}{4k-3}$ such that the initial value problem*

$$\begin{cases} iu_t + \Delta u = |u|^{2k}u, & k \geq 2, \\ u(0,x) = u_0(x) \end{cases} \tag{1.21}$$

is globally well-posed and scatters provided $s > s_k$. In particular, there exists $u_\pm \in H^s(\mathbb{R}^2)$ such that

$$\|u(t) - e^{it\Delta}u_\pm\|_{H^s(\mathbb{R}^2)} \to 0 \quad \text{as } t \to \pm\infty.$$

We note that estimates (1.18) and (1.17) come from the linear part of the solution and thus are true for any nonlinearity, while estimate (1.19) comes from the nonlinear part. Actually, the proof of Theorem 1.1 shows that the following estimate is true for any $n \geq 2$ (with the appropriate interpretations of course when the power of the derivative operator is positive or negative):

$$\|D^{-\frac{n-3}{2}}(|u|^2)\|_{L_t^2 L_x^2} \lesssim \|u\|_{L_t^\infty \dot{H}_x^{1/2}} \|u\|_{L_t^\infty L_x^2}.$$

The basic idea behind these new estimates is to view the evolution equations as describing the evolution of a compressible dispersive fluid whose pressure is a function of the density. In this case the mass and momentum conservation laws describe the conservation laws of an irrotational compressible and dispersive fluid. There is a difference, though, between one and two dimensions. In two and higher dimensions we use commutator vector operators that act on the conservation laws. In dimension 1 we use the heat kernel.

More precisely, we introduce into the Morawetz action the error function

$$\operatorname{erf}(x) = \int_0^x e^{-t^2}\,dt$$

scaled by ϵ whose derivative is the heat kernel in one dimension. We define the operator that is given as a convolution with the error function and apply it to the conservation laws of the equation. Integration by parts produces the solution of the one-dimensional heat equation. Sending ϵ to 0 we recover the estimates. This way the mass density plays the role of the initial data of the linear heat equation, and the method is nothing other than the Gauss-Weierstrass summability method in classical Fourier analysis. Again, for details the reader can consult Section 4.

The rest of the paper is organized as follows: In Section 2 we introduce some notation and state important propositions that we will use throughout the paper. In Section 3 we present the proofs of the correlation estimates in all dimensions and provide a general framework for obtaining similar estimates. In Section 4 we prove the H^1 scattering result for the L^2-supercritical nonlinear Schrödinger in two dimensions (Theorem 1.3). Finally, in Section 5 we prove global well-posedness and scattering below the energy space of the initial value problem (1.21) (Theorem 1.4.)

2 Notation

In this section, we introduce notation and some basic estimates we will invoke throughout this paper. We use $A \lesssim B$ to denote an estimate of the form $A \leq CB$ for some constant C. If $A \lesssim B$ and $B \lesssim A$ we say that $A \sim B$. We write $A \ll B$ to denote an estimate of the form $A \leq cB$ for some small constant $c > 0$. In addition, $\langle a \rangle := 1 + |a|$ and $a\pm := a \pm \epsilon$ with $0 < \epsilon \ll 1$.

We use $L_x^r(\mathbb{R}^n)$ to denote the Banach space of functions $f : \mathbb{R}^n \to \mathbb{C}$ whose norm

$$\|f\|_r := \left(\int_{\mathbb{R}^n} |f(x)|^r \, dx \right)^{\frac{1}{r}}$$

is finite, with the usual modifications when $r = \infty$.

We use $L_t^q L_x^r$ to denote the space-time norm

$$\|u\|_{q,r} := \|u\|_{L_t^q L_x^r(\mathbb{R}\times\mathbb{R}^n)} := \left(\int_{\mathbb{R}} \left(\int_{\mathbb{R}^n} |u(t,x)|^r \, dx \right)^{\frac{q}{r}} dt \right)^{\frac{1}{q}},$$

with the usual modifications when either q or r are infinity, or when the domain $\mathbb{R}\times\mathbb{R}^n$ is replaced by some smaller space-time region. When $q = r$, we abbreviate $L_t^q L_x^r$ by $L_{t,x}^q$. We define the Fourier transform of $f(x) \in L_x^1(\mathbb{R}^n)$ by

$$\widehat{f}(\xi) = \int_{\mathbb{R}^n} e^{-2\pi i \xi x} f(x) dx.$$

For an appropriate class of functions the following Fourier inversion formula holds:

$$f(x) = \int_{\mathbb{R}^n} e^{2\pi i \xi x} \widehat{f}(\xi)(d\xi).$$

Moreover, we know that the following identities are true:

(1) $\|f\|_{L^2} = \|\widehat{f}\|_{L^2}$ (Plancherel).
(2) $\int_{\mathbb{R}^n} f(x)\bar{g}(x)dx = \int_{\mathbb{R}^n} \widehat{f}(\xi)\bar{\widehat{g}}(\xi)(d\xi)$ (Parseval).
(3) $\widehat{fg}(\xi) = \widehat{f} \star \widehat{g}(\xi) = \int_{\mathbb{R}^n} \widehat{f}(\xi - \xi_1)\widehat{g}(\xi_1)d\xi_1$ (convolution).

We will also make use of the fractional differentiation operators $|\nabla|^s$ defined by

$$\widehat{|\nabla|^s f}(\xi) := |\xi|^s \widehat{f}(\xi).$$

These define the homogeneous Sobolev norms

$$\|f\|_{\dot{H}_x^s} := \||\nabla|^s f\|_{L_x^2}$$

and more general Sobolev norms

$$\|f\|_{H_x^{s,p}} := \|\langle\nabla\rangle^s f\|_p,$$

where $\langle\nabla\rangle = (1 + |\nabla|^2)^{1/2}$. Let $e^{it\Delta}$ be the free Schrödinger propagator. In physical space this is given by the formula

$$e^{it\Delta} f(x) = \frac{1}{(4\pi i t)^{n/2}} \int\limits_{\mathbb{R}^n} e^{\frac{i|x-y|^2}{4t}} f(y) dy$$

for $t \neq 0$ (using a suitable branch cut to define $(4\pi i t)^{d/2}$), while in frequency space one can write this as

$$\widehat{e^{it\Delta} f}(\xi) = e^{-4\pi^2 i t |\xi|^2} \widehat{f}(\xi). \tag{2.1}$$

In particular, the propagator obeys the *dispersive inequality*

$$\|e^{it\Delta} f\|_{L_x^\infty} \lesssim |t|^{-\frac{n}{2}} \|f\|_{L_x^1} \tag{2.2}$$

for all times $t \neq 0$. We also recall *Duhamel's formula*

$$u(t) = e^{i(t-t_0)\Delta} u(t_0) - i \int_{t_0}^t e^{i(t-s)\Delta} (i u_t + \Delta u)(s) ds. \tag{2.3}$$

DEFINITION 2.1 A pair of exponents (q, r) is called *Schrödinger-admissible* if $(q, r, n) \neq (2, \infty, 2)$,

$$\frac{2}{q} + \frac{n}{r} = \frac{n}{2}, \quad 2 \le r \le \infty.$$

For a space-time slab $I \times \mathbb{R}^n$, we define the Strichartz norm

$$\|f\|_{S^0(I)} := \sup_{(q,r) \text{ admissible}} \|f\|_{L_t^q L_x^r (I \times \mathbb{R}^n)}.$$

Then we have the following Strichartz estimates (for a proof, see [17] and the references therein):

LEMMA 2.2 *Let I be a compact time interval, $t_0 \in I$, $s \geq 0$, and let u be a solution to the forced Schrödinger equation*

$$i u_t + \Delta u = \sum_{i=1}^m F_i$$

for some functions $F_1, \dots, F_m$. Then,

$$\||\nabla|^s u\|_{S^0(I)} \lesssim \|u(t_0)\|_{\dot{H}_x^s} + \sum_{i=1}^m \||\nabla|^s F_i\|_{L_t^{q_i'} L_x^{r_i'} (I \times \mathbb{R}^n)} \tag{2.4}$$

for any admissible pairs (q_i, r_i), $1 \le i \le m$. Here p' denotes the conjugate exponent to p, that is, $\frac{1}{p} + \frac{1}{p'} = 1$.

The reader must have in mind that wherever in this paper we restrict the functions in frequency, we do it in a smooth way using the Littlewood-Paley projections. To address the frequency localization in a more precise way, we need some Littlewood-Paley theory. Specifically, let $\varphi(\xi)$ be a smooth bump supported in

$|\xi| \le 2$ and equaling 1 on $|\xi| \le 1$. For each dyadic number $N \in 2^{\mathbb{Z}}$, we define the Littlewood-Paley operators

$$\widehat{P_{\le N} f}(\xi) := \varphi\left(\frac{\xi}{N}\right)\hat{f}(\xi),$$
$$\widehat{P_{> N} f}(\xi) := \left[1 - \varphi\left(\frac{\xi}{N}\right)\right]\hat{f}(\xi),$$
$$\widehat{P_N f}(\xi) := \left[\varphi\left(\frac{\xi}{N}\right) - \varphi\left(2\frac{\xi}{N}\right)\right]\hat{f}(\xi).$$

Similarly, we can define $P_{<N}$, $P_{\ge N}$, and $P_{M<\cdots\le N} := P_{\le N} - P_{\le M}$, whenever M and N are dyadic numbers. We will frequently write $f_{\le N}$ for $P_{\le N} f$ and similarly for the other operators. Using the Littlewood-Paley decomposition we write, at least formally, $u = \sum_N P_N u$. We can write $u = \sum u_N$ and obtain bounds on each piece separately or by examining the interactions of the several pieces. We can recover information for the original function u by applying the Cauchy-Schwarz inequality and using the Littlewood-Paley theorem [23] or the cheap Littlewood-Paley inequality

$$\|P_N u\|_{L^p} \lesssim \|u\|_{L^p}$$

for any $1 \le p \le \infty$. Since this process is fairly standard, we will often omit the details of the argument throughout the paper. We also recall the following standard Bernstein and Sobolev type inequalities. The proofs can be found in [24].

LEMMA 2.3 *For any* $1 \le p \le q \le \infty$ *and* $s > 0$*, we have*

$$\|P_{\ge N} f\|_{L^p_x} \lesssim N^{-s} \||\nabla|^s P_{\ge N} f\|_{L^p_x},$$
$$\||\nabla|^s P_{\le N} f\|_{L^p_x} \lesssim N^s \|P_{\le N} f\|_{L^p_x},$$
$$\||\nabla|^{\pm s} P_N f\|_{L^p_x} \sim N^{\pm s} \|P_N f\|_{L^p_x},$$
$$\|P_{\le N} f\|_{L^q_x} \lesssim N^{\frac{1}{p}-\frac{1}{q}} \|P_{\le N} f\|_{L^p_x},$$
$$\|P_N f\|_{L^q_x} \lesssim N^{\frac{1}{p}-\frac{1}{q}} \|P_N f\|_{L^p_x}.$$

For $N > 1$, we define the Fourier multiplier $I := I_N$

$$\widehat{I_N u}(\xi) := m_N(\xi)\hat{u}(\xi),$$

where m_N is a smooth, radially decreasing function such that

$$m_N(\xi) = \begin{cases} 1 & \text{if } |\xi| \le N, \\ \left(\frac{|\xi|}{N}\right)^{s-1} & \text{if } |\xi| \ge 2N. \end{cases}$$

Thus, I is the identity operator on frequencies $|\xi| \le N$ and behaves like a fractional integral operator of order $1 - s$ on higher frequencies. In particular, I maps H^s_x to H^1_x. We collect the basic properties of the I operator as follows:

LEMMA 2.4 *Let* $1 < p < \infty$ *and* $0 \le \sigma \le s < 1$. *Then*

$$\|If\|_p \lesssim \|f\|_p, \tag{2.5}$$

$$\||\nabla|^\sigma P_{>N} f\|_p \lesssim N^{\sigma-1}\|\nabla I f\|_p, \tag{2.6}$$

$$\|f\|_{H_x^s} \lesssim \|If\|_{H_x^1} \lesssim N^{1-s}\|f\|_{H_x^s}. \tag{2.7}$$

PROOF: The estimate (2.5) is a direct consequence of Hörmander's multiplier theorem.

To prove (2.6), we write

$$\||\nabla|^\sigma P_{>N} f\|_p = \|P_{>N}|\nabla|^\sigma (\nabla I)^{-1}\nabla I f\|_p.$$

The claim follows again from Hörmander's multiplier theorem. Now we turn to (2.7). By the definition of the operator I and (2.6),

$$\begin{aligned}\|f\|_{H_x^s} &\lesssim \|P_{\le N} f\|_{H_x^s} + \|P_{>N} f\|_2 + \||\nabla|^s P_{>N} f\|_2 \\ &\lesssim \|P_{\le N} I f\|_{H_x^1} + N^{-1}\|\nabla I f\|_2 + N^{s-1}\,\|\nabla I f\|_2 \lesssim \|If\|_{H_x^1}.\end{aligned}$$

On the other hand, since the operator I commutes with $\langle\nabla\rangle^s$,

$$\|If\|_{H_x^1} = \|\langle\nabla\rangle^{1-s} I\langle\nabla\rangle^s f\|_2 \lesssim N^{1-s}\|\langle\nabla\rangle^s f\|_2 \lesssim N^{1-s}\|f\|_{H_x^s},$$

which proves the last inequality in (2.7). Note that a similar argument also yields

$$\|If\|_{\dot H_x^1} \lesssim N^{1-s}\|f\|_{\dot H_x^s}. \tag{2.8}$$

□

3 Correlation Estimates in All Dimensions

We consider solutions of the equation

$$iu_t + \Delta u = |u|^{p-1}u, \quad (x,t) \in \mathbb{R}^n \times [0,T]. \tag{3.1}$$

We want to obtain a monotonicity formula that takes advantage of the momentum conservation law of the equation

$$\vec{p}(t) = \int_{\mathbb{R}^n} \Im(\overline{u}(x,t)\nabla u(x,t))dx = \vec{p}(0).$$

We define the Morawetz action

$$M_a(t) = 2\int_{\mathbb{R}^n} \nabla a(x)\cdot \Im(\overline{u}(x)\nabla u(x))dx$$

where $a : \mathbb{R}^n \to \mathbb{R}$, a convex and locally integrable function of polynomial growth. By differentiating $M_a(t)$ with respect to time and using the conservation laws of the equation, we will obtain a priori estimates for solutions of (3.1). To accomplish that, we make a clever choice of the weight function $a(x)$. We note that in all of the cases that we will consider we pick $a(x) = f(|x|)$ where $f : \mathbb{R} \to \mathbb{R}$ is a convex function with the property that $f'(x) \geq 0$ for $x \geq 0$. Then a simple calculation shows that the second-derivative matrix of $a(x)$ is given by

$$\partial_j \partial_k a(x) = f''(|x|)\frac{x_j x_k}{|x|^2} + \frac{f'(|x|)}{|x|}\left(\delta_{kj} - \frac{x_j x_k}{|x|^2}\right).$$

But then the quadratic form $\langle y_j y_k \mid \partial_j \partial_k a(x)\rangle$ is positive definite since

$$\langle y_j y_k \mid \partial_j \partial_k a(x)\rangle = f''(|x|)\frac{(x \cdot y)^2}{|x|^2} + \frac{f'(|x|)}{|x|}\left(|y|^2 - \frac{(x \cdot y)^2}{|x|^2}\right) \geq 0$$

by the Cauchy-Schwarz inequality

$$|x \cdot y| \leq |x||y|.$$

As a final comment for the careful reader, we note that in all our arguments we will assume smooth solutions. This will simplify the calculations and enable us to justify the steps in the subsequent proofs. The local well-posedness theory and the perturbation theory [4] that has been established for this problem can then be applied to approximate the H^s solutions by smooth solutions and conclude the proofs. For most of the calculations in this section the reader can consult [12, 24].

The equation satisfies the following local conservation laws:

- local mass conservation

$$\partial_t \rho + \partial_j p^j = 0$$

- local momentum conservation

$$\partial_t p_k + \partial_j\left(\sigma_k^j + \delta_k^j\left(-\Delta\rho + 2^{\frac{p+1}{2}}\frac{p-1}{p+1}\rho^{\frac{p+1}{2}}\right)\right) = 0$$

where

$$\rho = \frac{1}{2}|u|^2$$

is the mass density,

$$p_j = \Im(\bar{u}\partial_j u)$$

is the momentum density, and

$$\sigma_{jk} = \frac{1}{\rho}(p_j p_k + \partial_j \rho \partial_k \rho)$$

is a stress tensor.

Using the identity

$$\Re(z_1 \bar{z}_2) = \Im z_1 \Im z_2 + \Re z_1 \Re z_2,$$

we can write

$$\sigma_{jk} = \frac{1}{\rho}(p_j p_k + \partial_j \rho \partial_k \rho) = 2\Re(\partial_k u \partial_j \bar{u}).$$

In what follows we will use both definitions of σ_{jk} according to what we find more appropriate with the situation at hand. Note that integration of the first equation leads to mass conservation while integration of the second leads to momentum conservation. We are ready to prove the *generalized virial identity* [18].

PROPOSITION 3.1 *If a is convex and u is a smooth solution to equation* (3.1) *on* $[0,T]\times\mathbb{R}^n$, *then the following inequality holds*:

$$\int_0^T \int_{\mathbb{R}^n} (-\Delta\Delta a)|u(x,t)|^2\,dx\,dt \lesssim \sup_{[0,T]} |M_a(t)|, \tag{3.2}$$

where $M_a(t)$ is the Morawetz action, which is given by

$$M_a(t) = 2\int_{\mathbb{R}^n} \nabla a(x)\cdot \Im(\overline{u}(x)\nabla u(x))dx. \tag{3.3}$$

PROOF: We can write the Morawetz action as

$$M_a(t) = 2\int_{\mathbb{R}^n} (\partial_j a) p_j\,dx.$$

Then

$$\begin{aligned}
\partial_t M_a(t) &= 2\int_{\mathbb{R}^n} (\partial_j a)\partial_t p_j\,dx \\
&= 2\int_{\mathbb{R}^n} \partial_j a\left(-\partial_k\left(\sigma_{jk} + \delta_{kj}\left(-\Delta\rho + 2^{\frac{p+1}{2}}\frac{p-1}{p+1}\rho^{\frac{p+1}{2}}\right)\right)\right)dx \\
&= 2\int_{\mathbb{R}^n} (\partial_j\partial_k a)\sigma_{jk}\,dx - 2\int_{\mathbb{R}^n} \partial_j a\partial_j\left(-\Delta\rho + 2^{\frac{p+1}{2}}\frac{p-1}{p+1}\rho^{\frac{p+1}{2}}\right)dx \\
&= 4\int_{\mathbb{R}^n} (\partial_j\partial_k a)\Re(\partial_k u\partial_j\bar{u})dx \\
&\quad + 2\int_{\mathbb{R}^n} \Delta a\left(-\Delta\rho + 2^{\frac{p+1}{2}}\frac{p-1}{p+1}\rho^{\frac{p+1}{2}}\right)dx \\
&= 4\int_{\mathbb{R}^n} (\partial_j\partial_k a)\Re(\partial_k u\partial_j\bar{u})dx + 2\int_{\mathbb{R}^n} (-\Delta\Delta a)\rho\,dx \\
&\quad + 2^{\frac{p+3}{2}}\frac{p-1}{p+1}\int_{\mathbb{R}^n} (\Delta a)\rho^{\frac{p+1}{2}}\,dx \\
&= 4\int_{\mathbb{R}^n} (\partial_j\partial_k a)\Re(\partial_k u\partial_j\bar{u})dx + \int_{\mathbb{R}^n} (-\Delta\Delta a)|u|^2\,dx \\
&\quad + \frac{2(p-1)}{p+1}\int_{\mathbb{R}^n} (\Delta a)|u|^{p+1}\,dx.
\end{aligned}$$

To prove this identity we used the local conservation of momentum law, integration by parts, and the definitions of ρ and σ_{jk}. But since a is convex, we have that

$$4(\partial_j \partial_k a)\Re(\partial_j \bar{u} \partial_k u) \geq 0.$$

In addition, the trace of the Hessian of $\partial_j \partial_k a$, which is Δa, is positive. Thus,

$$\int_{\mathbb{R}^n} (-\Delta\Delta a)|u|^2 \, dx \leq \partial_t M_a(t),$$

and by the fundamental theorem of calculus we have that

$$\int_0^T \int_{\mathbb{R}^n} (-\Delta\Delta a)|u(x,t)|^2 \, dx \, dt \lesssim \sup_{[0,T]} |M_a(t)|. \tag{3.4}$$

□

3.1 Interaction Morawetz Inequality in Dimension $n \geq 3$

Using the approach above we can derive correlation estimates that are very useful in studying the global well-posedness and the scattering properties of nonlinear dispersive partial differential equations. For clarity in this subsection we reproduce some calculations that have appeared in [8]. Let u_i, F_i be solutions to

$$i u_t + \Delta u = F(u) \tag{3.5}$$

in n_i spatial dimensions. Define the tensor product $u := (u_1 \otimes u_2)(t,x)$ for x in

$$\mathbb{R}^{n_1+n_2} = \{(x_1, x_2) : x_1 \in \mathbb{R}^{n_1},\ x_2 \in \mathbb{R}^{n_2}\}$$

by the formula

$$(u_1 \otimes u_2)(t,x) = u_1(x_1,t) u_2(x_2,t).$$

We abbreviate $u(x_i)$ by u_i and note that if u_1 solves (3.5) with forcing term F_1 and u_2 solves (3.5) with forcing term F_2, then $u_1 \otimes u_2$ solves (3.5) with forcing term $F = F_1 \otimes u_2 + F_2 \otimes u_1$. We have that

$$\begin{aligned} \rho &= \tfrac{1}{2}|u(x)|^2 = 2\rho_1\rho_2, \\ p_k &= \Im(\overline{u_1 u_2}\partial_k(u_1 u_2)), \\ \sigma_{jk} &= 2\Re(\partial_j(u_1u_2)\partial_k(\overline{u_1u_2})), \end{aligned}$$

where $\rho_i = \frac{1}{2}|u_i|^2$, $i = 1, 2$, and similarly for $p_k(u_i)$ and $\sigma_{jk}(u_i)$. Then the local conservation laws can be written in the following way:

$$\begin{aligned} \partial_t \rho + \partial_j p^j &= 0, \\ \partial_t p_k + \partial_j \big(\sigma_k^j + \delta_k^j(-\Delta\rho + G)\big) &= 0, \end{aligned}$$

where

$$G = 2^{\frac{p+1}{2}} \frac{p-1}{p+1} (G_1 \otimes |u_2|^2 + G_2 \otimes |u_1|^2) \geq 0$$

and $G_i = G(u_i) = \rho_i^{(p+1)/2}$. Of course, in this setting $\nabla = (\nabla_{x_1}, \nabla_{x_2})$ and $\Delta = \Delta_{x_1} + \Delta_{x_2}$. If we now apply Proposition 3.1 for the tensor product of the two solutions, we obtain for a convex function a that

$$\int_0^T \int_{\mathbb{R}^{n_1} \otimes \mathbb{R}^{n_2}} (-\Delta\Delta a)|u_1 \otimes u_2|^2(x,t)dx\,dt \lesssim \sup_{[0,T]} |M_a^{\otimes_2}(t)| \tag{3.6}$$

where again $\Delta = \Delta_{x_1} + \Delta_{x_2}$, the Laplacian in $\mathbb{R}^{n_1+n_2}$, and $M_a^{\otimes_2}(t)$ is the Morawetz action that corresponds to $u_1 \otimes u_2$ and thus

$$\begin{aligned} M_a^{\otimes_2}(t) &= 2 \int_{\mathbb{R}^{n_1} \otimes \mathbb{R}^{n_2}} \nabla a(x) \cdot \Im\left(\overline{u_1 \otimes u_2}(x) \nabla(u_1 \otimes u_2(x))\right) dx \\ &= M_a(u_1(t))\|u_2\|_{L^2}^2 + M_a(u_2(t))\|u_1\|_{L^2}^2. \end{aligned}$$

Now we pick $a(x) = a(x_1, x_2) = |x_1 - x_2|$ where $(x_1, x_2) \in \mathbb{R}^n \times \mathbb{R}^n$. Then an easy calculation shows that

$$-\Delta\Delta a(x_1, x_2) = \begin{cases} C_1 \delta(x_1 - x_2) & \text{if } n = 3, \\ \frac{C_2}{|x_1 - x_2|^3} & \text{if } n \geq 4, \end{cases}$$

where C_1, C_2 are constants. Applying equation (3.6) with this choice of a and choosing $u_1 = u_2$, we get that in the case that $n = 3$

$$\int_0^T \int_{\mathbb{R}^3} |u(x,t)|^4\,dx \lesssim \sup_{[0,T]} |M_a^{\otimes_2}(t)|,$$

and in the case that $n \geq 4$,

$$\int_0^T \int_{\mathbb{R}^n \otimes \mathbb{R}^n} |u(x_2,t)|^2 \frac{|u(x_1,t)|^2}{|x_1 - x_2|^3}\,dx_1\,dx_2\,dt \lesssim \sup_{[0,T]} |M_a^{\otimes_2}(t)|.$$

But

$$\int_0^T \int_{\mathbb{R}^n \otimes \mathbb{R}^n} |u(x_2,t)|^2 \frac{|u(x_1,t)|^2}{|x_1 - x_2|^3}\,dx_1\,dx_2\,dt = \int_0^T \int_{\mathbb{R}^n} \left(|u|^2 \star \frac{1}{|\cdot|^3}\right)(x)|u(x)|^2\,dx\,dt.$$

Now we define for $n \geq 4$ the integral operator

$$D^{-(n-3)} f(x) := \int_{\mathbb{R}^n} \frac{u(y)}{|x-y|^3}\,dy$$

where D stands for the derivative. This is indeed defined since for $n \geq 4$ the distributional Fourier transform of $|x|^{-3}$ is given by

$$\widehat{|\cdot|^{-3}}(\xi) = |\xi|^{-(n-3)}.$$

By applying Plancherel's theorem and distributing the derivatives, we obtain that

$$\int_0^T \int_{\mathbb{R}^n \otimes \mathbb{R}^n} |u(x_2,t)|^2 \frac{|u(x_1,t)|^2}{|x_1 - x_2|^3} \, dx_1 \, dx_2 \, dt = \int_0^T \int_{\mathbb{R}^n} |D^{-\frac{n-3}{2}}(|u(x)|^2)|^2 \, dx \, dt.$$

Thus we obtain that

$$\int_0^T \int_{\mathbb{R}^n} \left| D^{-\frac{n-3}{2}}(|u(x)|^2) \right|^2 dx \, dt \lesssim \sup_{[0,T]} |M_a^{\otimes_2}(t)|.$$

For simplicity, we combine the two estimates for $n \geq 3$, pretending that $D^0 = \mathbf{1}$, into

$$\left\| D^{-\frac{n-3}{2}}(|u(x)|^2) \right\|_{L_t^2 L_x^2}^2 \lesssim \sup_{[0,T]} |M_a^{\otimes_2}(t)|.$$

It can be shown using Hardy's inequality (for details, see [11]) that for $n \geq 3$

$$\sup_{[0,T]} |M_a(t)| \lesssim \sup_{[0,T]} \|u(t)\|_{\dot{H}^{1/2}}^2.$$

Since we have that

$$M_a^{\otimes_2}(t) = M_a(u_1(t)) \|u_2\|_{L^2}^2 + M_a(u_2(t)) \|u_1\|_{L^2}^2,$$

we obtain

$$\left\| D^{-\frac{n-3}{2}}(|u(x)|^2) \right\|_{L_t^2 L_x^2}^2 \lesssim \sup_{[0,T]} \|u(t)\|_{\dot{H}^{1/2}}^2 \|u(t)\|_{L^2}^2, \tag{3.7}$$

which is the interaction Morawetz estimates that appears in [11] and in [26].

Remark 3.2. The above method breaks down for $n < 3$ since the distribution $-\Delta\Delta(|x|)$ is not positive anymore.

3.2 Interaction Morawetz Inequality in Two Dimensions

In two dimensions, we follow an alternative approach [8]. In that case $(x_1, x_2) \in \mathbb{R}^2 \times \mathbb{R}^2$. The idea is again to consider the tensor product of two solutions but with a different weight function. We couldn't prove that $-\Delta\Delta a(x)$ is positive. Instead we obtained a difference of two positive functions and balanced the two terms by picking the constants in an appropriate way. The details are as follows:

Let $f : [0,\infty) \to [0,\infty)$ be such that

$$f(x) := \begin{cases} \frac{1}{2M} x^2 (1 - \log \frac{x}{M}) & \text{if } |x| < \frac{M}{\sqrt{e}}, \\ 100x & \text{if } |x| > M, \\ \text{smooth and convex for all } x, & \end{cases}$$

and M is a large parameter that we will choose later. It is obvious that the functions $\frac{1}{2M} x^2 (1 - \log \frac{x}{M})$ and $100x$ are convex in their domain, and the graph of either function lies strictly above the tangent lines of the other. Thus one can construct a function with the above properties. Note also that for $x \geq 0$ we have that $f'(x) \geq 0$. If we apply Proposition 3.1 with the weight $a(x_1, x_2) = f(|x_1 - x_2|)$ and tensoring again two functions, we conclude that

$$\int_0^T \int_{\mathbb{R}^2 \times \mathbb{R}^2} (-\Delta\Delta a(x_1, x_2)) |u(x_1, t)|^2 \, |u(x_2, t)|^2 \, dx_1 \, dx_2 \, dt \lesssim 2 \sup_{[0,T]} |M_a^{\otimes_2}(t)|.$$

But for $|x_1 - x_2| < M/\sqrt{e}$, we have that $\Delta a(x_1, x_2) = \frac{2}{M} \log(\frac{M}{|x_1 - x_2|})$ and thus

$$-\Delta\Delta a(x_1, x_2) = \frac{4\pi}{M} \, \delta_{\{x_1 = x_2\}}.$$

On the other hand, for $|x_1 - x_2| > M$ we have that

$$-\Delta\Delta a(x_1, x_2) = O\left(\frac{1}{|x_1 - x_2|^3}\right) = O\left(\frac{1}{M^3}\right).$$

We have a similar bound in the region in between just because $a(x_1, x_2)$ is smooth, so all in all, we have

$$-\Delta\Delta a(x_1, x_2) = \frac{4\pi}{M} \, \delta_{\{x_1 = x_2\}} + O\left(\frac{1}{M^3}\right).$$

Thus

$$\begin{aligned} &\int_0^T \int_{\mathbb{R}^2 \times \mathbb{R}^2} (-\Delta\Delta a(x_1, x_2)) |u(x_1, t)|^2 |u(x_2, t)|^2 \, dx_1 \, dx_2 \, dt \\ &\quad = O\left(\frac{1}{M}\right) \int_0^T \int_{\mathbb{R}^2} |u(x, t)|^4 \, dx \, dt \\ &\qquad + O\left(\frac{1}{M^3}\right) \int_0^T \int_{\mathbb{R}^2 \times \mathbb{R}^2} |u(x_1, t)|^2 \, |u(x_2, t)|^2 \, dx_1 \, dx_2 \, dt. \end{aligned}$$

By Fubini's theorem

$$\frac{C}{M^3} \int_0^T \int_{\mathbb{R}^2 \times \mathbb{R}^2} |u(x_1, t)|^2 \, |u(x_2, t)|^2 \, dx_1 \, dx_2 \, dt \lesssim \frac{CT}{M^3} \|u\|^4_{L_t^\infty L_x^2}. \tag{3.8}$$

On the other hand, we have

$$\sup_{[0,T]} |M_a^{\otimes_2}(t)| \lesssim \sup_{[0,T]} \|u\|^2_{L_t^\infty L_x^2} \|u\|^2_{L_t^\infty \dot{H}_x^{1/2}}.$$

Thus by applying Proposition 3.1,

$$\frac{1}{M}\int_0^T \int_{\mathbb{R}^2} |u(x,t)|^4 \, dx \, dt \lesssim \sup_{[0,T]} \|u\|^2_{L_t^\infty L_x^2} \|u\|^2_{L_t^\infty \dot{H}_x^{1/2}} + \frac{T}{M^3}\|u\|^4_{L_t^\infty L_x^2}.$$

Multiplying the above equation by M and balancing the two terms on the right-hand side by picking

$$M \sim T^{\frac{1}{3}} \left(\frac{\|u\|_{L_t^\infty L_x^2}}{\|u\|_{L_t^\infty \dot{H}_x^{1/2}}} \right)^{\frac{2}{3}},$$

we get a better estimate,

$$\|u\|^4_{L^4_{t\in[0,T]} L_x^4} \lesssim T^{\frac{1}{3}} \|u\|^{8/3}_{L_t^\infty L_x^2} \|u\|^{4/3}_{L_t^\infty \dot{H}_x^{1/2}}$$

than the one obtained in [14].

3.3 A New Correlation Estimate in Two Dimensions: Proof of Theorem 1.1

We can refine the tensor product approach of the previous subsection and prove a new estimate. Notice that so far we have used $a(r = |x|)$ such that $a(r) \sim r^2 \log \frac{1}{r}$ for $r \sim 0$ and $a(r) \sim r$ for large values of r. In between we didn't provide an explicit formula but used only the quantitative properties of the function. We would like to follow this path one more time and implicitly define a radial function $a : \mathbb{R}^2 \to \mathbb{R}$ such that

$$\Delta a(r) = \int_r^\infty s \log\left(\frac{s}{r}\right) w_{r_0}(s) ds$$

where

$$w_{r_0}(s) := \begin{cases} \frac{1}{s^3} & \text{if } s \geq r_0 \\ 0 & \text{otherwise} \end{cases}$$

and $r_0 > 0$ and small.

In addition, by the definition of $w(x)$ and $a(x)$, we have that

$$\Delta a \geq 0$$

and

$$\int_{\mathbb{R}^2} w_{r_0}(|\vec{x}|) dx = \frac{2\pi}{r_0} \quad \text{or} \quad \int_0^\infty s w_{r_0}(s) ds = \frac{1}{r_0}.$$

Δa can be rewritten as

$$\begin{aligned}\Delta a &= \int_0^\infty s w_{r_0}(s) \log\left(\frac{s}{r}\right) ds - \int_0^r s w_{r_0}(s) \log\left(\frac{s}{r}\right) ds \\ &= -\frac{1}{r_0}\log(r) + \int_0^\infty s w_{r_0}(s) \log(s) ds + \int_0^r s w_{r_0}(s) \log\left(\frac{r}{s}\right) ds.\end{aligned}$$

By setting $\log C = r_0 \int_0^\infty s w_{r_0}(s) \log(s) ds$, we can write

$$\Delta a = \frac{1}{r_0} \log\left(\frac{C}{r}\right) + p(r)$$

where

$$p(r) = \int_0^r w_{r_0}(s) s \log\left(\frac{r}{s}\right) ds.$$

It is immediately clear that the Laplacian of the radial function p is $w_{r_0}(r)$, since an explicit calculation shows this if we use the fact that

$$\Delta p = p_{rr} + \frac{1}{r} p_r.$$

Thus $\Delta p = w_{r_0}$ and

$$-\Delta\Delta a(|x|) = \frac{2\pi}{r_0} \delta(|x|) - w_{r_0}(|x|).$$

We want to apply Proposition 3.1 with $a(\vec{x_1}, \vec{x_2}) = a(|\vec{x_1} - \vec{x_2}|)$ to a tensor product of two functions. We need to prove that $a(r)$ is convex, and as we have already mentioned, this will be immediate if we establish that $a_{rr} \geq 0$ and $a_r \geq 0$. Assuming this is true, we obtain

$$\begin{aligned} &\int_0^T \frac{2\pi}{r_0} \int_{\mathbb{R}^2} |u(\vec{x})|^4 \, d\vec{x} \, dt \\ &- \int_0^T \int_{\mathbb{R}^2 \times \mathbb{R}^2} w_{r_0}(|\vec{x_1} - \vec{x_2}|) |u(\vec{x_1})|^2 |u(\vec{x_2})|^2 \, d\vec{x_1} \, d\vec{x_2} \, dt \\ &\lesssim \sup_{[0,T]} |M_a^{\otimes_2}(t)|. \end{aligned} \tag{3.9}$$

The left-hand side can be rewritten as

$$\begin{aligned} &\int_0^T \frac{2\pi}{r_0} \int_{\mathbb{R}^2} |u(x)|^4 \, dx \, dt \\ &- \int_0^T \int_{\mathbb{R}^2 \times \mathbb{R}^2} w_{r_0}(|x_1 - x_2|) |u(x_1)|^2 |u(x_2)|^2 \, dx_1 \, dx_2 \, dt \\ &= \frac{1}{2} \int_0^T \frac{2\pi}{r_0} \int_{\mathbb{R}^2} |u(x_1)|^4 \, dx_1 \, dt + \frac{1}{2} \int_0^T \frac{2\pi}{r_0} \int_{\mathbb{R}^2} |u(x_2)|^4 \, dx_2 \, dt \\ &- \int_0^T \int_{\mathbb{R}^2 \times \mathbb{R}^2} w_{r_0}(|x_1 - x_2|) |u(x_1)|^2 |u(x_2)|^2 \, dx_1 \, dx_2 \, dt. \end{aligned}$$

Taking into account that

$$\int_{\mathbb{R}^2} w_{r_0}(|x_1 - x_2|)dx_1 = \frac{2\pi}{r_0},$$

we can rewrite (3.9) as

$$\int_0^T \int_{\mathbb{R}^2\times\mathbb{R}^2} \{|u(t,x_1)|^2 - |u(t,x_2)|^2\}^2 w_{r_0}(|x_1 - x_2|)dx_1\, dx_2\, dt \lesssim \sup_{[0,T]} |M_a^{\otimes_2}(t)|.$$

Since this last estimate is true for every $r_0 > 0$, by taking the limit as $r_0 \to 0$ we obtain

$$\int_0^T \int_{\mathbb{R}^2\times\mathbb{R}^2} \frac{\{|u(t,x_1)|^2 - |u(t,x_2)|^2\}^2}{|x_1 - x_2|^3}\, dx_1\, dx_2\, dt \lesssim \sup_{[0,T]} |M_a^{\otimes_2}(t)|.$$

But

$$\int_{\mathbb{R}^2\times\mathbb{R}^2} \frac{\{|u(t,x_1)|^2 - |u(t,x_2)|^2\}^2}{|x_1 - x_2|^3}\, dx_1\, dx_2 \sim \||u|^2\|^2_{\dot{H}^{1/2}};$$

see, for example, [1, exercise 7, p. 162]. Thus we get

$$\|D^{1/2}|u|^2\|^2_{L_t^2 L_x^2} \lesssim \sup_{[0,T]} |M_a^{\otimes_2}(t)|.$$

If $\vec{\nabla} a = (\vec{x}/|\vec{x}|)/a_r$ is bounded, we can estimate $M_a^{\otimes_2}(t)$ as before and obtain the new a priori correlation estimate for solutions of (3.1)

$$\|D^{1/2}|u|^2\|^2_{L_t^2 L_x^2} \lesssim \|u\|^2_{L_t^\infty L_x^2}\, \|u\|^2_{L_t^\infty \dot{H}_x^{1/2}}.$$

Thus it remains to establish that $a(r)$ is convex and that $a_r(r)$ is bounded. For r near 0, we have that $a(r) \sim r^2 \log(\frac{1}{r})$ and thus a_r is bounded for small values of r. In particular, $a_r(0) = 0$. Using this as an initial condition, we can solve in terms of a_r the equation $a_{rr} + \frac{1}{r}a_r = \Delta a$ and obtain

$$a_r(r) = \frac{1}{r}\int_0^r s(\Delta a)(s)ds \geq 0.$$

Thus $a_r \geq 0$. We will shortly show that $a_{rr} \geq 0$ for any $r \geq r_0$ and thus $a_r(r)$ is a positive increasing function. Because of this, it is enough to consider the values of a_r for large values of r. Recall that

$$\Delta a(r) = \int_r^\infty s \log\left(\frac{s}{r}\right) w_{r_0}(s)ds$$

and that for $s \geq r_0$, $w_{r_0}(s) = \frac{1}{s^3}$. Thus

$$\Delta a(r) = \int_r^\infty \frac{1}{s^2} \log\left(\frac{s}{r}\right) ds = \frac{1}{r}.$$

Since for $s \geq r_0$ we have $\Delta a(s) = \frac{1}{s}$, for $s \geq r \geq r_0$ we obtain

$$a_r(r) = \frac{1}{r}\int_0^r ds = 1$$

and a_r is bounded.

It remains to show that $a_{rr} \geq 0$ for $r \geq r_0$. This will be enough since in effect we consider the limit as r_0 tends to 0. To this end notice that

$$a_{rr}(r) = \Delta a(r) - \frac{1}{r}a_r(r) = \Delta a(r) - \frac{1}{r^2}\int_0^r s(\Delta a)(s)ds = \frac{q(r)}{r^2}$$

where

$$q(r) = \int_0^r [2\Delta a(r) - \Delta a(s)]s\,ds.$$

Now we must show that $q(r) \geq 0$. Since $q(0) = 0$, it is enough to show that $q_r(r) \geq 0$. An elementary calculation shows that

$$q_r(r) = r[\Delta a(r) + r(\Delta a)'(r)].$$

Thus we must show that

$$\Delta a(r) + r(\Delta a)'(r) \geq 0.$$

Again, recall that

$$\Delta a(r) = \int_r^\infty s \log\left(\frac{s}{r}\right) w_s(s)ds.$$

If we differentiate with respect to r, we obtain

$$(\Delta a)'(r) = -\frac{1}{r}\int_r^\infty s w_{r_0}(s)ds.$$

Thus

$$\Delta a(r) + r(\Delta a)'(r) = \int_r^\infty s w_{r_0}(s) \log\frac{s}{re}\,ds.$$

A calculation shows that

$$\Delta a(r) + r(\Delta a)'(r) = 0$$

for any $r \geq r_0$ and thus we are done.

3.4 Commutator Vector Operators and Correlation Estimates: An Alternative Proof of Theorem 1.1

In this subsection we derive correlation estimates by using commutator vector operators acting on the conservation laws of the equation. It turns out that this method is more flexible and can also be generalized. Recall that

$$M_a^{\otimes_2}(t) = 2\int_{\mathbb{R}^{n_1}\otimes\mathbb{R}^{n_2}} \nabla a(x)\cdot \Im\left(\overline{u_1\otimes u_2}(x)\nabla(u_1\otimes u_2(x))\right)dx$$

is the Morawetz action for the tensor product of two solutions $u := (u_1 \otimes u_2)(t, x)$ where $x = (x_1, x_2) \in \mathbb{R}^n \otimes \mathbb{R}^n$. If we specialize to the case that $u_1 = u_2$, $a(x) = |x|$, $n \geq 2$, and observe that

$$\partial_{x_1} a(x_1, x_2) = \frac{x_1 - x_2}{|x_1 - x_2|} = -\frac{x_2 - x_1}{|x_1 - x_2|} = -\partial_{x_2} a(x_1, x_2),$$

we can view $M_a^{\otimes_2}(t) := M(t)$ as

$$M(t) = \int_{\mathbb{R}^n \otimes \mathbb{R}^n} \frac{x_1 - x_2}{|x_1 - x_2|} \cdot \{\vec{p}(x_1, t)\rho(x_2, t) - \vec{p}(x_2, t)\rho(x_1, t)\} dx_1\, dx_2 \tag{3.10}$$

where $\rho = \frac{1}{2}|u|^2$ is the mass density and $p_j = \Im(\bar{u}\partial_j u)$ is the momentum density.

Now let's define the integral operator

$$D^{-(n-1)} f(x) = \int_{\mathbb{R}^n} \frac{1}{|x - y|} f(y) dy$$

where D stands for the derivative. This is indeed justified because for $n \geq 2$ the distributional transform of $1/|x|$ is $1/|\xi|^{n-1}$. The main observation is that we can write the action term $M(t)$ using a commutator in the following manner:

$$M(t) = \langle [x, D^{-(n-1)}]\rho(t) \mid \vec{p}(t) \rangle.$$

This equation follows from an elementary rearrangement of the terms of (3.10). This suggests that the estimate is derived using the vector operator, which we will denote by $\vec{X}$, defined by

$$\vec{X} = [x, D^{-(n-1)}].$$

We change notation and write $x_1 := x$ and $x_2 := y$. The crucial property is that the derivatives of this operator $\partial_j X^k$ form a positive definite operator. Note that in physical space

$$\vec{X} f(x) = \int_{\mathbb{R}^n} \frac{x - y}{|x - y|} f(y) dy,$$

and a calculation shows that

$$\partial_j X^k = D^{-(n-1)} \delta_j^k + [x^k, R_j]$$

where R_j is the singular integral operator corresponding to the symbol $\xi_j / |\xi|^{n-1}$.

Thus we have that

$$R_j = \partial_j D^{-(n-1)}$$

and acts on a function in the following manner:

$$R_j f(x) = -\int_{\mathbb{R}^n} \frac{x_j - y_j}{|x - y|^3} f(y) dy.$$

To see how $\partial_j X^k$ acts on functions, we associate a kernel with the commutator $[x_k, R_j]$; let's call it $r_{jk}(x,y)$ and thus

$$[x_k, R_j]f(x) = \int_{\mathbb{R}^n} r_{jk}(x,y) f(y) dy$$

where

$$r_{kj}(x,y) = -\frac{(x_k - y_k)(x_j - y_j)}{|x-y|^3}.$$

Thus

$$(\partial_j X^k) f(x) = \int_{\mathbb{R}^n} \eta_{kj}(x,y) f(y) dy$$

where

$$\eta_{kj}(x,y) = \frac{\delta_{kj}|x-y|^2 - (x_j - y_j)(x_k - y_k)}{|x-y|^3},$$

and thus the derivatives of the vector operator $\vec{X}$ form a positive definite operator. Note also that the divergence of the vector field $\vec{X}$ is given by

$$\nabla \cdot \vec{X} = \partial_j X^j = n D^{-(n-1)} + [x^j, R_j] = (n-1) D^{-(n-1)}.$$

Now if we differentiate

$$M(t) = \langle [x, D^{-(n-1)}]\rho(t) \mid \vec{p}(t) \rangle = \langle \vec{X}\rho(t) \mid \vec{p}(t) \rangle,$$

we obtain that

$$\partial_t M(t) = \langle \vec{X} \partial_t \rho(t) \mid \vec{p}(t) \rangle - \langle \vec{X} \cdot \partial_t \vec{p}(t) \mid \rho(t) \rangle \tag{3.11}$$

where we have used the fact that $\vec{X}$ is an antisymmetric operator.

Now recall the local conservation laws

$$\partial_t \rho + \partial_j p^j = 0, \tag{3.12}$$

$$\partial_t p_k + \partial_j \left(\sigma_k^j + \delta_k^j \left(-\Delta\rho + 2^{\frac{p+1}{2}} \frac{p-1}{p+1} \rho^{\frac{p+1}{2}} \right) \right) = 0. \tag{3.13}$$

To simplify the calculations, we will treat the cubic nonlinearity ($p = 3$), but the method is general and gives the same results for the general nonlinearity $|u|^{p-1}u$. Thus we have

$$\partial_t p_k + \partial_j \big(\sigma_k^j + \delta_k^j (-\Delta\rho + 2\rho^2) \big) = 0. \tag{3.14}$$

Applying the operator to the equation (3.12) and contracting with p_k and similarly applying the operator to equation (3.14) and contracting with ρ, we obtain that

$$\begin{aligned} \partial_t M(t) = {} & \langle \sigma_k^j(t) \mid (\partial_j X^k)\rho(t) \rangle - \langle p^j(t) \mid (\partial_j X^k) p_k(t) \rangle \\ & + \langle (-\Delta\rho(t) + 2\rho^2(t)) \mid (\partial_j X^j)\rho(t) \rangle. \end{aligned}$$

Now, recalling that

$$\sigma_{jk} = \frac{1}{\rho}(p_j p_k + \partial_j \rho \partial_k \rho)$$

we have that

$$\partial_t M(t) = P_1 + P_2 + P_3 + P_4$$

where

$$P_1 := \langle \rho^{-1}\partial_k\rho\partial_j\rho \mid (\partial_j X^k)\rho(t)\rangle, \tag{3.15}$$

$$P_2 := \langle \rho^{-1} p_k p_j \mid (\partial_j X^k)\rho(t)\rangle - \langle p_j \mid (\partial_j X^k) p_k\rangle, \tag{3.16}$$

$$P_3 := \langle (-\Delta\rho) \mid (\partial_j X^j)\rho\rangle = \langle (-\Delta\rho) \mid (\nabla\cdot\vec{X})\rho\rangle \tag{3.17}$$

$$P_4 := 2\langle \rho^2 \mid (\partial_j X^j)\rho\rangle = 2\langle \rho^2 \mid (\nabla\cdot\vec{X})\rho\rangle. \tag{3.18}$$

The term P_1 is clearly positive since $\partial_j X^k$ is a positive definite operator.

Let's analyze P_3. Recalling that $-\Delta = D^2$ we have that

$$\begin{aligned} P_3 = \langle (-\Delta\rho) \mid (\nabla\cdot\vec{X})\rho\rangle &= (n-1)\langle (D^2\rho) \mid D^{-(n-1)}\rho\rangle \\ &= (n-1)\big\langle D^{-\frac{n-3}{2}}\rho \mid D^{-\frac{n-3}{2}}\rho\big\rangle \\ &= \frac{n-1}{2}\big\| D^{-\frac{n-3}{2}}(|u|^2)\big\|_{L^2}^2. \end{aligned}$$

P_4 is also positive since

$$P_4 = 2\int_{\mathbb{R}^n\times\mathbb{R}^n} \frac{\rho^2(x)\rho(y)}{|x-y|}\,dx\,dy = \frac{1}{4}\int_{\mathbb{R}^n\times\mathbb{R}^n} \frac{|u(x)|^4|u(y)|^2}{|x-y|}\,dx\,dy \geq 0.$$

The only term whose positivity is not immediate is term P_2. Recall that

$$(\partial_j X^k) f(x) = \int_{\mathbb{R}^n} \eta_{kj}(x,y) f(y)dy$$

where the kernel $\eta_{kj}(x,y)$ is symmetric. Then

$$P_2 = \int_{\mathbb{R}^n\times\mathbb{R}^n} \left\{\frac{\rho(y)}{\rho(x)} p_k(x)p_j(y) - p_k(y)p_j(x)\right\}\eta_{kj}(x,y)dx\,dy.$$

By changing variables we get

$$P_2 = \int_{\mathbb{R}^n\times\mathbb{R}^n} \left\{\frac{\rho(x)}{\rho(y)} p_k(y)p_j(x) - p_k(x)p_j(y)\right\}\eta_{kj}(x,y)dx\,dy,$$

and thus

$$P_2 = \frac{1}{2}\int_{\mathbb{R}^n\times\mathbb{R}^n}\left\{\frac{\rho(y)}{\rho(x)}p_k(x)p_j(y) + \frac{\rho(x)}{\rho(y)}p_k(y)p_j(x) - p_j(x)p_k(y) - p_j(y)p_k(x)\right\}\eta_{kj}(x,y)dx\,dy$$

$$= \frac{1}{2}\int_{\mathbb{R}^n\times\mathbb{R}^n}\left\{\sqrt{\frac{\rho(y)}{\rho(x)}}p_k(x) - \sqrt{\frac{\rho(x)}{\rho(y)}}p_j(y)\right\} \cdot \left\{\sqrt{\frac{\rho(y)}{\rho(x)}}p_j(x) - \sqrt{\frac{\rho(x)}{\rho(y)}}p_k(y)\right\}\eta_{kj}(x,y)dx\,dy.$$

Thus if we define the two-point momentum vector

$$\vec{J}(x,y) = \sqrt{\frac{\rho(y)}{\rho(x)}}\,\vec{p}(x) - \sqrt{\frac{\rho(x)}{\rho(y)}}\,\vec{p}(y),$$

we can write

$$P_2 = \frac{1}{2}\langle J^j J_k \mid (\partial_j X^k)\rangle \geq 0$$

since $\partial_j X^k$ is positive definite. We keep only P_3, and after integrating in time we have the main estimate of this paper, which reads

$$\left\| D^{-\frac{n-3}{2}}(|u|^2)\right\|^2_{L^2_t L^2_x} \lesssim \sup_t M(t).$$

It remains to show that $M(t)$ is bounded by the appropriate norms. But

$$\begin{aligned} M(t) &= \langle [x, D^{-(n-1)}]_j\rho(t) \mid p_j(t)\rangle \\ &\lesssim \|p_j\|_{L^1}\,\|[x, D^{-(n-1)}]_j\rho(t)\|_{L^\infty} \\ &\lesssim \|p_j\|_{L^1}\,\|\rho\|_{L^1}\,\|[x, D^{-(n-1)}]_j\|_{L^1\to L^\infty}. \end{aligned}$$

Now by Hardy's inequality we have

$$\|p_j\|_{L^1} \lesssim \|u\|^2_{\dot{H}^{1/2}} \quad \text{while} \quad \|\rho\|_{L^1} = \frac{1}{2}\|u\|^2_{L^2}.$$

Finally, the operator norm $\|[x; D^{-(n-1)}]_j\|_{L^1\to L^\infty}$ is bounded by 1 since

$$\vec{X}f(x) = \int_{\mathbb{R}^n}\frac{x-y}{|x-y|}f(y)dy \quad \text{for } f \in L^1.$$

Thus all in all we have that

$$\left\| D^{-\frac{n-3}{2}}(|u|^2)\right\|^2_{L^2_t L^2_x} \lesssim \|u\|^2_{L^\infty_t \dot{H}^{1/2}}\,\|u\|^2_{L^\infty_t L^2_x}$$

is valid for all $n \geq 2$. In particular, for $n = 2$ the estimate reads

$$\left\| D^{\frac{1}{2}}(|u|^2) \right\|^2_{L^2_t L^2_x} \lesssim \|u\|^2_{L^\infty_t \dot{H}^{1/2}} \|u\|^2_{L^\infty_t L^2_x},$$

which corresponds to the nonlinear diagonal case analogue of Bourgain's bilinear refinement of the Strichartz estimate [2]. In this paper, we will use the following estimate in dimension 2:

$$\|u\|^4_{L^4_t L^8_x} \lesssim \|u\|^2_{L^\infty_t \dot{H}^{1/2}} \|u\|^2_{L^\infty_t L^2_x} \tag{3.19}$$

which can be obtained by the previous estimate and the Sobolev embedding in two dimensions, since

$$\|u\|^4_{L^4_t L^8_x} = \||u|^2\|^2_{L^2_t L^4_x} \lesssim \|D^{\frac{1}{2}}(|u|^2)\|^2_{L^2_t L^2_x} \lesssim \|u\|^2_{L^\infty_t \dot{H}^{1/2}} \|u\|^2_{L^\infty_t L^2_x}.$$

Note that the method we used is quite general. Thus we can consider operators of the form

$$\vec{X} := [x, H]$$

where H is a self-adjoint operator. The two crucial properties that we need is that $\partial_j X^k$ is positive and that we can bound the action $M(t)$ for a weight function $a(x)$. We will exploit these in a subsequent paper.

3.5 Correlation Estimates in One Dimension: Proof of Theorem 1.2

In this subsection we would like to prove the analogue of (3.7) in one dimension. Thus we show that

$$\|\partial_x(|u|^2)\|^2_{L^\infty_t L^2_x} \lesssim \|u\|^3_{L^\infty_t L^2_x} \|u\|_{L^\infty_t \dot{H}^1_x} \tag{3.20}$$

for solutions of the one-dimensional NLS $iu_t + u_{xx} = |u|^{p-1}u$ for any p. Since this is a linear estimate as the proof will show, the estimate is true for any power nonlinearity. We will do the calculations for $p = 3$, but the same calculations establish (3.20) for any power nonlinearity. We will follow the Gauss-Weierstrass summability method. The local conservation laws in one dimension can be written in the following form:

$$\partial_t \rho + \partial_x p = 0 \qquad \text{mass conservation,} \tag{3.21}$$

$$\partial_t p + \partial_x \left\{ 2\rho^2 - \rho_{xx} + \frac{1}{\rho}(p^2 + \rho_x^2) \right\} = 0 \qquad \text{momentum conservation} \tag{3.22}$$

where $\rho = \frac{1}{2}|u|^2$ and $p = \Im(\bar{u} u_x)$.

Define the action

$$M(t) = \int\int_{\mathbb{R}\times\mathbb{R}} a(x-y)\rho(y)p(x)dx\,dy$$

where

$$a(x-y) = \operatorname{erf}\left(\frac{x-y}{\epsilon}\right) = \int_0^{\frac{x-y}{\epsilon}} e^{-t^2}\,dt$$

is the scaled error function. This function is bounded. Its derivative is

$$\partial_x \operatorname{erf}\left(\frac{x-y}{\epsilon}\right) = \frac{1}{\epsilon}\, e^{-\frac{(x-y)^2}{\epsilon^2}} \geq 0,$$

which is the heat kernel in one dimension. It is immediate that

$$\sup_t |M(t)| \lesssim \|u\|^3_{L^\infty_t L^2_x}\, \|u\|_{L^\infty_t \dot{H}^1_x}.$$

Notice that the action $M(t)$ can be written as

$$M(t) = \langle X\rho \mid p\rangle,$$

where X is the antisymmetric operator acting on functions as

$$Xf(x) = \left(\operatorname{erf}\left(\frac{\cdot}{\epsilon}\right) \star f\right)(x) = \int_{\mathbb{R}} \operatorname{erf}\left(\frac{x-y}{\epsilon}\right) f(y)dy.$$

The derivative of this operator is the solution of the heat equation in one dimension

$$X^{'} f(x) = \frac{1}{\epsilon}\int_{\mathbb{R}} e^{-\frac{(x-y)^2}{\epsilon^2}} f(y)dy$$

with initial data the function $f(x)$. Since X is antisymmetric and thus $\langle Xf \mid g\rangle = -\langle f \mid Xg\rangle$ by differentiating the action with respect to time, we obtain

$$\dot{M}(t) = \langle X\partial_t\rho \mid p\rangle + \langle X\rho \mid \partial_t p\rangle = -\langle \partial_t\rho \mid Xp\rangle + \langle X\rho \mid \partial_t p\rangle.$$

If we use the conservation laws (3.21) and (3.22) and integrate by parts, we have that

$$\dot{M}(t) = P_1 + P_2 + P_3 + P_4$$

where

$$P_1 = \left\langle X^{'}\rho \;\middle|\; \frac{1}{\rho}\rho_x^2\right\rangle, \quad P_4 = \langle X^{'}\rho \mid 2\rho^2\rangle,$$
$$P_2 = \left\langle X^{'}\rho \;\middle|\; \frac{1}{\rho}p^2\right\rangle - \langle X^{'}p \mid p\rangle.$$

But

$$P_1 = \iint \frac{1}{\epsilon}\, e^{-\frac{(x-y)^2}{\epsilon^2}}\, \frac{\rho(y)}{\rho(x)}\, \rho_x^2(x)dx\, dy \geq 0,$$
$$P_4 = \iint \frac{2}{\epsilon}\, e^{-\frac{(x-y)^2}{\epsilon^2}}\, \rho(y)\rho(x)^2\, dx\, dy \geq 0,$$
$$P_2 = \iint \frac{1}{\epsilon}\, e^{-\frac{(x-y)^2}{\epsilon^2}} \left(\frac{\rho(y)}{\rho(x)} p^2(x) - p(x)p(y)\right) dx\, dy,$$

and thus

$$2P_2 = \iint \frac{1}{\epsilon}\, e^{-\frac{(x-y)^2}{\epsilon^2}} \left(\frac{\rho(y)}{\rho(x)}\, p^2(x) + \frac{\rho(x)}{\rho(y)}\, p^2(y) - 2p(x)p(y)\right) dx\, dy$$
$$= \iint \frac{1}{\epsilon}\, e^{-\frac{(x-y)^2}{\epsilon^2}} \left(\sqrt{\frac{\rho(y)}{\rho(x)}}\, p(x) - \sqrt{\frac{\rho(x)}{\rho(y)}}\, p(y)\right)^2 dx\, dy \geq 0.$$

Thus we have that

$$P_3 \leq \dot{M}(t).$$

But

$$P_3 = \iint \frac{1}{\epsilon}\, e^{-\frac{(x-y)^2}{\epsilon^2}} \rho(y)(-\rho_{xx}(x)) dx\, dy =$$
$$\int \left(\frac{1}{\epsilon}\, e^{-(\frac{\cdot}{\epsilon})^2} \star \rho\right)(x)(-\rho_{xx}(x)) dx = \int \xi^2 \hat{\rho}^2(\xi) e^{-\epsilon \xi^2}\, d\xi \geq 0$$

by Plancherel's theorem. Sending $\epsilon \downarrow 0$ and integrating in time, we obtain (3.20).

Actually more is true. Notice that since

$$\lim_{\epsilon \to 0} \left(\frac{1}{\epsilon}\, e^{-(\frac{\cdot}{\epsilon})^2} \star \rho\right)(x) = \rho(x),$$

we have that

$$\lim_{\epsilon \to 0} P_1 = \int \rho_x^2(x) dx = \frac{1}{4} \|\partial_x(|u|^2)\|_{L_x^2}^2,$$
$$\lim_{\epsilon \to 0} P_2 = 0, \qquad \lim_{\epsilon \to 0} P_4 = \frac{1}{4} \|u\|_{L_x^6}^6.$$

Notice that P_1 and P_3 are linear estimates while P_4 is a nonlinear estimate. Thus if we consider a nonlinearity of the form $|u|^{p-1}u$, we have that

$$\lim_{\epsilon \to 0} P_4 = \frac{1}{2^{\frac{p+1}{2}}} \|u\|_{L_x^{p+3}}^{p+3}.$$

This implies that for the solutions of $iu_t + u_{xx} = |u|^{p-1}u$, we obtain the following a priori one-dimensional estimate:

$$\|u\|_{L_t^{p+3} L_x^{p+3}}^{p+3} \lesssim \|u\|_{L_t^\infty L_x^2}^3 \|u\|_{L_t^\infty \dot{H}_x^1}.$$

Recalling that the scaling is

$$u^\lambda(x,t) = \lambda^{-\frac{2}{p-1}} u\left(\frac{x}{\lambda}, \frac{t}{\lambda^2}\right),$$

we can easily verify that the above estimate is scale invariant.

4 H^1 Scattering: Proof of Theorem 1.3

In this section we prove Theorem 1.3. As we have said, the first proof of this result was obtained in [20] with a more complicated argument using induction on energy. An analogous simplified proof of scattering for the L^2-supercritical NLS problems in one space dimension appeared in [10]. What we have shown so far is that for solutions of (1.1) in two dimensions, the following global a priori estimate is true:

$$\|D^{\frac{1}{2}}(|u|^2)\|_{L_t^2 L_x^2} \lesssim \|u\|_{L_t^\infty \dot{H}_x^{1/2}} \|u\|_{L_t^\infty L_x^2}. \tag{4.1}$$

As we have already mentioned, by Sobolev embedding and using (4.1), we obtain that

$$\|u\|_{L_t^4 L_x^8}^4 \lesssim \|u\|_{L_t^\infty \dot{H}_x^{1/2}}^2 \|u\|_{L_t^\infty L_x^2}^2. \tag{4.2}$$

By conservation of energy and mass, the estimate implies that

$$\|u\|_{L_t^4 L_x^8} \lesssim C_{E(u_0)}. \tag{4.3}$$

To prove scattering, we have to upgrade this control to Strichartz control. Define the norms

$$\|u\|_{S^1} := \sup_{\frac{1}{q}+\frac{1}{r}=\frac{1}{2}} \|\langle\nabla\rangle u\|_{S^0}.$$

Assume that we have

$$\|u\|_{L_t^4 L_x^8} \lesssim C_{E(u_0)}.$$

Divide the real line into finitely many subintervals I_j such that on each I_j we have that

$$\|u\|_{L_{t\in I_j}^4 L_x^8} \sim \delta.$$

We will show that on each I_j we have the bound

$$\|u\|_{S^1(I_j)} \lesssim \|u_0\|_{H^1}. \tag{4.4}$$

Since there are only finitely many I_j's, we have

$$\|u\|_{S^1} \lesssim C_E,$$

and thus scattering follows by standard arguments. Thus it remains to prove (4.4).

We will suppress the I_j notation for what follows. By Duhamel's formula we have

$$u(x,t) = e^{it\Delta}u_0 - i\int_0^T e^{i(t-s)\Delta}(|u|^{p-1}u)(s)ds.$$

By Lemma 2.2 and Hölder's inequality we have that

$$\begin{aligned}\|u\|_{S^1} &\lesssim \|u_0\|_{H^1} + \|\langle\nabla\rangle(|u|^{p-1}u)\|_{L_t^{4/3}L_x^{4/3}}\\ &\lesssim \|u_0\|_{H^1} + \||u|^{p-1}(\langle\nabla\rangle u)\|_{L_t^{4/3}L_x^{4/3}}\\ &\lesssim \|u_0\|_{H^1} + \|\langle\nabla\rangle u\|_{L_t^\infty L_x^2}\|u^{p-1}\|_{L_t^{4/3}L_x^4} \lesssim\end{aligned}$$

$$\lesssim \|u_0\|_{H^1} + \|u\|_{S^1} \, \|u^{p-1}\|_{L_t^{4/3} L_x^4}$$
$$\lesssim \|u_0\|_{H^1} + \|u\|_{S^1} \, \|u\|^{\epsilon}_{L_t^4 L_x^8} \, \|u\|^{p-1-\epsilon}_{L_t^{\frac{4(p-1-\epsilon)}{3-\epsilon}} L_x^{\frac{8(p-1-\epsilon)}{2-\epsilon}}}.$$

This last inequality follows by the interpolation of the L_p spaces. Thus

$$\|u\|_{S^1} \lesssim \|u_0\|_{H^1} + \delta^{\epsilon} \, \|u\|_{S^1} \, \|u\|^{p-1-\epsilon}_{L_t^{\frac{4(p-1-\epsilon)}{3-\epsilon}} L_x^{\frac{8(p-1-\epsilon)}{2-\epsilon}}}.$$

Now we apply Sobolev embedding

$$\|u\|_{L_t^{\frac{4(p-1-\epsilon)}{3-\epsilon}} L_x^{\frac{8(p-1-\epsilon)}{2-\epsilon}}} \lesssim \||\nabla|^{\alpha} u\|_{L_t^{\frac{4(p-1-\epsilon)}{3-\epsilon}} L_x^{\frac{4(p-1-\epsilon)}{2p-5-\epsilon}}}$$

where

$$\alpha = \frac{p-3-\frac{\epsilon}{4}}{p-1-\epsilon}.$$

Note to apply the Sobolev embedding we must have

$$\frac{8(p-1-\epsilon)}{2-\epsilon} > \frac{4(p-1-\epsilon)}{2p-5-\epsilon},$$

a restriction that gives $p > 3 + \frac{\epsilon}{4}$, which is acceptable. For the same reason $\alpha > 0$.

Finally, note that the pair

$$\left(\frac{4(p-1-\epsilon)}{3-\epsilon}, \frac{4(p-1-\epsilon)}{2p-5-\epsilon} \right)$$

is Strichartz admissible, and thus since $\alpha < 1$ we have that

$$\||\nabla|^{\alpha} u\|_{L_t^{\frac{4(p-1-\epsilon)}{3-\epsilon}} L_x^{\frac{4(p-1-\epsilon)}{2p-5-\epsilon}}} \lesssim \|u\|_{S^1}.$$

All in all we have

$$\|u\|_{S^1} \lesssim \|u_0\|_{H^1} + \delta^{\epsilon} \, \|u\|^{p-\epsilon}_{S^1},$$

and by a continuity argument for ϵ small we obtain

$$\|u\|_{S^1} \lesssim C_E. \tag{4.5}$$

We now use this estimate to prove asymptotic completeness, that is, there exist unique $u_{\pm}$ such that

$$\|u(t) - e^{it\Delta} u_{\pm}\|_{H^1(\mathbb{R}^2)} \to 0 \quad \text{as } t \to \pm\infty. \tag{4.6}$$

By time reversal symmetry, it suffices to prove the claim for positive times only. For $t > 0$, we define $v(t) := e^{-it\Delta} u(t)$. We will show that $v(t)$ converges in H_x^1 as $t \to +\infty$, and define u_+ to be the limit. Indeed, by Duhamel's formula,

$$v(t) = u_0 - i \int_0^t e^{-is\Delta} (|u|^{p-1} u)(s) ds. \tag{4.7}$$

Therefore, for $0 < \tau < t$,

$$v(t) - v(\tau) = -i \int_{\tau}^t e^{-is\Delta} (|u|^{p-1} u)(s) ds.$$

Arguing as above, by Lemma 2.2 and Sobolev embedding,

$$\begin{aligned}\|v(t)-v(\tau)\|_{H^1(\mathbb{R}^2)} &\lesssim \|\langle\nabla\rangle(|u|^{p-1}u)\|_{L_t^{4/3}L_x^{4/3}([t,\tau]\times\mathbb{R}^2)} \\ &\lesssim \|u\|^{\epsilon}_{L_{t\in[t,\tau]}{}^4 L_x^8}\|\langle\nabla\rangle u\|^{p-\epsilon}_{S^0([t,\tau])}.\end{aligned}$$

Thus, by (4.3) and (4.5),

$$\|v(t)-v(\tau)\|_{H^1(\mathbb{R})} \to 0 \quad \text{as } \tau, t \to \infty.$$

In particular, this implies u_+ is well-defined. Inspecting (4.7), we find

$$u_+ = u_0 - i\int_0^\infty e^{-is\Delta}(|u|^{p-1}u)(s)\,ds.$$

Using the same estimates as above, it is now an easy matter to derive (4.6). This completes the proof of Theorem 1.3.

5 Proof of Theorem 1.4 and Comments on Further Refinements

There is a problem when one tries to employ the strategy of Section 4 to prove Theorem 1.4. To prove that the problem is globally well-posed and that it scatters, we have to obtain a priori control on the Strichartz norms. The idea is to upgrade (4.2) to obtain control on all the relevant Strichartz norms. The problem is that for solutions below the energy space, the right-hand side of (4.2) is not bounded anymore. Recall that to prove Theorem 1.3 we used the fact that the H^1 norm of the solutions was bounded. Then we used this bound along with estimate (4.2) to bound the S^1 norm of the solutions. Thus to prove Theorem 1.4 we have to bound the H^s norm of the solution uniformly in time for $s < 1$ and then use this bound along with (4.2). The H^1 bound came from conservation of energy, and we do not have at the moment a conserved quantity at the H^s level. But we can define a new functional

$$E(Iu)(t) = \frac{1}{2}\int |\nabla Iu(t)|^2\,dx + \frac{1}{p+1}\int |Iu(t)|^{p+1}\,dx = E(Iu_0), \tag{5.1}$$

where Iu is a solution to the initial value problem

$$\begin{cases} iIu_t + \Delta Iu - I(|u|^{2k}u) = 0, & x\in\mathbb{R}^2,\ t\in\mathbb{R},\\ Iu(x,0) = Iu_0(x)\in H^s(\mathbb{R}^2). \end{cases} \tag{5.2}$$

Note that Iu solves the original equation (1.1) up to an error

$$I(|u|^{2k}u) - |Iu|^{2k}Iu.$$

Because of this we expect the functional $E(Iu)$ to be "almost conserved" in the sense that its derivative will decay with respect to a large parameter. This will allow us to control $E(Iu)$ in time intervals where the local solutions are well-posed, and we can iterate this control to obtain control globally in time. Then immediately we obtain a bound for the H^1 norm of Iu, which by Lemma 2.4 will give us an H^s bound for the solutions u. In this process we will use (4.2). On the other hand, to be able to use (4.2) we need to have H^s control on the norm of u. This feedback argument can be successfully implemented with the help of a standard continuity argument, and this will be the content of this section. We will closely follow the argument in [10].

We start by showing that the functional $E(Iu)$ is almost conserved. We need to define new norms. We fix $t \in [t_0, T]$ and define

$$\|u\|_{Z(t)} := \sup_{(q,r) \text{ admissible}} \Big(\sum_{N \geq 1} \|\nabla P_N u\|^2_{L_t^q L_x^r([t_0,t]\times\mathbb{R})} \Big)^{\frac{1}{2}}$$

with the convention that $P_1 u = P_{\leq 1} u$. We observe the inequality

$$\Big\| \Big(\sum_{N\in 2^{\mathbb{Z}}} |f_N|^2 \Big)^{\frac{1}{2}} \Big\|_{L_t^q L_x^r} \leq \Big(\sum_{N\in 2^{\mathbb{Z}}} \|f_N\|^2_{L_t^q L_x^r} \Big)^{\frac{1}{2}} \tag{5.3}$$

for all $2 \leq q, r \leq \infty$ and arbitrary functions f_N, which one proves by interpolating between the trivial cases $(2,2)$, $(2,\infty)$, $(\infty,2)$, and (∞,∞). In particular, (5.3) holds for all admissible exponents (q,r). Combining this with the Littlewood-Paley inequality, we find

$$\|u\|_{L_t^q L_x^r} \lesssim \Big\| \Big(\sum_{N\in 2^{\mathbb{Z}}} |P_N u|^2 \Big)^{\frac{1}{2}} \Big\|_{L_t^q L_x^r} \lesssim \Big(\sum_{N\in 2^{\mathbb{Z}}} \|P_N u\|^2_{L_t^q L_x^r} \Big)^{\frac{1}{2}}.$$

In particular,

$$\|\nabla u\|_{S^0([t_0,t])} \lesssim \|u\|_{Z(t)}.$$

The appearance of the homogeneous derivative in our definition of the space $Z(t)$ instead of the nonhomogeneous derivative operator $\langle\nabla\rangle$ that we used in [8] is imposed by the level of the criticality. That means that as the problem is L^2-supercritical, the L^2 norm of Iu^λ grows as λ grows. Thus using scaling we cannot control the full H^1 norm of the rescaled solution. This is the reason that we define the Z norm as the homogeneous part of the H^1 norm. The reader can notice that we control all subsequent quantities by the homogeneous part of the H^1 norm where scaling works in our favor.

The dual estimate of (5.3) is

$$\Big(\sum_{N\in 2^{\mathbb{Z}}} \|f_N\|^2_{L_t^{q'} L_x^{r'}} \Big)^{\frac{1}{2}} \leq \Big\| \Big(\sum_{N\in 2^{\mathbb{Z}}} |f_N|^2 \Big)^{\frac{1}{2}} \Big\|_{L_t^{q'} L_x^{r'}}. \tag{5.4}$$

Since the Littlewood-Paley operators commute with $i\partial_t + \Delta$ by Lemma 2.2, we have that

$$\||\nabla| P_N u\|_{S^0(I)} \lesssim \|u(t_0)\|_{\dot{H}_x^s} + \||\nabla| P_N F\|_{L_t^{q_i'} L_x^{r_i'}(I\times\mathbb{R}^n)}. \tag{5.5}$$

Thus

$$\begin{aligned}\|u\|_{Z(t)} &:= \sup_{(q,r)\text{ admissible}} \Big(\sum_{N\geq 1} \|\nabla P_N u\|^2_{L^q_t L^r_x([t_0,t]\times\mathbb{R})}\Big)^{\frac12}\\ &\lesssim \|u(t_0)\|_{\dot H^s_x} + \Big(\sum_{N\in 2^{\mathbb{Z}}} \||\nabla| P_N(i\partial_t+\Delta)\|^2_{L^{q'}_t L^{r'}_x}\Big)^{\frac12}\\ &\lesssim \|u(t_0)\|_{\dot H^s_x} + \Big\|\Big(\sum_{N\in 2^{\mathbb{Z}}} |P_N|\nabla|(i\partial_t+\Delta)|^2\Big)^{\frac12}\Big\|_{L^{q'}_t L^{r'}_x}\end{aligned}$$

where in the last inequality we applied (5.4). Thus if we apply the Littlewood-Paley theorem to this last inequality, we obtain

$$\|u\|_{Z(t)} \lesssim \|u(t_0)\|_{\dot H^s_x} + \||\nabla|(i\partial_t+\Delta)\|_{L^{q'}_t L^{r'}_x}. \tag{5.6}$$

Now we define $Z_I(t) = \|Iu\|_{Z(t)}$.

PROPOSITION 5.1 *Let $s > 1 - \frac{1}{2k-1}$, $k \geq 2$, $k \in \mathbb{N}$, and let u be an H^s_x solution to* (1.1) *on the space-time slab $[t_0, T] \times \mathbb{R}^2$ with $E(Iu(t_0)) \leq 1$. Suppose, in addition, that*

$$\|u\|_{L^4_{t\in[t_0,T]} L^8_x} \leq \eta \tag{5.7}$$

for a sufficiently small $\eta > 0$ (depending on k and on $E(Iu(t_0))$). Then we have

$$\begin{aligned}Z_I(t) &\lesssim \|\nabla I u(t_0)\|_2 + N^{-2} Z_I(t)^{2k+1} + \eta^2 Z_I(t)^{2k-1}\\ &\quad + \eta^2 \sup_{s\in[t_0,t]} E(I_N u(s))^{\frac{k-1}{k+1}} Z_I(t).\end{aligned} \tag{5.8}$$

PROOF: Throughout this proof, all space-time norms are on $[t_0, t] \times \mathbb{R}^2$. By (5.6) and Hölder's inequality, combined with the fact that ∇I acts as a derivative (because the multiplier of ∇I is increasing in $|\xi|$), we estimate

$$\begin{aligned}Z_I(t) &\lesssim \|\nabla I u(t_0)\|_2 + \|\nabla I(|u|^{2k}u)\|_{L^{4/3}_t L^{4/3}_x}\\ &\lesssim \|\nabla I u(t_0)\|_2 + \|u\|^{2k}_{4k,4k} \|\nabla I u\|_{4,4}\\ &\lesssim \|\nabla I u(t_0)\|_2 + \|u\|^{2k}_{4k,4k} Z_I(t).\end{aligned} \tag{5.9}$$

To estimate $\|u\|_{4k,4k}$, we decompose $u := u_{\leq 1} + u_{1<\cdots\leq N} + u_{>N}$. To estimate the low frequencies, we use interpolation and obtain

$$\|u_{\leq 1}\|^{2k}_{4k,4k} \lesssim \|u\|^2_{L^4_t L^8_x} \|u_{\leq 1}\|^{2(k-1)}_{L^\infty_t L^{8(k-1)}_x}.$$

Since for $k \geq 2$ we have that $8(k-1) > 2k+2$ by Bernstein's inequality, we have that

$$\|u_{\leq 1}\|_{L^\infty_t L^{8(k-1)}_x} \lesssim \|u_{\leq 1}\|_{L^\infty_t L^{2k+2}_x} \lesssim E(Iu)^{\frac{1}{2k+2}}$$

where we use the energy bound. Thus

$$\|u_{\leq 1}\|_{4k,4k}^{2k} \lesssim \eta^2 \sup_{s\in[t_0,t]} E(Iu)^{\frac{k-1}{k+1}}. \tag{5.10}$$

For the medium frequencies, again by interpolation we have

$$\|u_{1<\cdots\leq N}\|_{4k,4k}^{2k} \lesssim \|u\|_{L_t^4 L_x^8}^2 \|u_{1<\cdots\leq N}\|_{L_t^\infty L_x^{8(k-1)}}^{2(k-1)}.$$

But by Sobolev embedding

$$\|u_{1<\cdots\leq N}\|_{L_t^\infty L_x^{8(k-1)}}^{2(k-1)} \lesssim \||\nabla|^{\frac{4k-5}{4k-4}} u_{1<\cdots\leq N}\|_{L_t^\infty L_x^2}^{2(k-1)} \lesssim \|\nabla I u\|_{L_t^\infty L_x^2}^{2(k-1)},$$

and thus

$$\|u_{1<\cdots\leq N}\|_{4k,4k}^{2k} \lesssim \eta^2 Z_I(t)^{2(k-1)}. \tag{5.11}$$

Finally, to estimate the high frequencies we apply Lemma 2.4 and Sobolev embedding to obtain

$$\|u_{>N}\|_{4k,4k}^{2k} \lesssim \||\nabla|^{1-\frac{1}{k}} u_{>N}\|_{L_t^{4k} L_x^{\frac{4k}{2k-1}}}^{2k} \lesssim N^{-2} \|\nabla I u\|_{L_t^{4k} L_x^{\frac{4k}{2k-1}}}^{2k}.$$

Since the pair $(4k, \frac{4k}{2k-1})$ is admissible, we obtain

$$\|u_{>N}\|_{4k,4k}^{2k} \lesssim N^{-2} Z_I(t)^{2k}. \tag{5.12}$$

Using (5.9), (5.10), (5.11), and (5.12) we obtain the proposition. □

PROPOSITION 5.2 *Let $s > 1 - \frac{1}{2k-1}$, $k \geq 2$, $k \in \mathbb{N}$, and let u be an H_x^s solution to* (1.1) *on the space-time slab $[t_0, T] \times \mathbb{R}^2$ with $E(Iu(t_0)) \leq 1$. Suppose, in addition, that*

$$\|u\|_{L^4_{t\in[t_0,T]} L_x^8} \leq \eta \tag{5.13}$$

for a sufficiently small $\eta > 0$ (depending on k and on $E(Iu(t_0))$). Then we have

$$\begin{aligned}
&\Big|\sup_{s\in[t_0,t]} E(Iu(s)) - E(Iu(t_0))\Big| \\
&\quad\lesssim N^{-1+}\Bigg(Z_I(t)^{2k+2} + \eta^2 Z_I(t)^2 \sup_{s\in[t_0,t]} E(Iu(s))^{\frac{k-1}{k+1}} \\
&\qquad\qquad + \sum_{J=3}^{2k+2} \eta^{\frac{2k+2-J}{2k-1}} Z_I(t)^J \sup_{s\in[t_0,t]} E(Iu(s))^{\frac{(k-1)(2k+2-J)}{(2k-1)(k+1)}}\Bigg) \\
&\qquad + N^{-1+}\Big(Z_I(t)^{2k+1} + \eta^2 Z_I(t) \sup_{s\in[t_0,t]} E(Iu(s))^{\frac{k-1}{k+1}}\Big) \\
&\qquad\quad \times \Big(Z_I(t)^{2k+1} + \eta \sup_{s\in[t_0,t]} E(Iu(s))^{\frac{k}{k+1}}\Big) +
\end{aligned} \tag{5.14}$$

$$+ N^{-1+} \sum_{J=3}^{2k+2} \eta^{\frac{2k+2-J}{2k-1}} Z_I(t)^{J-1} \sup_{s\in[t_0,t]} E(Iu(s))^{\frac{(k-1)(2k+2-J)}{(2k-1)(k+1)}}$$
$$\times \Big(Z_I(t)^{2k+1} + \eta \sup_{s\in[t_0,t]} E(Iu(s))^{\frac{k}{k+1}} \Big).$$

PROOF: From

$$\frac{d}{dt} E(u(t)) = \Re \int \bar{u}_t (|u|^{2k} u - \Delta u) dx = \Re \int \bar{u}_t (|u|^{2k} u - \Delta u - i u_t) dx,$$

we obtain

$$\begin{aligned} \frac{d}{dt} E(Iu(t)) &= \Re \int I\bar{u}_t (|Iu|^{2k} Iu - \Delta Iu - iIu_t) dx \\ &= \Re \int I\bar{u}_t (|Iu|^{2k} Iu - I(|u|^{2k} u)) dx. \end{aligned}$$

Using the fundamental theorem of calculus and Plancherel, we write

$$\begin{aligned} &E(Iu(t)) - E(Iu(t_0)) \\ &\quad = \Re \int_{t_0}^{t} \int_{\sum_{i=1}^{2k+2} \xi_i = 0} \left(1 - \frac{m(\xi_2 + \xi_3 + \cdots + \xi_{2k+2})}{m(\xi_2) m(\xi_3) \cdots m(\xi_{2k+2})} \right) \\ &\qquad \cdot \widehat{\overline{I\partial_t u}}(\xi_1) \widehat{Iu}(\xi_2) \cdots \widehat{\overline{Iu}}(\xi_{2k+1}) \widehat{Iu}(\xi_{2k+2}) d\sigma(\xi) ds. \end{aligned}$$

Since $iu_t = -\Delta u + |u|^{2k} u$, we need to control

$$\left| \int_{t_0}^{t} \int_{\sum_{i=1}^{2k+2} \xi_i = 0} \left(1 - \frac{m(\xi_2 + \xi_3 + \cdots + \xi_{2k+2})}{m(\xi_2) m(\xi_3) \cdots m(\xi_{2k+2})} \right) \Delta\widehat{\overline{Iu}}(\xi_1) \widehat{Iu}(\xi_2) \cdots \widehat{\overline{Iu}}(\xi_{2k+1}) \widehat{Iu}(\xi_{2k+2}) d\sigma(\xi) ds \right| \tag{5.15}$$

and

$$\left| \int_{t_0}^{t} \int_{\sum_{i=1}^{2k+2} \xi_i = 0} \left(1 - \frac{m(\xi_2 + \xi_3 + \cdots + \xi_{2k+2})}{m(\xi_2) m(\xi_3) \cdots m(\xi_{2k+2})} \right) \widehat{\overline{I(|u|^{2k} u)}}(\xi_1) \widehat{Iu}(\xi_2) \cdots \widehat{\overline{Iu}}(\xi_{2k+1}) \widehat{Iu}(\xi_{2k+2}) d\sigma(\xi) ds \right|. \tag{5.16}$$

We first estimate (5.15). To this end, we decompose

$$u := \sum_{N \geq 1} P_N u$$

with the convention that $P_1 u := P_{\leq 1} u$. Using this notation and symmetry, we estimate

$$(5.17) \qquad (5.15) \lesssim \sum_{\substack{N_1,\dots,N_{2k+2}\geq 1 \\ N_2\geq N_3\geq\cdots\geq N_{2k+2}}} B(N_1,\dots,N_{2k+2}),$$

where

$$\begin{aligned} &B(N_1,\dots,N_{2k+2}) \\ &\quad := \bigg| \int_{t_0}^{t} \int_{\sum_{i=1}^{2k+2}\xi_i=0} \left(1-\frac{m(\xi_2+\xi_3+\cdots+\xi_{2k+2})}{m(\xi_2)m(\xi_3)\cdots m(\xi_{2k+2})}\right) \\ &\qquad \Delta\widehat{Iu_{N_1}}(\xi_1)\widehat{Iu_{N_2}}(\xi_2)\cdots\widehat{\overline{Iu_{N_{2k+1}}}}(\xi_{2k+2})\widehat{Iu_{N_{2k+2}}}(\xi_{2k+2})\,d\sigma(\xi)\,ds\bigg|. \end{aligned}$$

Case I: $N_1 > 1$, $N_2 \geq \cdots \geq N_{2k+2} > 1$.

Case I$_a$: $N \gg N_2$. In this case,

$$m(\xi_2+\xi_3+\cdots+\xi_{2k+2}) = m(\xi_2) = \cdots = m(\xi_{2k+2}) = 1.$$

Thus,

$$B(N_1,\dots,N_{2k+2}) = 0,$$

and the contribution to the right-hand side of (5.17) is 0.

Case I$_b$: $N_2 \gtrsim N \gg N_3$. Since $\sum_{i=1}^{2k+2}\xi_i = 0$, we must have $N_1 \sim N_2$. Thus, by the fundamental theorem of calculus,

$$\begin{aligned} \left|1-\frac{m(\xi_2+\xi_3+\cdots+\xi_{2k+2})}{m(\xi_2)m(\xi_3)\cdots m(\xi_{2k+2})}\right| &= \left|1-\frac{m(\xi_2+\cdots+\xi_{2k+2})}{m(\xi_2)}\right| \\ &\lesssim \left|\frac{\nabla m(\xi_2)(\xi_3+\cdots+\xi_{2k+2})}{m(\xi_2)}\right| \lesssim \frac{N_3}{N_2}. \end{aligned}$$

Applying the multilinear multiplier theorem of Coifman and Meyer (cf. [6, 7]), Sobolev embedding, and Bernstein, and recalling that $N_j > 1$, we estimate

$$\begin{aligned} &B(N_1,\dots,N_{2k+2}) \\ &\quad \lesssim \frac{N_3}{N_2}\|\Delta Iu_{N_1}\|_{4,4}\,\|Iu_{N_2}\|_{4,4}\,\|Iu_{N_3}\|_{4,4}\prod_{j=4}^{2k+2}\|Iu_{N_j}\|_{4(2k-1),4(2k-1)} \\ &\quad \lesssim \frac{N_1}{N_2^2}\prod_{j=1}^{3}\|\nabla Iu_{N_j}\|_{4,4}\prod_{j=4}^{2k+2}\||\nabla|^{\frac{k-2}{2k-1}}Iu_{N_j}\|_{4(2k-1),\frac{4(2k-1)}{4k-3}} \\ &\quad \lesssim \frac{1}{N_2}Z_I(t)^{2k+2} \lesssim N^{-1+}N_2^{0-}Z_I(t)^{2k+2}. \end{aligned}$$

The factor N_2^{0-} allows us to sum in $N_1, N_2, \dots, N_{2k+2}$; this case contributes at most $N^{-1+}Z_I(t)^{2k+2}$ to the right-hand side of (5.17).

Case I_c: $N_2 \gg N_3 \gtrsim N$. Because $\sum_{i=1}^{2k+2} \xi_i = 0$, we must have $N_1 \sim N_2$. Thus, since m is decreasing,

$$\left|1 - \frac{m(\xi_2 + \xi_3 + \cdots + \xi_{2k+2})}{m(\xi_2)m(\xi_3)\cdots m(\xi_{2k+2})}\right| \lesssim \frac{m(\xi_1)}{m(\xi_2)m(\xi_3)\cdots m(\xi_{2k+2})}.$$

Using again the multilinear multiplier theorem, Sobolev embedding, Bernstein, and the fact that $m(\xi)|\xi|^{1/(2k-1)}$ is increasing for $s > 1 - \frac{1}{2k-1}$, we estimate

$$\begin{aligned}
&B(N_1, \dots, N_{2k+2})\\
&\quad\lesssim \frac{m(N_1)}{m(N_2)\cdots m(N_{2k+2})} \frac{N_1}{N_2 N_3}\\
&\qquad \times \prod_{j=1}^{3} \|\nabla I u_{N_j}\|_{4,4} \prod_{j=4}^{2k+2} \||\nabla|^{\frac{2(k-1)}{2k-1}} I u_{N_j}\|_{4(2k-1), \frac{4(2k-1)}{4k-3}}\\
&\quad\lesssim \frac{1}{N_3 m(N_3) \prod_{j=4}^{2k+2} m(N_j) N_j^{\frac{1}{2k-1}}}\\
&\qquad \times \prod_{j=1}^{3} \|\nabla I u_{N_j}\|_{4,4} \prod_{j=4}^{2k+2} \|\nabla I u_{N_j}\|_{4(2k-1), \frac{4(2k-1)}{4k-3}} \lesssim\\
&\quad\lesssim \frac{1}{N_3 m(N_3)} \|\nabla I u_{N_1}\|_{4,4} \|\nabla I u_{N_2}\|_{4,4} Z_I(t)^{2k}\\
&\quad\lesssim N^{-1+} N_3^{0-} \|\nabla I u_{N_1}\|_{4,4} \|\nabla I u_{N_2}\|_{4,4} Z_I(t)^{2k}..
\end{aligned}$$

The factor N_3^{0-} allows us to sum over $N_3, N_4, \dots, N_{2k+2}$. To sum over N_1 and N_2, we use the fact that $N_1 \sim N_2$ and Cauchy-Schwarz to estimate the contribution to the right-hand side of (5.17) by

$$N^{-1+}\Big(\sum_{N_1>1} \|\nabla I u_{N_1}\|_{4,4}^2\Big)^{\frac{1}{2}} \Big(\sum_{N_2>1} \|\nabla I u_{N_2}\|_{4,4}^2\Big)^{\frac{1}{2}} Z_I(t)^{2k} \lesssim N^{-1+} Z_I(t)^{2k+2}.$$

Case I_d: $N_2 \sim N_3 \gtrsim N$. As $\sum_{i=1}^{2k+2} \xi_i = 0$, we obtain $N_1 \lesssim N_2$, and hence $m(N_1) \gtrsim m(N_2)$ and $m(N_1)N_1 \lesssim m(N_2)N_2$. Thus,

$$\left|1 - \frac{m(\xi_2 + \xi_3 + \cdots + \xi_{2k+2})}{m(\xi_2)m(\xi_3)\cdots m(\xi_{2k+2})}\right| \lesssim \frac{m(N_1)}{m(N_2)m(N_3)\cdots m(N_{2k+2})}.$$

Arguing as for Case I_c, we estimate

$$
\begin{aligned}
B(N_1, \ldots, N_{2k+2}) &\lesssim \frac{m(N_1)N_1}{m(N_2)N_2 m(N_3)N_3 \prod_{j=4}^{2k+2} m(N_j)N_j^{\frac{1}{2k-1}}} Z_I(t)^{2k+2} \\
&\lesssim \frac{1}{m(N_3)N_3} Z_I(t)^{2k+2} \\
&\lesssim N^{-1+} N_3^{0-} Z_I(t)^{2k+2}.
\end{aligned}
$$

The factor N_3^{0-} allows us to sum over $N_1, \ldots, N_{2k+2}$. This case contributes at most $N^{-1+} Z_I(t)^{2k+2}$ to the right-hand side of (5.17).

Case II: There exists $1 \leq j_0 \leq 2k+2$ such that $N_{j_0} = 1$. Recall that by our convention, $P_1 := P_{\leq 1}$.

Case II_a: $N_1 = 1$. Let J be such that $N_2 \geq \cdots \geq N_J > 1 = N_{J+1} = \cdots = N_{2k+2}$. Note that we may assume $J \geq 3$ since otherwise

$$B(N_1, \ldots, N_{2k+2}) = 0.$$

Also, arguing as for Case I_a, if $N \gg N_2$ then

$$B(N_1, \ldots, N_{2k+2}) = 0.$$

Thus, we may assume $N_2 \gtrsim N$. In this case we cannot have $N_2 \gg N_3$ since it would contradict $\sum_{i=1}^{2k+2} \xi_i = 0$ and $N_1 = 1$. Hence, we must have

$$N_2 \sim N_3 \gtrsim N.$$

Because

$$\left| 1 - \frac{m(\xi_2 + \xi_3 + \cdots + \xi_{2k+2})}{m(\xi_2)m(\xi_3)\cdots m(\xi_{2k+2})} \right| \lesssim \frac{1}{m(N_2)m(N_3)\cdots m(N_{2k+2})},$$

we use the multilinear multiplier theorem and Sobolev embedding to estimate

$$
\begin{aligned}
&B(N_1, \ldots, N_{2k+2}) \\
&\quad \lesssim \frac{N_1}{m(N_2)N_2 m(N_3)N_3 m(N_4)\cdots m(N_{2k+2})} \prod_{j=1}^{3} \|\nabla I u_{N_j}\|_{4,4} \\
&\qquad \times \prod_{j=4}^{J} \||\nabla|^{\frac{2(k-1)}{2k-1}} I u_{N_j}\|_{4(2k-1), \frac{4(2k-1)}{4k-3}} \prod_{j=J+1}^{2k+2} \|I u_{N_j}\|_{4(2k-1), 4(2k-1)} \lesssim
\end{aligned}
$$

$$\lesssim \frac{1}{m(N_2)N_2 m(N_3)N_3 \prod_{j=4}^{J} m(N_j) N_j^{\frac{1}{2k-1}}} Z_I(t)^J \times \prod_{j=J+1}^{2k+2} \|Iu_{N_j}\|_{4(2k-1),4(2k-1)}$$

$$\lesssim N^{-2+} N_2^{0-} Z_I(t)^J \prod_{j=J+1}^{2k+2} \|Iu_{N_j}\|_{4(2k-1),4(2k-1)}.$$

Applying interpolation, the bound for the $L_t^4 L_x^8$ norm of u that we assumed in (5.13), and Bernstein, we bound

$$\begin{aligned} \|Iu_{\leq 1}\|_{4(2k-1),4(2k-1)} &\lesssim \|Iu_{\leq 1}\|_{L_t^4 L_x^8}^{\frac{1}{2k-1}} \|Iu_{\leq 1}\|_{L_t^\infty L_x^{16(k-1)}}^{\frac{2(k-1)}{2k-1}} \\ &\lesssim \|Iu_{\leq 1}\|_{L_t^4 L_x^8}^{\frac{1}{2k-1}} \|Iu_{\leq 1}\|_{L_t^\infty L_x^{2k+2}}^{\frac{2(k-1)}{2k-1}} \\ &\lesssim \eta^{\frac{1}{2k-1}} \sup_{s\in[t_0,t]} E(Iu(s))^{\frac{k-1}{(2k-1)(k+1)}}. \end{aligned}$$

Thus,

$$B(N_1,\dots,N_{2k+2}) \lesssim N^{-2+} N_2^{0-} \eta^{\frac{2k+2-J}{2k-1}} Z_I(t)^J \sup_{s\in[t_0,t]} E(Iu(s))^{\frac{(k-1)(2k+2-J)}{(2k-1)(k+1)}}.$$

The factor N_2^{0-} allows us to sum in $N_2,\dots,N_J$. This case contributes at most

$$N^{-2+} \sum_{J=3}^{2k+2} \eta^{\frac{2k+2-J}{2k-1}} Z_I(t)^J \sup_{s\in[t_0,t]} E(Iu(s))^{\frac{(k-1)(2k+2-J)}{(2k-1)(k+1)}}$$

to the right-hand side of (5.17).

Case II_b: $N_1 > 1$ and $N_2 = \cdots = N_{2k+2} = 1$. As $\sum_{i=1}^{2k+2} \xi_i = 0$, we obtain $N_1 \lesssim 1$ and thus, taking N sufficiently large depending on k, we get

$$1 - \frac{m(\xi_2+\xi_3+\cdots+\xi_{2k+2})}{m(\xi_2)m(\xi_3)\cdots m(\xi_{2k+2})} = 0.$$

This case contributes 0 to the right-hand side of (5.17).

Case II_c: $N_1 > 1$ and $N_2 > 1 = N_3 = \cdots = N_{2k+2}$. Because $\sum_{i=1}^{2k+2} \xi_i = 0$, we must have $N_1 \sim N_2$. If $N_1 \sim N_2 \ll N$, then

$$1 - \frac{m(\xi_2+\xi_3+\cdots+\xi_{2k+2})}{m(\xi_2)m(\xi_3)\cdots m(\xi_{2k+2})} = 0$$

and the contribution is 0. Thus we may assume $N_1 \sim N_2 \gtrsim N$. Applying the fundamental theorem of calculus,

$$\left|1 - \frac{m(\xi_2 + \xi_3 + \cdots + \xi_{2k+2})}{m(\xi_2)m(\xi_3)\cdots m(\xi_{2k+2})}\right| = \left|1 - \frac{m(\xi_2 + \cdots + \xi_{2k+2})}{m(\xi_2)}\right| \lesssim \left|\frac{\nabla m(\xi_2)}{m(\xi_2)}\right| \lesssim \frac{1}{N_2}.$$

By the multilinear multiplier theorem,

$$\begin{aligned} B(N_1, \dots, N_{2k+2}) &\lesssim \frac{1}{N_2} \|\Delta I u_{N_1}\|_{4,4} \, \|I u_{N_2}\|_{4,4} \prod_{j=3}^{2k+2} \|I u_{N_j}\|_{4k,4k} \\ &\lesssim \frac{N_1}{N_2^2} \|\nabla I u_{N_1}\|_{4,4} \, \|\nabla I u_{N_2}\|_{4,4} \, \|I u_{\le 1}\|_{4k,4k}^{2k} \\ &\lesssim N^{-1+} N_2^{0-} Z_I(t)^2 \|I u_{\le 1}\|_{4k,4k}^{2k}. \end{aligned}$$

The factor N_2^{0-} allows us to sum in N_1 and N_2. Using interpolation, (2.5), (5.13), and Bernstein, we estimate

$$\begin{aligned} \|I u_{\le 1}\|_{4k,4k} &\lesssim \|I u_{\le 1}\|_{L_t^4 L_x^8}^{1/k} \|I u_{\le 1}\|_{L_t^\infty L_x^{8(k-1)}}^{1-1/k} \\ &\lesssim \eta^{\frac{1}{k}} \|I u_{\le 1}\|_{L_t^\infty L_x^{2k+2}}^{1-1/k} \\ &\lesssim \eta^{\frac{1}{k}} \sup_{s \in [t_0,t]} E(I u(s))^{\frac{k-1}{2k(k+1)}}. \end{aligned}$$

Thus, this case contributes at most

$$N^{-1+} \eta^2 Z_I(t)^2 \sup_{s \in [t_0,t]} E(I u(s))^{\frac{k-1}{k+1}}$$

to the right-hand side of (5.17).

Case II$_d$: $N_1 > 1$ and there exists $J \ge 3$ such that $N_2 \ge \cdots \ge N_J > 1 = N_{J+1} = \cdots = N_{2k+2}$. To estimate the contribution of this case, we argue as for Case I; the only new ingredient is that the low frequencies are estimated via (5.18). This case contributes at most

$$N^{-1+} \sum_{J=3}^{2k+2} \eta^{\frac{2k+2-J}{2k-1}} Z_I(t)^J \sup_{s \in [t_0,t]} E(I u(s))^{\frac{(k-1)(2k+2-J)}{(2k-1)(k+1)}}$$

to the right-hand side of (5.17). Putting everything together, we get

$$\begin{aligned} (5.15) &\lesssim N^{-1+} Z_I(t)^{2k+2} + N^{-1+} \eta^2 Z_I(t)^2 \sup_{s \in [t_0,t]} E(I u(s))^{\frac{k-1}{k+1}} \\ &\quad + N^{-1+} \sum_{J=3}^{2k+2} \eta^{\frac{2k+2-J}{2k-1}} Z_I(t)^J \sup_{s \in [t_0,t]} E(I u(s))^{\frac{(k-1)(2k+2-J)}{(2k-1)(k+1)}}. \end{aligned} \tag{5.18}$$

We turn now to estimating (5.16). Again we decompose

$$u := \sum_{N\geq 1} P_N u$$

with the convention that $P_1 u := P_{\leq 1} u$. Using this notation and symmetry, we estimate

$$(5.16) \lesssim \sum_{\substack{N_1,\ldots,N_{2k+2}\geq 1\\ N_2\geq\cdots\geq N_{2k+2}}} C(N_1,\ldots,N_{2k+2}),$$

where

$$\begin{aligned}
&C(N_1,\ldots,N_{2k+2})\\
&:= \left| \int_{t_0}^{t} \int_{\sum_{i=1}^{2k+2}\xi_i=0} \left(1-\frac{m(\xi_2+\xi_3+\cdots+\xi_{2k+2})}{m(\xi_2)m(\xi_3)\cdots m(\xi_{2k+2})}\right)\right.\\
&\left.\quad \widehat{P_{N_1} I(|u|^{2k}u)}(\xi_1)\widehat{Iu_{N_2}}(\xi_2)\cdots\widehat{Iu_{N_{2k+1}}}(\xi_{2k+1})\widehat{Iu_{N_{2k+2}}}(\xi_{2k+2})d\sigma(\xi)ds\right|.
\end{aligned}$$

In order to estimate $C(N_1,\ldots,N_{2k+2})$, we make the observation that in estimating $B(N_1,\ldots,N_{2k+2})$, for the term involving the N_1 frequency we only use the bound

$$\|P_{N_1} I\Delta u\|_{4,4} \lesssim N_1 \|\nabla I u_{N_1}\|_{4,4} \lesssim N_1 Z_I(t). \tag{5.19}$$

Thus, to estimate (5.16) it suffices to prove

$$\|P_{N_1} I(|u|^{2k}u)\|_{4,4} \lesssim Z_I(t)^{2k+1} + \eta \sup_{s\in[t_0,t]} E(Iu(s))^{\frac{k}{k+1}}, \tag{5.20}$$

for then, arguing as for (5.15) and substituting (5.20) for (5.19), we obtain

$$\begin{aligned}
(5.16) \lesssim N^{-1+}&\Big(Z_I(t)^{2k+1} + \eta^2 Z_I(t) \sup_{s\in[t_0,t]} E(Iu(s))^{\frac{k-1}{k+1}}\Big)\\
&\times\Big(Z_I(t)^{2k+1} + \eta \sup_{s\in[t_0,t]} E(Iu(s))^{\frac{k}{k+1}}\Big)\\
&+ N^{-1+}\sum_{J=3}^{2k+2} \eta^{\frac{2k+2-J}{2k-1}} Z_I(t)^{J-1} \sup_{s\in[t_0,t]} E(Iu(s))^{\frac{(k-1)(2k+2-J)}{(2k-1)(k+1}}\\
&\times\Big(Z_I(t)^{2k+1} + \eta \sup_{s\in[t_0,t]} E(Iu(s))^{\frac{k}{k+1}}\Big).
\end{aligned}$$

Thus, we are left to proving (5.20). Using (2.5) and the boundedness of the Littlewood-Paley operators, and decomposing $u := u_{\leq 1} + u_{>1}$, we estimate

$$\begin{aligned}
\|P_{N_1} I(|u|^{2k}u)\|_{4,4} &\lesssim \|u\|_{4(2k+1),4(2k+1)}^{2k+1}\\
&\lesssim \|u_{\leq 1}\|_{4(2k+1),4(2k+1)}^{2k+1} + \|u_{>1}\|_{4(2k+1),4(2k+1)}^{2k+1}.
\end{aligned}$$

Applying interpolation, (5.13), and Bernstein, we estimate

$$\|u_{\le 1}\|_{4(2k+1),4(2k+1)}^{2k+1} \lesssim \|u_{\le 1}\|_{L_t^4 L_x^8} \|u_{\le 1}\|_{L_t^\infty L_x^{16k}}^{2k}$$
$$\lesssim \eta \|u_{\le 1}\|_{L_t^\infty L_x^{2k+2}}^{2k} \lesssim \eta \sup_{s\in[t_0,t]} E(Iu(s))^{\frac{k}{k+1}}.$$

Finally, by Sobolev embedding and (2.6),

$$\|u_{>1}\|_{4(2k+1),4(2k+1)}^{2k+1} \lesssim \||\nabla|^{\frac{2k}{2k=1}} u_{>1}\|_{4(2k+1),\frac{4(2k+1)}{4k+1}}^{2k+1} \lesssim Z_I(t)^{2k+1}.$$

Putting things together, we derive (5.20). This completes the proof of Proposition 5.2. □

Now we will combine Propositions 5.1 and 5.2 and prove that the quantity $E(Iu)(t)$ is "almost conserved."

PROPOSITION 5.3 *Let $s > \frac{1}{2k-1}$ and let u be an H_x^s solution to* (1.1) *on the space-time slab* $[t_0, T] \times \mathbb{R}^2$ *with* $E(I_N u(t_0)) \le 1$. *Suppose in addition that*

$$\|u\|_{L^4_{t\in[t_0,T]} L_x^8} \le \eta \tag{5.21}$$

for a sufficiently small $\eta > 0$ (*depending on k and on $E(I_N u(t_0))$*). *Then, for N sufficiently large* (*depending on k and on $E(I_N u(t_0))$*),

$$\sup_{t\in[t_0,T]} E(I_N u(t)) = E(I_N u(t_0)) + N^{-1+}. \tag{5.22}$$

PROOF: Indeed, Proposition 5.3 follows immediately from Propositions 5.1 and 5.2, if we establish

$$Z_I(t) \lesssim 1 \quad \text{and} \quad \sup_{s\in[t_0,t]} E(I_N u(s)) \lesssim 1 \quad \text{for all } t \in [t_0, T].$$

Given the assumption that $E(I_N u(t_0)) \lesssim 1$, it suffices to show that

$$Z_I(t) \lesssim \|\nabla I_N u(t_0)\|_2 \quad \text{for all } t \in [t_0, T] \tag{5.23}$$

and

$$\sup_{s\in[t_0,t]} E(I_N u(s)) \lesssim E(I_N u(t_0)) \quad \text{for all } t \in [t_0, T]. \tag{5.24}$$

We achieve this via a bootstrap argument. We want to show that the set of times that those two properties hold is the set $[0, \infty)$. We define

$$\Omega_1 := \Big\{t \in [t_0, T] : Z_I(t) \le C_1 \|\nabla I_N u(t_0)\|_2,$$
$$\sup_{s\in[t_0,t]} E(I_N u(s)) \le C_2 E(I_N u(t_0))\Big\}$$
$$\Omega_2 := \Big\{t \in [t_0, T] : Z_I(t) \le 2C_1 \|\nabla I_N u(t_0)\|_2,$$
$$\sup_{s\in[t_0,t]} E(I_N u(s)) \le 2C_2 E(I_N u(t_0))\Big\}.$$

If we can prove that Ω_1 is nonempty, open, and closed, then since the set $[0,\infty)$ is connected, we must have that $\Omega_1 = [0,\infty)$. Thus in order to run the bootstrap argument successfully, we need to check four things:

(1) $\Omega_1 \neq \varnothing$. This is satisfied since $t_0 \in \Omega_1$ if we take C_1 and C_2 sufficiently large.
(2) Ω_1 is a closed set. This follows from Fatou's lemma.
(3) If $t \in \Omega_1$, then there exists $\epsilon > 0$ such that $[t, t+\epsilon] \in \Omega_2$. This follows from the dominated convergence theorem combined with (5.8) and (5.14).
(4) $\Omega_2 \subset \Omega_1$. This follows from (5.8) and (5.14) taking C_1 and C_2 sufficiently large depending on absolute constants (like the Strichartz constant) and choosing N sufficiently large and η sufficiently small depending on C_1, C_2, k, and $E(I_N u(t_0))$.

The last two points prove that Ω_1 is open and so Proposition 5.3 is proved. □

Finally, we are ready to prove Theorem 1.4.

PROOF OF THEOREM 1.4: Given Proposition 5.3, the proof of global well-posedness for (1.1) is reduced to showing

$$\|u\|_{L_t^4 L_x^8} \leq C(\|u_0\|_{H_x^s}). \tag{5.25}$$

This also implies scattering, as we will see later by an argument close to what we used to obtain Theorem 1.3. We have proved that

$$\|u\|_{L_t^4 L_x^8} \lesssim \|u_0\|_2^{1/2} \, \|u\|^{1/2}_{L_t^\infty \dot{H}_x^{1/2}(I\times\mathbb{R})} \tag{5.26}$$

on any space-time slab $I \times \mathbb{R}^2$ on which the solution to (1.1) exists and lies in $H_x^{1/2}$. However, the $H_x^{1/2}$ norm of the solution is not a conserved quantity either, and in order to control it we must resort to the H_x^s bound on the solution. As we remarked at the beginning of this section, this will be achieved by controlling $\|Iu\|_{\dot{H}^1}$. Thus, in order to obtain a global Morawetz estimate, we need a global bound for $\|Iu\|_{\dot{H}^1}$. This will be done by patching together time intervals where the norm $\|u\|_{L_t^4 L_x^8}$ is very small.

This sets us up for a bootstrap argument. Let u be the solution to (1.1). Because $E(Iu_0)$ is not necessarily small, we first rescale the solution such that the energy of the rescaled initial data satisfies the conditions in Proposition 5.3. By scaling,

$$u^\lambda(x,t) := \lambda^{-\frac{1}{k}} u(\lambda^{-2}t, \lambda^{-1}x)$$

is also a solution to (1.1) with initial data

$$u_0^\lambda(x) := \lambda^{-\frac{1}{k}} u_0(\lambda^{-1}x).$$

By (2.8) and Sobolev embedding for $s \geq 1 - \frac{1}{k+1}$,

$$\|\nabla I u_0^\lambda\|_2 \lesssim N^{1-s} \|u_0^\lambda\|_{\dot{H}_x^s} = N^{1-s} \lambda^{1-\frac{1}{k}-s} \|u_0\|_{\dot{H}_x^s},$$
$$\|I_N u_0^\lambda\|_{2k+2} \lesssim \|u_0^\lambda\|_{2k+2} = \lambda^{\frac{1}{k+1}-\frac{1}{k}} \|u_0\|_{2k+2} \lesssim \lambda^{\frac{1}{k+1}-\frac{1}{k}} \|u_0\|_{H_x^s}.$$

Since $s > 1-\frac{1}{4k-3} > 1-\frac{1}{k+1} > 1-\frac{1}{k}$, choosing λ sufficiently large (depending on $\|u_0\|_{H_x^s}$ and N) such that

$$N^{1-s} \lambda^{1-\frac{1}{k}-s} \|u_0\|_{H_x^s} \ll 1 \quad \text{and} \quad \lambda^{\frac{1}{k+1}-\frac{1}{k}} \|u_0\|_{H_x^s} \ll 1, \tag{5.27}$$

we get

$$E(I_N u_0^\lambda) \ll 1.$$

Thus

$$\lambda \sim N^{\frac{s-1}{1-s-1/k}}.$$

We now show that there exists an absolute constant C_1 such that

$$\|u^\lambda\|_{L_t^4 L_x^8} \leq C_1 \lambda^{\frac{3}{4}(1-\frac{1}{k})}. \tag{5.28}$$

Undoing the scaling, this yields (5.25). We prove (5.28) via a bootstrap argument. By time reversal symmetry, it suffices to argue for positive times only. Define

$$\Omega_1 := \{t \in [0,\infty) : \|u^\lambda\|_{L_t^4 L_x^8([0,t]\times\mathbb{R}^2)} \leq C_1 \lambda^{\frac{3}{4}(1-\frac{1}{k})}\},$$
$$\Omega_2 := \{t \in [0,\infty) : \|u^\lambda\|_{L_{t\in[0,t]}^4 L_x^8([0,t]\times\mathbb{R}^2)} \leq 2C_1 \lambda^{\frac{3}{4}(1-\frac{1}{k})}\}.$$

In order to run the bootstrap argument, we need to verify four things:

(1) $\Omega_1 \neq \varnothing$. This is obvious since $0 \in \Omega_1$.
(2) Ω_1 is closed. This follows from Fatou's lemma.
(3) $\Omega_2 \subset \Omega_1$.
(4) If $T \in \Omega_1$, then there exists $\epsilon > 0$ such that $[T, T+\epsilon) \subset \Omega_2$. This is a consequence of the local well-posedness theory and the proof of (3). We skip the details.

Thus, we need to prove (3). Fix $T \in \Omega_2$; we will show that in fact, $T \in \Omega_1$. By (5.26) and the conservation of mass,

$$\begin{aligned}\|u^\lambda\|_{L_t^4 L_x^8([0,t]\times\mathbb{R}^2)} &\lesssim \|u_0^\lambda\|_2^{1/2} \|u^\lambda\|_{L_t^\infty \dot{H}_x^{1/2}([0,T]\times\mathbb{R}^2)}^{1/2} \\ &\lesssim \lambda^{\frac{1}{2}(1-\frac{1}{k})} C(\|u_0\|_2) \|u^\lambda\|_{L_t^\infty \dot{H}_x^{1/2}([0,T]\times\mathbb{R}^2)}^{1/2}.\end{aligned}$$

To control the factor $\|u^\lambda\|_{L_t^\infty \dot{H}_x^{1/2}([0,T]\times\mathbb{R}^2)}$, we decompose

$$u^\lambda(t) := P_{\leq N} u^\lambda(t) + P_{>N} u^\lambda(t).$$

To estimate the low frequencies, we interpolate between the L^2_x norm and the $\dot{H}^1_x$ norm and use the fact that I is the identity on frequencies $|\xi| \le N$:

$$\begin{aligned}\|P_{\le N} u^\lambda(t)\|_{\dot{H}^{1/2}_x} &\lesssim \|P_{\le N} u^\lambda(t)\|_2^{1/2} \|P_{\le N} u^\lambda(t)\|_{\dot{H}^1_x}^{1/2} \\ &\lesssim \lambda^{\frac{1}{2}(1-\frac{1}{k})} C(\|u_0\|_2) \|I_N u^\lambda(t)\|_{\dot{H}^1_x}^{1/2}.\end{aligned}$$

To control the high frequencies, we interpolate between the L^2_x norm and the $\dot{H}^s_x$ norm and use Lemma 2.4 and the relation between N and λ to get

$$\begin{aligned}\|P_{>N} u^\lambda(t)\|_{\dot{H}^{1/2}_x} &\lesssim \|P_{>N} u^\lambda(t)\|_{L^2_x}^{1-1/2s} \|P_{>N} u^\lambda(t)\|_{\dot{H}^s_x}^{1/2s} \\ &\lesssim \lambda^{(1-\frac{1}{2s})(1-\frac{1}{k})} N^{\frac{s-1}{2s}} \|I u^\lambda(t)\|_{\dot{H}^1_x}^{1/2s} \\ &\lesssim \lambda^{\frac{1}{2}-\frac{1}{k}} \|I u^\lambda(t)\|_{\dot{H}^1_x}^{1/2s}.\end{aligned}$$

Collecting all these estimates, we get

$$\begin{aligned}\|u^\lambda\|_{L^4_t L^8_x([0,t]\times\mathbb{R}^2)} &\lesssim \\ &\lambda^{\frac{3}{4}(1-\frac{1}{k})} C(\|u_0\|_2) \sup_{t\in[0,T]} \big(\|\nabla I u^\lambda(t)\|_2^{1/4} + \|\nabla I u^\lambda(t)\|_2^{1/4s}\big).\end{aligned} \tag{5.29}$$

Thus, taking C_1 sufficiently large depending on $\|u_0\|_2$, we obtain $T \in \Omega_1$, provided

$$\sup_{t\in[0,T]} \|\nabla I u^\lambda(t)\|_2 \le 1. \tag{5.30}$$

We now prove that $T \in \Omega_2$ implies (5.30). Indeed, let $\eta > 0$ be a sufficiently small constant as in Proposition 5.3 and divide $[0,T]$ into

$$L \sim \left(\frac{\lambda^{\frac{3}{4}(1-\frac{1}{k})}}{\eta}\right)^4$$

subintervals $I_j = [t_j, t_{j+1}]$ such that

$$\|u^\lambda\|_{L^4_t L^8_x(I_j\times\mathbb{R}^2)} \le \eta.$$

Applying Proposition 5.3 on each of the subintervals I_j, we get

$$\sup_{t\in[0,T]} E(I_N u^\lambda(t)) \le E(I_N u_0^\lambda) + E(I_N u_0^\lambda) L N^{-1+}.$$

To maintain small energy during the iteration, we need

$$LN^{-1+} \sim \lambda^{3(1-\frac{1}{k})} N^{-1+} \ll 1,$$

which, combined with (5.27), leads to

$$\left(N^{\frac{1-s}{s-1+\frac{1}{k}}}\right)^{3(1-\frac{1}{k})} N^{-1+} \leq c(\|u_0\|_{H_x^s}) \ll 1.$$

This may be ensured by taking N large enough (depending only on $\|u_0\|_{H^s(\mathbb{R})}$ and k), provided that

$$s > s(k) := 1 - \frac{1}{4k-3}.$$

As can be easily seen, $s(k) \to 1$ as $k \to \infty$.

This completes the bootstrap argument and hence (5.28), and moreover (5.25), follows. Therefore (5.30) holds for all $T \in \mathbb{R}$, and the conservation of mass and Lemma 2.4 imply

$$\begin{aligned}
\|u(T)\|_{H_x^s} &\lesssim \|u_0\|_{L_x^2} + \|u(T)\|_{\dot{H}_x^s} \\
&\lesssim \|u_0\|_{L_x^2} + \lambda^{s-(1-\frac{1}{k})} \|u^\lambda(\lambda^2 T)\|_{\dot{H}_x^s} \\
&\lesssim \|u_0\|_{L_x^2} + \lambda^{s-(1-\frac{1}{k})} \|I u^\lambda(\lambda^2 T)\|_{H_x^1} \\
&\lesssim \|u_0\|_{L_x^2} + \lambda^{s-(1-\frac{1}{k})} \big(\|u^\lambda(\lambda^2 T)\|_{L_x^2} + \|\nabla I u^\lambda(\lambda^2 T)\|_{L_x^2}\big) \\
&\lesssim \|u_0\|_{L_x^2} + \lambda^{s-(1-\frac{1}{k})} (\lambda^{1-\frac{1}{k}} \|u_0\|_{L_x^2} + 1) \\
&\lesssim C(\|u_0\|_{H_x^s})
\end{aligned}$$

for all $T \in \mathbb{R}$. Hence,

$$\|u\|_{L_t^\infty H_x^s} \leq C(\|u_0\|_{H_x^s}). \tag{5.31}$$

Finally, we prove that scattering holds in H_x^s for $s > s_k$. The construction of the wave operators is standard and follows by a fixed point argument (see [4]). Here we show only asymptotic completeness.

The first step is to upgrade the global Morawetz estimate to global Strichartz control. Let u be a global H_x^s solution to (1.1). Then u satisfies (5.25). Let $\delta > 0$ be a small constant to be chosen momentarily and split $\mathbb{R}$ into $L = L(\|u_0\|_{H_x^s})$ subintervals $I_j = [t_j, t_{j+1}]$ such that

$$\|u\|_{L_t^4 L_x^8(I_j \times \mathbb{R}^2)} \leq \delta.$$

By Lemma 2.2, (5.31), and the fractional chain rule [5], we estimate

$$\begin{aligned}\|\langle\nabla\rangle^s u\|_{S^0(I_j)} &\lesssim \|u(t_j)\|_{H^s_x} + \|\langle\nabla\rangle^s(|u|^{2k}u)\|_{L^{4/3}_{t,x}(I_j\times\mathbb{R}^2)}\\ &\lesssim C(\|u_0\|_{H^s_x}) + \|u\|^{2k}_{L^{4k}_{t,x}}\|\langle\nabla\rangle^s u\|_{L^4_{t,x}(I_j\times\mathbb{R}^2)},\end{aligned}$$

while by Hölder and Sobolev embedding,

$$\begin{aligned}\|u\|^{2k}_{L^{4k}_{t,x}(I_j\times\mathbb{R}^2)} &\lesssim \|u\|^{\frac{2k}{2k-1}}_{L^4_tL^8_x(I_j\times\mathbb{R}^2)}\|u\|^{\frac{4k(k-1)}{2k-1}}_{L^{8k}_tL_x^{\frac{16k(k-1)}{3k-2}}(I_j\times\mathbb{R}^2)}\\ &\lesssim \delta^{\frac{2k}{2k-1}}\||\nabla|^{\frac{8k^2-13k+4}{8k^2-8k}}u\|^{\frac{4k(k-1)}{2k-1}}_{L^{8k}_tL_x^{\frac{8k}{4k-1}}(I_j\times\mathbb{R}^2)}\\ &\lesssim \delta^{\frac{2k}{2k-1}}\|\langle\nabla\rangle^s u\|^{\frac{4k(k-1)}{2k-1}}_{S^0(I_j)}.\end{aligned}$$

The last inequality follows from the fact that for any $k \geq 2$ we have that

$$s_k = 1 - \frac{1}{4k-3} > \frac{8k^2-13k+4}{8k^2-8k}.$$

Therefore,

$$\|\langle\nabla\rangle^s u\|_{S^0(I_j)} \lesssim C(\|u_0\|_{H^s_x}) + \delta^{\frac{2k}{2k-1}}\|\langle\nabla\rangle^s u\|^{1+\frac{4k(k-1)}{2k-1}}_{S^0(I_j)}.$$

A standard continuity argument yields

$$\|\langle\nabla\rangle^s u\|_{S^0(I_j)} \leq C(\|u_0\|_{H^s_x}),$$

provided we choose δ sufficiently small depending on k and $\|u_0\|_{H^s_x}$. Summing over all subintervals I_j, we obtain

$$\|\langle\nabla\rangle^s u\|_{S^0(\mathbb{R})} \leq C(\|u_0\|_{H^s_x}). \tag{5.32}$$

We now use (5.32) to prove asymptotic completeness; that is, there exist unique $u_\pm$ such that

$$\lim_{t\to\pm\infty}\|u(t) - e^{it\Delta}u_\pm\|_{H^s_x} = 0. \tag{5.33}$$

Arguing as in Section 4, it suffices to see that

$$\left\|\int_t^\infty e^{-is\Delta}(|u|^{2k}u)(s)ds\right\|_{H^s_x} \to 0 \quad \text{as } t\to\infty. \tag{5.34}$$

The estimates above yield

$$\left\|\int_t^\infty e^{-is\Delta}(|u|^{2k}u)(s)ds\right\|_{H^s_x} \lesssim \|u\|^{\frac{2k}{2k-1}}_{L^4_tL^8_x(I_j\times\mathbb{R}^2)}\|\langle\nabla\rangle^s u\|^{1+\frac{4k(k-1)}{2k-1}}_{S^0([t,\infty]\times\mathbb{R})}.$$

Using (5.25) and (5.32) we derive (5.34) and conclude the proof of Theorem 1.4. □

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JAMES COLLIANDER
University of Toronto
Department of Mathematics
6110 Bahen Center
40 St. George Street
Toronto, Ontario M5S 3G3
CANADA
E-mail: `colliand@math.toronto.edu`

MANOUSSOS G. GRILLAKIS
University of Maryland
Department of Mathematics
College Park, MD 20742
E-mail: `mng@math.umd.edu`

NIKOLAOS TZIRAKIS
University of Illinois
at Urbana-Champaign
Department of Mathematics
1409 W. Green Street
Urbana, IL 61801
E-mail: `tzirakis@math.uiuc.edu`

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