Tensor Products of Analytic Continuations of Holomorphic Discrete Series

Article *in* Canadian Journal of Mathematics · December 1997

DOI: 10.4153/CJM-1997-060-5

Bent Ørsted
Aarhus University

Genkai Zhang
Chalmers University of Technology

Can. J. Math. Vol. **49** (6), 1997 pp. 1224–1241

TENSOR PRODUCTS OF ANALYTIC CONTINUATIONS OF HOLOMORPHIC DISCRETE SERIES

BENT ØRSTED AND GENKAI ZHANG

ABSTRACT. We give the irreducible decomposition of the tensor product of an analytic continuation of the holomorphic discrete series of $SU(2,2)$ with its conjugate.

0. **Introduction.** The work of Segal [IES] and Mautner [M] established the abstract Plancherel theorem for type I groups. This meant that for an arbitrary unitary representation, one could find its spectral decomposition into irreducibles and a corresponding spectral measure. To make this program explicit on L^2-spaces on homogeneous spaces is one of the main subjects of harmonic analysis. Another interesting case is that of decomposing a tensor product of irreducible representations; our aim in this paper is to consider this for certain holomorphic representations.

The problem of finding the irreducible decomposition of tensor products of holomorphic discrete series of the group $SL(2,\mathbb{R})$ has been studied by Repka [Re1]. The results there were used by Howe [How] to give the decomposition of the metaplectic representation for certain dual pairs. See also [OZ]. For a general semisimple Lie group G of Hermitian type a similar problem is studied in [Re2]. It is shown that the tensor product $\pi_\nu \otimes \overline{\pi_\nu}$ of a scalar holomorphic discrete series π_ν with its conjugate $\overline{\pi_\nu}$ is unitarily equivalent to the L^2-space on the corresponding Hermitian symmetric space, $L^2(G/K)$. Therefore we know its decomposition from the known theory of Harish-Chandra; namely

$$\pi_\nu \otimes \overline{\pi_\nu} \cong L^2(G/K) \cong \int_{\mathfrak{a}^*/W}^{\oplus} \mathcal{H}(\underline{\lambda})|C(\underline{\lambda})|^{-2}\, d\underline{\lambda}$$

for $\nu > p-1$ where p is an integer, also called genus of G/K. Here $\mathcal{H}(\underline{\lambda})$ is the induced representation $\mathrm{Ind}_P(1 \otimes i\underline{\lambda} \otimes 1)$ from the minimal parabolic P. In the above formula the isomorphisms are realized by explicit intertwining operators (the second from Harish-Chandra's and Helgason's theory of spherical functions), and our aim in this paper is to study variants of these isomorphisms and explicit Plancherel measures for a larger range of the parameter ν.

In this paper we take $G = SU(2,2)$ with genus $p = 4$. The scalar discrete series π_ν has an analytic continuation, namely for $\nu \in \{0,1\} \cup (1,\infty)$ we still get a unitary representation of $SU(2,2)$; this set of ν is called the Wallach set. See for example the

Received by the editors April 20, 1995.

AMS subject classification: 22E45, 43A85, 32M15, 33A65.

Key words and phrases: Holomorphic discrete series, tensor product, spherical function, Clebsch-Gordan coefficient, Plancherel theorem.

references [EHW] and [Wa]. Our problem in this paper is to study the irreducible decomposition of $\pi_\nu \otimes \overline{\pi_\nu}$ for all ν in the Wallach set. For $\nu = 0$, this is a trivial problem since π_0 is the trivial representation.

In [OZ] we proved the somewhat surprising fact, that the tensor product $\pi_\nu \otimes \overline{\pi_\nu}$ of the analytic continuation π_ν of the holomorphic discrete series of SU(1, 1) (or rather its universal covering group) with it conjugate $\overline{\pi_\nu}$ has a complementary series as a discrete part when $\nu < \frac{1}{2}$. (The parameter ν is normalized there so that π_ν is the Hardy space when $\nu = 1$.) The idea there was to study the action of the Casimir operator on the orthogonal basis of K-invariant subspace and to use certain orthogonality relation of a class of hypergeometric orthogonal polynomials, also called the continuous dual Hahn polynomials.

It was observed in [PZ] that the Plancherel measure of the decomposition is the symbol of the Berezin transform viewed as function of the invariant differential operators. Recently Unterberger and Upmeier [UU] have found the symbol function of the Berezin transform on any Hermitian symmetric space. This gives us an interesting new aspect of the tensor product problem.

In this paper we will use similar techniques to study the tensor products $\pi_\nu \otimes \overline{\pi_\nu}$ of analytic continuations of (scalar) holomorphic discrete series π_ν of SU(2, 2) with its conjugate $\overline{\pi_\nu}$. We give a complete answer to the question of finding the explicit Plancherel decomposition of this tensor product. Let $\mathfrak{a}$ be a maximal non-compact abelian subspace of $\mathfrak{g}$, the Lie algebra of SU(2, 2), and let f_1 and f_2 be some basis of $\mathfrak{a}^*$. We prove, with explicit decomposition of the invariant differential operators and measures on the right-hand side, see Theorem 2.13 and Theorem 3.1, that

$$\pi_\nu \otimes \overline{\pi_\nu} \cong \int_{\lambda_1>\lambda_2>0}^{\oplus} H\,(\lambda_1 f_1 + \lambda_2 f_2)\, d\underline{\lambda}, \tag{0.1}$$

if $\nu \geq \frac{3}{2}$; and

$$\begin{aligned} \pi_\nu \otimes \overline{\pi_\nu} \cong &\int_{\lambda_1>\lambda_2>0}^{\oplus} H\,(\lambda_1 f_1 + \lambda_2 f_2)\, d\underline{\lambda} \\ &+ \int_{\mathbb{R}^+}^{\oplus} H\left(i(2\nu-3)f_1 + \lambda_2 f_2\right) d\lambda_2 \end{aligned} \tag{0.2}$$

if $1 < \nu < \frac{3}{2}$; here $H\left(i(2\nu-3)f_1 + \lambda_2 f_2\right)$ is a subrepresentation of the induced representation $\mathrm{Ind}_P\Big(1 \otimes i\big(i(2\nu-3)f_1 + \lambda_2 f_2\big) \otimes 1\Big)$ with $\underline{\lambda} = i(2\nu-3)f_1 + \lambda_2 f_2$; and

$$\pi_\nu \otimes \overline{\pi_\nu} \cong \int_{\mathbb{R}^+}^{\oplus} H\,(-if_1 + \lambda_2 f_2)\, d\lambda_2 \tag{0.3}$$

if $\nu = 1$. This distinction between the cases of ν intervals comes about using explicit densities on the real line, see Lemma 1.1 and Lemma 2.10. One can furthermore identify our representations appearing above with those in the Knapp-Speh classification [KS].

The discrete series here are realized as Hilbert spaces of holomorphic functions over the corresponding bounded symmetric domain D. The diagonal operator from the tensor product $\pi_\nu \otimes \overline{\pi_\nu}$, realized as the space of functions $f(z, w)$ that are analytic in z and anti-analytic in w, to $C^\infty(D)$ is a formal intertwining operator ([Re2] and [PZ]). When π_ν is a discrete series (*i.e.*, $\nu > 3$) this operator is bounded from $\pi_\nu \otimes \overline{\pi_\nu}$ to $L^2(D)$ and has dense image; thus by polar decomposition we get a unitary intertwining operator onto $L^2(D)$. However for non-discrete π_ν the diagonal operator is no longer a bounded operator. Our idea is, roughly speaking, to use the diagonal operator to study the action of the invariant differential operator on the representation space $\pi_\nu \otimes \overline{\pi_\nu}$. We proceed to explain our method.

The ring of invariant differential operators on $C^\infty(D)$ has two generators. By conjugating by the diagonal operator we find a pair of invariant differential operators $(\Box_1, \Box_2)$ on the tensor space $\pi_\nu \otimes \overline{\pi_\nu}$. Now the K-invariant eigenfunctions have an expansion in terms of the K-invariant polynomials $e_{\underline{\mathbf{m}}}$ with coefficients $\widetilde{e_{\underline{\mathbf{m}}}}$, and $e_{\underline{\mathbf{m}}}$ form an orthogonal basis of $(\pi_\nu \otimes \overline{\pi_\nu})_0$, the space of K-invariant functions in $\pi_\nu \otimes \overline{\pi_\nu}$. See (2.7) below. We prove that $(\Box_1, \Box_2)$ acts on $e_{\underline{\mathbf{m}}}$ by the same formula as their "symbols" $\big(\Box_1(\underline{\lambda}), \Box_2(\underline{\lambda})\big)$ on the polynomials $\widetilde{e_{\underline{\mathbf{m}}}}(\underline{\lambda})$. We prove further that $\widetilde{e_{\underline{\mathbf{m}}}}(\underline{\lambda})$ are orthonormal basis in a L^2-space of a suitable measure on $\mathfrak{a}_{\mathbb{C}}^*$. Thus the spectrum of $(\Box_1, \Box_2)$ is given by the support of the measure!

An interesting phenomenon occurs when $\nu = 1$. The representation π_1 has one-dimensional K-types and the operator $\Box_2$ acting on $\pi_1 \otimes \overline{\pi_1}$ as a scalar 0. Thus the spectral decomposition of $(\Box_1, \Box_2)$ is the same as that of $\Box_1$, which is of course one-dimensional. This gives us (0.3) above.

The paper is organized as follows. In Section 1 we recall some results about spectral decomposition of some second order differential operators, which appear in the study of tensor products of analytic continuations of holomorphic discrete series of $SU(1, 1)$, and we recall some facts about $SU(2, 2)$. Section 2 is devoted to the study of irreducible decomposition of the tensor product $\pi_\nu \otimes \overline{\pi_\nu}$ for good values of the parameter ν, $\nu \geq \frac{3}{2}$. Lemma 2.7 and Corollary 2.8, which gives an expansion of the spherical functions in terms of the K-invariant polynomials, may be of independent interests. Sections 3 and 4 are devoted to the cases $1 < \nu \leq \frac{3}{2}$ and $\nu = 1$ respectively. In Section 5 we give a generalization of our results to general holomorphic discrete series of a semisimple Lie group, though the result is less complete.

There remains an open question relating to our work, that is to prove directly (without using the decomposition of invariant differential operators on K-invariant vectors) the intertwining property from the tensor product to the spherical principal series or their submodule. When $\nu > 3$, namely when π_ν is holomorphic discrete, then this can be done via the polar decomposition of diagonal map and the spherical transform, see [PZ]. However for $\nu \leq 3$ the diagonal map does not map into $L^2(D)$ and we can not directly use the spherical transform defined on $L^2(D)$ ([He], Chapter IV, Section 5) as intertwining operator; see also the remark after Lemma 2.3.

ACKNOWLEDGEMENTS. It is a pleasure to thank the referee for a number of clarifying comments and useful suggestions, in particular for pointing out that some of our results are valid in the general setting of Hermitian symmetric spaces. This we have added as

the last section 5, where we also point out that if we had Lemma 2.7 in the general case, then that could be used to give the required generalization of Proposition 2.9. As it stands, Lemma 2.7 is somewhat of an aside (perhaps of independent interest). We also would like to thank Jaak Peetre for some helpful discussions. The second author thanks the Magnuson fund, Royal Swedish Academy of Sciences for supporting his research during April, 1992 at Department of Mathematics, Stockholm University where part of this work was done.

1. **Preliminaries:** SU(1, 1) **and** SU(2, 2).

1.1 *Analysis on* SU(1, 1) *and related orthogonal polynomials.* To begin with we recall some results obtained in [Z], [PZ] and [OZ].

The spherical functions on the unit disk $\{z \in \mathbb{C}\,;\ |z| < 1\}$ are given by

$$\phi_\lambda(z) = (1-t)^{\frac{1-i\lambda}{2}}\, {}_2F_1\left(\frac{1-i\lambda}{2}, \frac{1-i\lambda}{2}; 1; t\right),$$

where $t = |z|^2$. See [He].

In [Z] we introduced the function $(1-|z|^2)^{-\kappa}\phi_\lambda(z)$, with an extra parameter κ. Then we have the following expansion

$$(1-|z|^2)^{-\kappa}\phi_\lambda(z) = \sum_{n=0}^{\infty} p_{n,\kappa}(\Lambda)t^n,$$

where

$$p_{n,\kappa}(\Lambda) = \frac{1}{(n!)^2} S_n\left(-\left(\frac{\lambda}{2}\right)^2; \frac{1}{2}, \frac{1}{2}, \frac{2\kappa-1}{2}\right),$$

and

$$\Lambda = \Lambda(\lambda) = \kappa - \frac{1}{4}(1+\lambda^2). \tag{1.0}$$

It follows from the proof in [Z] that the series converges uniformly on compact sets in D. Here S_n are the continuous dual Hahn polynomials,

$$S_n(x^2; a, b, c) = (a+b)_n(a+c)_n\ {}_3F_2(-n, a+ix, a-ix; a+b, a+c; 1).$$

For further details see the above references.

Consider the following operator,

$$\begin{aligned} C_\kappa &= (1-t)^2\left(t\frac{d^2}{dt^2} + \frac{d}{dt}\right) - 2\kappa t(1-t)\frac{d}{dt} + \kappa^2 t \\ &= (1-t)^{-\kappa}\left((1-t)^2\left(t\frac{d^2}{dt^2} + \frac{d}{dt}\right) + \kappa\right)(1-t)^{\kappa}, \end{aligned}$$

and the functions

$$e_n(t) = \frac{\Gamma(n+\kappa)}{\Gamma(n+1)\,\Gamma(\kappa)}t^n. \tag{1.1}$$

It is proved in [Z] and [PZ] that C_κ has the following matrix form on the basis vectors e_n

$$C_\kappa e_n = a_n e_{n+1} + b_n e_n + c_n e_{n-1} \tag{1.2}$$

where

$$a_n = (n+1)(n+\kappa),\ b_n = -n(2n+2\kappa),\ c_n = a_{n-1};$$

and that the multiplication operator by Λ has the same matrix form on the functions $\widetilde{e_n}$

$$\Lambda\widetilde{e_n}(\lambda) = a_n\widetilde{e_{n+1}}(\lambda) + b_n\widetilde{e_n}(\lambda) + c_n\widetilde{e_{n-1}}(\lambda), \tag{1.3}$$

where

$$\widetilde{e_n}(\lambda) = \frac{n!}{\Gamma(\kappa+n)}p_{n,\kappa}(\Lambda) = \frac{1}{n!\,\Gamma(\kappa+n)}S_n\left(-\left(\frac{\lambda}{2}\right)^2; \frac{1}{2}, \frac{1}{2}, \frac{2\kappa-1}{2}\right).$$

The above expansion of $(1-|z|^2)^{-\kappa}\phi_\lambda(z)$ now reads as follows

$$(1-|z|^2)^{-\kappa}\phi_\lambda(z) = \sum_{n=0}^{\infty} e_n(|z|^2)\widetilde{e_n}(\lambda). \tag{1.4}$$

We let

$$c(\lambda) = \frac{2^{i\lambda}\Gamma(i\lambda)}{\Gamma(\frac{1+i\lambda}{2})\Gamma(\frac{1+i\lambda}{2})}$$

be the Harish-Chandra c-function of the group $\mathrm{SU}(1,1)$ and let

$$b_\kappa = \left|\Gamma\left(\frac{2\kappa-1+i\lambda}{2}\right)\right|^2.$$

(b_κ is the symbol of Berezin transform, see [PZ].)

By using the Askey-Wilson orthogonality relation [AW] we have the following

LEMMA 1.1. *Let H be the Hilbert space of functions on $[0,1]$ with $\{e_n\}_{n=0}^{\infty}$ as an orthonormal basis. Let $d\mu_\kappa$ be the measure on $\mathbb{R}$ defined by*

$$d\mu_\kappa(\lambda) = \frac{1}{2}b_\kappa(\lambda)|c(\lambda)|^{-2}\,d\lambda, \quad \lambda\in\mathbb{R},$$

if $\kappa \ge \frac{1}{2}$, and let $d\mu_\kappa$ be the measure on $\mathbb{R}\cup\{i(1-2\kappa)\}$ with $d\mu_\kappa(\lambda)$ given as above on $\mathbb{R}$ and

$$d\mu_\kappa\big(\pm i(1-2\kappa)\big) = 2\pi\frac{\Gamma(1-\kappa)\,\Gamma^2(\kappa)}{\Gamma(1-2\kappa)},$$

if $0<\kappa\le\frac{1}{2}$. Then $\widetilde{e_n}$ form an orthonormal basis in the subspace $L^2(\mathbb{R},\mu_\kappa)_0$ of even functions in $L^2(\mathbb{R},\mu_\kappa)$ if $\kappa\ge\frac{1}{2}$, and in the subspace $L^2\big(\mathbb{R}\cup\{\pm i(1-2\kappa)\},\mu_\kappa\big)_0$ of even functions in $L^2\big(\mathbb{R}\cup\{\pm i(1-2\kappa)\},\mu_\kappa\big)$ if $0<\kappa<\frac{1}{2}$. Moreover the map $e_n\longmapsto\widetilde{e_n}$ gives a unitary equivalence between the operator C_κ on H and the multiplication operator by the function $\Lambda(\lambda)=\kappa-\frac{1}{4}(\lambda^2+1)$ on $L^2\big(\mathbb{R}\cup\{i(1-2\kappa)\},\mu_\kappa\big)_0$.

PROOF. The orthogonality relation of Wilson (see [W] p. 697 and [OZ]) shows that indeed $\widetilde{e_n}$ are orthogonal and have norm 1 in the space $L^2(\mathbb{R},\mu_\kappa)_0$ if $\kappa\ge\frac{1}{2}$, and in

$L^2\big(\mathbb{R}\cup\{\pm i(1-2\kappa)\},\mu_\kappa\big)_0$ if $0<\kappa<\frac{1}{2}$. We need to prove that they are dense in the Hilbert spaces. This will follow from some elementary argument which we sketch below.

Suppose $\kappa\geq\frac{1}{2}$ and suppose f in $L^2(\mathbb{R},\mu_\kappa)_0$ is perpendicular to all $\tilde{e}_n$. Then it is perpendicular to all λ^{2n} since they are linear combinations of $\tilde{e}_n$, that is,

$$\int_{\mathbb{R}} f(\lambda)\lambda^m\,d\mu_\kappa(\lambda)=0,$$

if m is an nonnegative even integer. However since $f(\lambda)$ is an even function and the measure $d\mu_\kappa$ is symmetric under $\lambda\longrightarrow-\lambda$ we see that the above is true for all nonnegative integers m. We define

$$F(z)=\int_{\mathbb{R}} e^{i\lambda z}f(\lambda)\,d\mu_\kappa(\lambda). \tag{1.5}$$

We claim that $F(z)$ is a holomorphic function on the strip $-\pi<\Im(z)<\pi$. In fact, using simple formulas for the Gamma function (see [He., p. 42]), we have

$$d\mu_\kappa(\lambda)=C\lambda\tanh\left(\frac{\pi\lambda}{2}\right)\left|\Gamma\left(\frac{2\kappa-1+i\lambda}{2}\right)\right|^2 d\lambda,$$

and (see [Ma], Chapter I)

$$\left|\Gamma\left(\frac{2\kappa-1+i\lambda}{2}\right)\right|^2=O(|\lambda|^{\kappa-2}e^{-\pi|\lambda|})$$

if $|\lambda|\longrightarrow\infty$. By using Hölder's inequality we see that (1.5) is absolute convergent if $-\pi<\Im(\lambda)<\pi$ and thus that $F(z)$ is analytic in this strip.

Now differentiating $F(z)$ we see that all its derivatives are zero at $z=0$, thus $F(z)=0$. In particular taking $z\in\mathbb{R}$ we see that the Fourier transform of

$$f(\lambda)\lambda\tanh\left(\frac{\pi\lambda}{2}\right)\left|\Gamma\left(\frac{2\kappa-1+i\lambda}{2}\right)\right|^2 \tag{1.6}$$

is zero; this function is easily seen in $L^2(\mathbb{R},d\lambda)$. Thus it is zero and consequently $f(\lambda)=0$. That is the map $e_n\longmapsto\widetilde{e_n}$ is onto.

For the case $0<\kappa<\frac{1}{2}$ we proceed similarly and define the function $F(z)$. $F(z)=0$ implies that (1.6) has, up to a nonzero constant, Fourier transform

$$f\big(i(1-2\kappa)\big)(e^{z(1-2\kappa)}+e^{-z(1-2\kappa)})$$

which is in $L^2(\mathbb{R})$ if and only if $f\big(i(1-2\kappa)\big)=0$ and consequently $f(\lambda)=0$.

The remaining claim of our lemma follows from (1.2) and (1.3) and the unitarity of the map $e_n\longmapsto\widetilde{e_n}$. ∎

REMARK. When $\kappa\geq\frac{1}{2}$ the above claim is also equivalent to that the map $e_n\longrightarrow\widetilde{e_n}$ establishes a unitary equivalence between the operator C_κ on H and the multiplication operator by the function $\Lambda(\lambda)=\kappa-\frac{1}{4}(1+\lambda^2)$ on the space $L^2(\mathbb{R}^+,\mu_\kappa)$. When $0<\kappa<\frac{1}{2}$ it is equivalent to that the map $e_n\longrightarrow\widetilde{e_n}$ establishes a unitary equivalence between the operator C_κ on H and the multiplication operator by the function $\Lambda(\lambda) = \kappa - \frac{1}{4}(1+\lambda^2)$ on the space $L^2\big(\mathbb{R}^+ \cup \{i(1-2\nu)\}, \mu_\kappa\big)$.

1.2 *Representations of* SU(2, 2). We proceed to recall some known facts about irreducible representations of SU(2, 2).

Let $G = \mathrm{SU}(2,2)$ be the group of linear matrices on $\mathbb{C}^4$ of determinant one and keeping the indefinite metric $|z_1|^2 + |z_2|^2 - |z_3|^2 - |z_4|^2$ invariant. Let $\mathfrak{g}$ be the Lie algebra of G. Let $K = S\big(U(2), U(2)\big)$, a maximal compact subgroup and $\mathfrak{k}$ its Lie algebra and $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ the corresponding Cartan decomposition. We take a maximal abelian subspace $\mathfrak{a}$ of $\mathfrak{p}$ as follows

$$\mathfrak{a} = \left\{ a = \begin{pmatrix} 0 & \mathrm{diag}(s,t) \\ \mathrm{diag}(s,t) & 0 \end{pmatrix},\ (s,t) \in \mathbb{R}^2 \right\},$$

and define linear functionals on $\mathfrak{a}$ by

$$f_1(a) = s,\ \ f_2(a) = t.$$

The root system of $(\mathfrak{g}, \mathfrak{a})$ consists of $\pm(f_1 \pm f_2)$, $\pm 2f_1$ and $\pm 2f_2$ with multiplicities 2, 1 and 1 respectively. Let $\mathfrak{a}^*$ be the dual space of $\mathfrak{a}$ and let $\mathfrak{a}^*_{\mathbb{C}}$ be its complexification. An element of $\mathfrak{a}^*_{\mathbb{C}}$, the complexification of the dual of $\mathfrak{a}$, will be written as $\underline{\lambda} = \lambda_1 f_1 + \lambda_2 f_2$. We choose the ordering on $\mathfrak{a}^*$ defined by $f_1 > f_2 > 0$ and let $\mathfrak{a}^*_+$ be the set of positive elements. For $\underline{\lambda} \in \mathfrak{a}^*_{\mathbb{C}}$ we let $\phi_{\underline{\lambda}}$ be the corresponding spherical function. The space of G-translates of $\phi_{\underline{\lambda}}$ can be completed to a Hilbert space $H(\underline{\lambda})$ and in such a way that $H(\underline{\lambda})$ forms a unitary representation of G, which is equivalent to the induced representation $\mathrm{Ind}_P^G(1 \otimes i\underline{\lambda} \otimes 1)$, where P is the minimal parabolic corresponding to $\mathfrak{a}$. See [Kn, p. 168] and [He, Chapter IV, Theorem 3.7].

2. **The tensor products** $\pi_\nu \otimes \overline{\pi_\nu}$ **for** $\nu \geq \frac{3}{2}$ **for** SU(2, 2). Let D be the bounded symmetric domain of 2×2 of contractive matrices, that is

$$D = \left\{ Z \in M_{2\times 2}(\mathbb{C}),\ Z^*Z < I_2 \right\}.$$

We consider the Hilbert space H_ν of holomorphic functions on D with the norm

$$c_\nu \int_D |f(Z)|^2 \det(1 - Z^*Z)^{\nu-4}\, dm(Z) < \infty$$

where $\nu > 3$, dm is the Lebesgue measure and c_ν is a constant so that the function 1 is a unit vector. The group $G = \mathrm{SU}(2,2)$ (or rather, its universal covering group) acts unitarily on the Hilbert space H_ν via

$$\pi_\nu(g)\colon f(Z) \longmapsto \big(\det(CZ+D)\big)^{-\nu} f(g^{-1}Z), \quad g^{-1} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in G, \tag{2.1}$$

The reproducing kernel of this space is

$$K_\nu(Z, W) = \det(1 - W^*Z)^{-\nu}.$$

Furthermore, (H_ν, π_ν) has an analytic continuation in ν and for $\nu \geq 1$ we still get a unitary representation. See [FK]. Moreover (H_ν, π_ν) is a highest weight representation of $\mathfrak{g}^{\mathbb{C}}$ with highest weight vector the function 1.

The objective of this paper is to give the explicit irreducible decomposition of the tensor product $H_\nu \otimes \overline{H_\nu}$. (It follows from the general theory that such an irreducible decomposition exists as a direct integral, see [Pu] and [M].) Here $\overline{H_\nu}$ is the complex conjugate of H_ν and the group G acts on it by taking the formal complex conjugate of (2.1).

Denote $(H_\nu \otimes \overline{H_\nu})_0$ the space of K-fixed vectors. We will study the spectral decomposition of a pair of generators of invariant differential operators on $(H_\nu \otimes \overline{H_\nu})_0$. We will prove below that this is indeed sufficient to obtain the decomposition for $H_\nu \otimes \overline{H_\nu}$.

So let temporarily $\mathfrak{g}$ be a semisimple Lie algebra of Hermitian type and let $\mathfrak{g} = \mathfrak{p} + \mathfrak{k}$ be the Cartan decomposition. Let $\mathfrak{h}$ be a Cartan algebra of $\mathfrak{k}$. Let $\Delta(\mathfrak{g}^{\mathbb{C}}, \mathfrak{h}^{\mathbb{C}})$ be the set of positive roots with respect to an ordering on $(i\mathfrak{h})^*$, $\mathfrak{u}^+$ and $\mathfrak{u}^-$ be the sum of positive and negative root spaces, respectively. Let M be a $(\mathfrak{g}^{\mathbb{C}}, \mathfrak{k}^{\mathbb{C}})$ module. M is called a highest weight module if there exists a weight vector $v \in M$ of $\mathfrak{k}^{\mathbb{C}}$ so that $\mathfrak{u}^+ v = 0$. Thus, assuming this is a cyclic vector, $M = U(\mathfrak{u}^-)v$, by the Birkhoff-Witt theorem, where $U(\mathfrak{u}^-)$ stands for the universal enveloping algebra of $\mathfrak{u}^-$. Similarly one can define lowest weight modules.

LEMMA 2.1. *Let M_1 and M_2 be a highest and a lowest weight modules of $\mathfrak{g}^{\mathbb{C}}$ with the highest weight and lowest weight vectors v_1, v_2, respectively. Then $M_1 \otimes M_2$ is a cyclic module with $v_1 \otimes v_2$ as a cyclic vector. That is $U(\mathfrak{g}^{\mathbb{C}})(v_1 \otimes v_2) = M_1 \otimes M_2$.*

PROOF. For any $a \in M_1$ there exists an element $X \in U(\mathfrak{u}^-)$, such that $Xv_1 = a$. Therefore, notice that $Yv_2 = 0$ for any $Y \in U(\mathfrak{u}^-)$,

$$X(v_1 \otimes v_2) = Xv_1 \otimes v_2 = a \otimes v_2.$$

That is $a \otimes v_2 \in U(\mathfrak{g}^{\mathbb{C}})(v_1 \otimes v_2)$. Similarly using the elements from $U(\mathfrak{u}^+)$ we see that $v_1 \otimes b \in U(\mathfrak{g}^{\mathbb{C}})(v_1 \otimes v_2)$ for any $b \in M_2$.

Now we fix $b \in M_2$. If $X \in \mathfrak{u}^-$ then

$$X(v_1 \otimes b) = Xv_1 \otimes b + v_1 \otimes Xb.$$

Since $v_1 \otimes Xb \in U(\mathfrak{g}^{\mathbb{C}})(v_1 \otimes v_2)$ we have that

$$Xv_1 \otimes b = X(v_1 \otimes b) - v_1 \otimes Xb \in U(\mathfrak{g}^{\mathbb{C}})(v_1 \otimes v_2)$$

Thus $\mathfrak{u}^- v_1 \otimes b \subset U(\mathfrak{g}^{\mathbb{C}})(v_1 \otimes v_2)$. Continuing we get $(\mathfrak{u}^-)^n v_1 \otimes b \subset U(\mathfrak{g}^{\mathbb{C}})(v_1 \otimes v_2)$ for all $n = 1, 2, \ldots$. Since $M_1 = U(\mathfrak{u}^-)v_1 = \sum_{n=0}^{\infty} (\mathfrak{u}^-)^n v_1$ we see that for any $a \in M_1$ we get $a \otimes b \in U(\mathfrak{g}^{\mathbb{C}})(v_1 \otimes v_2)$. This finishes the proof. ∎

LEMMA 2.2. *Suppose*

$$\pi_\nu \otimes \overline{\pi_\nu} = \int_S^{\oplus} \pi(s)\, d\sigma(s)$$

is the irreducible decomposition of $\pi_\nu \otimes \overline{\pi_\nu}$*. Then each* $\pi(s)$ *a.e. in* S *has an one-dimensional subspace of* K*-fixed vectors.*

PROOF. Suppose there exists a set $S_0 \subset S$ such that $\sigma(S_0) \neq 0$ and that for all $s \in S_0$ the space $\pi(s)$ has no nonzero K-fixed vectors. The function $1 \otimes 1$ in $\pi_\nu \otimes \overline{\pi_\nu}$ is K-invariant and thus

$$1 \otimes 1 = \int_{S \setminus S_0}^{\oplus} 1 \widetilde{\otimes} 1(s)\, d\sigma(s).$$

Let $U(\mathfrak{g}^{\mathbb{C}})$ act on the equality. It follows from Lemma 2.1 that

$$\pi_\nu \otimes \overline{\pi_\nu} = \int_{S \setminus S_0}^{\oplus} \pi(s)\, d\sigma(s),$$

which is a contradiction. That is each $s \in S$ a.e has K-fixed vectors. Now each $\pi(s)$ is irreducible, thus the K-fixed vectors form an one-dimensional subspace ([He, Theorem 3.4]). ∎

Now we briefly recall some results obtained in [PZ]. Define an operator R

$$R: H_\nu \otimes \overline{H_\nu} \longrightarrow C^\infty(D)$$
$$R: f(Z, W) \longmapsto f(Z, Z) K_\nu(Z, Z)^{-1} = f(Z, Z) \det(1 - Z^* Z)^\nu.$$

Let U_0 be the regular action of G on $C^\infty(D)$. We have as in [PZ] by direct calculation

LEMMA 2.3. *R intertwines the action of* $\pi_\nu \otimes \overline{\pi_\nu}$ *with* U_0*.*

It is shown in [PZ] that if $\nu > 3$ then R is a bounded operator from $H_\nu \otimes \overline{H_\nu}$ to $L^2(D)$ with respect to the invariant measure. The decomposition of $L^2(D)$ under G is done by the well-known theory of Harish-Chandra of spherical functions. However for $\nu \leq 3$ the image of R is not all in $L^2(D)$.

We will however, as mentioned in the introduction, calculate the actions of invariant differential operators on $(H_\nu \otimes \overline{H_\nu})_0$.

The algebra of invariant differential operators on $C^\infty(D)$ is generated by two operators. We recall these formulas here. See [Hoo]. Here we are using slightly different coordinates.

The polar coordinates of D are given by $KA^+ \cdot 0$, and $A^+ \cdot 0 = \{(s_1, s_2);\, 1 \geq s_1 \geq s_2 \geq 0\}$. We perform change of variables $t_1 = s_1^2$, $t_2 = s_2^2$. Denote

$$\Omega(t) = (t_1 - t_2), \quad h(t) = (1 - t_1)(1 - t_2).$$

Let

$$L_i = (1 - t_i)^2 \left(t_i \frac{\partial^2}{\partial t_i^2} + \frac{\partial}{\partial t_i} \right),$$

and

$$\Delta_1 = \Omega^{-1} h (L_1 + L_2) \Omega h^{-1}, \;\; \Delta_2 = \Omega^{-1} h (L_1 L_2) \Omega h^{-1}.$$

Then Δ_1 and Δ_2 generate the algebra of radial parts of invariant differential operators on $C^\infty(D)$.

We define

$$\Box_1 = \frac{1}{4}R^{-1}\Delta_1 R + 2(\nu-1), \quad \Box_2 = \frac{1}{4}R^{-1}\Delta_2 R + (\nu-1)\Box_1.$$

From Lemma 2.3 we see that $(\Box_1, \Box_2)$ generates the algebra of invariant differential operators on $(H_\nu \otimes \overline{H_\nu})_0$. Our objective is to give the spectral decomposition of $(\Box_1, \Box_2)$ on $(H_\nu \otimes \overline{H_\nu})_0$.

A direct calculation shows that

$$\Box_1 = \Omega^{-1}(C_{1,\nu-1} + C_{2,\nu-1})\Omega, \tag{2.1}$$

and

$$\Box_2 = \Omega^{-1}(C_{1,\nu-1}C_{2,\nu-1})\Omega, \tag{2.2}$$

where $C_{1,\nu-1}$ and $C_{2,\nu-1}$ are the operator $C_{\nu-1}$ in Section 1 with $\frac{d}{dt}$ replaced by $\frac{d}{dt_1}$ and $\frac{d}{dt_2}$ respectively.

The following fact is now standard.

LEMMA 2.4. *Suppose*

$$\pi_\nu \otimes \overline{\pi_\nu} = \int_S^{\otimes} \pi(s)\, d\sigma(s)$$

is the irreducible decomposition of $\pi_\nu \otimes \overline{\pi_\nu}$. Then $(\Box_1, \Box_2)$ acts on $\pi(s)$ by a pair of scalars $\big(\Box_1(s), \Box_2(s)\big)$ and we have the corresponding spectral decomposition of $(\Box_1, \Box_2)$,

$$(\Box_1, \Box_2) = \int_S \big(\Box_1(s), \Box_2(s)\big)\, d\sigma(s)$$

counting the multiplicities.

For a pair of commuting selfadjoint operators (A, B) of a Hilbert space H we denote $\sigma\big((A,B), H\big)$ its spectrum (counting the multiplicities). Clearly $(H_\nu \otimes \overline{H_\nu})_0$ is an invariant subspace for $(\Box_1, \Box_2)$.

From Lemma 2.2 we can easily derive

LEMMA 2.5. *We have the following formula*

$$\sigma\big((\Box_1, \Box_2), H_\nu \otimes \overline{H_\nu}\big) = \sigma\big((\Box_1, \Box_2), (H_\nu \otimes \overline{H_\nu})_0\big)$$

That is, restriction to the subspace of K-invariant functions does not change the spectrum of $(\Box_1, \Box_2)$.

We therefore proceed to study the spectrum of $(\Box_1, \Box_2)$ on $(H_\nu \otimes \overline{H_\nu})_0$.

For this purpose we recall some known facts about the Hilbert structure of H_ν. Let $\mathcal{P}$ be the space of polynomials on D. The group $K = S\big(U(2), U(2)\big)$ acts on it by changing of variables. The space $\mathcal{P}$ is decomposed as

$$\mathcal{P} = \sum_{\underline{\mathbf{m}} \geq 0} \mathcal{P}_{\underline{\mathbf{m}}}, \tag{2.3}$$

where $P_{\underline{\mathbf{m}}}$ has highest weight $\underline{\mathbf{m}} \otimes \underline{\mathbf{m}}^*$. Here $\underline{\mathbf{m}} = (m_1, m_2)$, $m_1 \geq m_2 \geq 0$ and $\underline{\mathbf{m}}^*$ is the contragredient of $\underline{\mathbf{m}}$. See [FK] and [Or]. We denote $d_{\underline{\mathbf{m}}} = (m_1 - m_2 + 1)^2$ the dimension of the representation space of $P_{\underline{\mathbf{m}}}$. Let $\chi_{\underline{\mathbf{m}}}$ on $U(2)$ be the character of this representation $\underline{\mathbf{m}}$. It extends to a polynomial on the space of 2×2 matrices. On the diagonal matrices $t = \operatorname{diag}(t_1, t_2)$ it is given by

$$\chi_{\underline{\mathbf{m}}}(t) = \frac{(t_1 t_2)^{m_2}(t_1^{m_1-m_2+1} - t_2^{m_1-m_2+1})}{\Omega(t)} = \frac{(t_1 t_2)^{m_2}(t_1^{m_1-m_2+1} - t_2^{m_1-m_2+1})}{t_1 - t_2}$$

Following the notation in [FK], we let

$$(\alpha)_{\underline{\mathbf{m}}} = \frac{\Gamma(\alpha + m_1)\Gamma(\alpha + m_2 - 1)}{\Gamma(\alpha)\Gamma(\alpha - 1)} = (\alpha)_{m_1}(\alpha - 1)_{m_2} \tag{2.4}$$

be the generalized Pochhammer's symbol.

LEMMA 2.6. *If $\nu > 1$ The functions*

$$e_{\underline{\mathbf{m}}}(W^*Z) = \frac{(\nu)_{\underline{\mathbf{m}}}}{(2)_{\underline{\mathbf{m}}}}\chi_{\underline{\mathbf{m}}}(W^*Z), \ \ \underline{\mathbf{m}} \geq 0$$

form an orthonormal basis of $(H_\nu \otimes \overline{H_\nu})_0$; if $\nu = 1$ the functions

$$e_{\underline{\mathbf{m}}}(Z, W) = \frac{1}{m_1 + 1}\chi_{\underline{\mathbf{m}}}(W^*Z), \ \ \underline{\mathbf{m}} = (m_1, 0), \ \ m_1 \geq 0$$

form an orthonormal basis of $(H_\nu \otimes \overline{H_\nu})_0$.

PROOF. Consider the case $\nu > 1$ first. In fact, from Theorem 3.4 and Lemma 3.1 in [FK] we know that the $K_{\underline{\mathbf{m}}}$ function there is given by

$$K_{\underline{\mathbf{m}}}(Z, W) = \frac{d_{\underline{\mathbf{m}}}^{\frac{1}{2}}}{(2)_{\underline{\mathbf{m}}}}\chi_{\underline{\mathbf{m}}}(W^*Z).$$

The fact that $K_{\underline{\mathbf{m}}}(Z, W)$ span a dense subspace of $(H_\nu \otimes \overline{H_\nu})_0$ follows easily from the Schur's Lemma. Moreover they are orthogonal. It follows from Corollary 3.7 in [FK] that

$$\|K_{\underline{\mathbf{m}}}\|^2_{H_\nu \otimes \overline{H_\nu}} = \frac{d_{\underline{\mathbf{m}}}}{(\nu)^2_{\underline{\mathbf{m}}}},$$

substituting the above formula we find that $e_{\underline{\mathbf{m}}}$ has norm 1. This proves the case for $\nu > 1$.

If $\nu = 1$ the K-types appearing in the space (H_1, π_1) are of the type $\underline{\mathbf{m}} = (m_1, 0)$ with $m_1 \geq 0$. The rest of the proof can be done similarly. ∎

On diagonal matrices the function $e_{\underline{\mathbf{m}}}$ is given by

$$e_{\underline{\mathbf{m}}}(t) = \Omega(t)^{-1} \det\left(e_{m_j+2-j}(t_k)\right). \tag{2.5}$$

Here e_n is defined in (1.1) in Section 1 with $\kappa = \nu - 1$.

We define

$$P_{\underline{\mathbf{m}},\nu}(\underline{\lambda}) = \frac{\det\left(p_{m_j+2-j,\nu-1}(\lambda_k)\right)}{\lambda_2^2-\lambda_1^2}$$

and

$$\widetilde{e_{\underline{\mathbf{m}},\nu}}(\underline{\lambda}) = \frac{(2)_{\underline{\mathbf{m}}}}{(\nu)_{\underline{\mathbf{m}}}} P_{\underline{\mathbf{m}},\nu}(\underline{\lambda}) = \frac{1}{\lambda_2^2-\lambda_1^2} \det\left(\widetilde{e_{m_j+2-j}}(\lambda_k)\right). \tag{2.6}$$

where $\widetilde{e_m}$ is defined in (1.3) with $\kappa = \nu - 1$. This is the analogue of $\widetilde{e_m}$ in the case of the unit disk. Note that the above definition makes sense for all $\nu \geq 1$.

We will find a similar expansion as (1.4) in Section 1. For this purpose we recall more known facts about spherical functions.

The spherical function on type I domains has been obtained explicitly in [Hoo]. From [Hoo] we have the spherical function on D is given by

$$\phi_{\underline{\lambda}}(s_1,s_2) = \frac{-2^4 h(t)}{(\lambda_1^2-\lambda_2^2)\Omega(t)} \det\left(\phi_{\lambda_j}(s_k)\right),$$

where $t_1 = s_1^2$, $t_2 = s_2^2$ and where the determinant is that of the 2×2 matrix in j,k. It satisfies

$$\Delta_1\phi_{\underline{\lambda}} = (\lambda_1^2+\lambda_2^2+2)\phi_{\underline{\lambda}}, \quad \Delta_2\phi_{\underline{\lambda}} = (\lambda_1^2+1)(\lambda_2^2+1)\phi_{\underline{\lambda}}$$

Lemma 2.7 and Corollary 2.8 below can be used to prove Proposition 2.9, though we give a more direct proof. We present them since they might be of independent interests. In view of our main results below (Theorem 2.13, Theorem 3.1 and Theorem 4.2) it gives the Clebsch-Gordan coefficients of the tensor product decompositions.

LEMMA 2.7. *The following formula holds*

$$K_\nu(Z,Z)\phi_{\underline{\lambda}}(Z) = \sum_{\underline{\mathbf{m}}\geq 0} \widetilde{e_{\underline{\mathbf{m}},\nu}}(\underline{\lambda}) e_{\underline{\mathbf{m}}}(Z^*Z),$$

and the series converges uniformly on compact sets of D.

PROOF. Both sides are K-invariant so it suffices to prove the Lemma for diagonal matrices $Z = \{(s_1,s_2); 1 > s_1 \geq s_2 \geq 0\}$. Now the result follows from Theorem 1.2.1 in [Hu] and (1.4) in Section 1. We omit the details. ∎

When $\nu = 0$ the above expansion gives the expansion of the spherical function.

COROLLARY 2.8. *The spherical function has the following expansion*

$$\phi_{\underline{\lambda}}(Z) = \sum_{\underline{\mathbf{m}}\geq 0} P_{\underline{\mathbf{m}},0}(\underline{\lambda})\chi_{\underline{\mathbf{m}}}(Z^*Z)$$

where

$$P_{\underline{\mathbf{m}},0}(\underline{\lambda}) = \frac{1}{2^4}\prod_{j=1}^{2}\left(\frac{1+i\lambda_j}{2}\right)_2\left(\frac{1-i\lambda_j}{2}\right)_2 \times \frac{\det\Big({}_3F_2\big(-(m_j-2-j)+2, \frac{1+i\lambda_k}{2}+2, \frac{1-i\lambda_k}{2}+2; 3,3;1\big)\Big)}{\lambda_2^2-\lambda_1^2}$$

We study now the matrix form of the pair $(\Box_1, \Box_2)$ on the orthonormal basis $e_{\underline{\mathbf{m}}}(W^*Z)$ and find similar formulas as (1.2) and (1.3). We thus define the "symbol" of the differential operators $\Box_1$, $\Box_2$ by

$$n_1(\underline{\lambda}) = \Lambda_1 + \Lambda_2 = 2(\nu - 1) - \frac{1}{4}(1 + \lambda_1^2) - \frac{1}{4}(1 + \lambda_2)^2,$$

and

$$n_2(\underline{\lambda}) = \Lambda_1\Lambda_2 = \left((\nu - 1) - \frac{1}{4}(1 + \lambda_1^2)\right)\left(\nu - 1 - \frac{1}{4}(1 + \lambda_2)^2\right),$$

where $\Lambda_i = (\nu - 1) - \frac{1}{4}(1 + \lambda_i^2)$, $i = 1, 2$ as in (1.0).

PROPOSITION 2.9. *The pair* $(\Box_1, \Box_2)$ *on* $e_{\underline{\mathbf{m}}}$ *and the pair of multiplication operators* (n_1, n_2) *on* $\widetilde{e_{\underline{\mathbf{m}},\nu}}$ *have the same matrix form.*

PROOF. This is almost trivial from (1.2)–(1.3) in Section 1, (2.1)–(2.2) and (2.5)–(2.6) in Section 2. We only check that for $\Box_1$ and n_1. We have

$$\begin{aligned}
&\Box_1 e_{\underline{\mathbf{m}}}(t_1, t_2)\\
&= \Omega(t)^{-1}(C_{1,\nu-1} + C_{2,\nu-1})\big(e_{m_1+1,\nu-1}(t_1)e_{m_2,\nu-1}(t_2) - e_{m_2,\nu-1}(t_1)e_{m_1+1,\nu-1}(t_2)\big)\\
&= a_{m_1+1}e_{\underline{\mathbf{m}}+\varepsilon_1}(t_1, t_2) + (b_{m_1+1} + b_{m_2})e_{\underline{\mathbf{m}}}(t_1, t_2) + c_{m_1+1}e_{\underline{\mathbf{m}}-\varepsilon_1}(t_1, t_2)\\
&\quad + a_{m_2}e_{\underline{\mathbf{m}}+\varepsilon_2}(t_1, t_2) + c_{m_2}e_{\underline{\mathbf{m}}-\varepsilon_2}(t_1, t_2)
\end{aligned}$$

where in the second equality we use (1.2). Here $\varepsilon_1 = (1, 0)$ and $\varepsilon_2 = (0, 1)$ and a_n, b_n and c_n are given in Section 1. Note that those terms $e_{\underline{\mathbf{m}}\pm\varepsilon_j}$ have zero coefficients if $\underline{\mathbf{m}} \pm \varepsilon_j$ violate the condition for the highest weight. Similarly

$$\begin{aligned}
&n_1(\underline{\lambda})\widetilde{e_{\underline{\mathbf{m}},\nu}}(\underline{\lambda})\\
&\quad= \frac{1}{\lambda_2^2 - \lambda_1^2}(\Lambda_1 + \Lambda_2)\big(\widetilde{e_{m_1+1}}(\lambda_1)\widetilde{e_{m_2}}(\lambda_2) - \widetilde{e_{m_2}}(\lambda_1)\widetilde{e_{m_1+1}}(\lambda_1)\big)\\
&\quad= a_{m_1+1}\widetilde{e_{\underline{\mathbf{m}}+\varepsilon_1,\nu}}(\underline{\lambda}) + (b_{m_1+1} + b_{m_2})\widetilde{e_{\underline{\mathbf{m}},\nu}}(\underline{\lambda}) + c_{m_1+1}\widetilde{e_{\underline{\mathbf{m}}-\varepsilon_1,\nu}}(\underline{\lambda})\\
&\qquad + a_{m_2}\widetilde{e_{\underline{\mathbf{m}}+\varepsilon_2,\mu}}(\underline{\lambda}) + c_{m_2}\widetilde{e_{\underline{\mathbf{m}}-\varepsilon_2,\nu}}(\underline{\lambda}),
\end{aligned}$$

where in the second equality we use (1.3). Thus $\Box_1$ and n_1 have the same matrix form. ∎

Another proof is to use the expansion (2.7), which we omit.

Note that when $\nu = 1$ the operator $\Box_2$ acts on $e_{\underline{\mathbf{m}}}$ as the zero operator.

We now recall a result from [KM] about determinants of orthogonal polynomials, which can also be proved directly.

LEMMA 2.10. *Let* $a \in \mathbb{R}$ *and let* $\psi(x)$ *be a distribution function on the semi-axis* $[a, \infty)$ *with infinitely many points of increase, and with finite moment of all orders. Let* $Q_n(x)$, $n = 0, 1, 2, \ldots$ *be the orthogonal polynomials for the distribution* ψ *with*

$$\int_a^\infty Q_i(x)Q_j(x)\,d\psi(x) = \pi_i\delta_{i,j}.$$

Taking integers $0 \le i_1 < i_2$ *we form the determinant*

$$Q_{i_1,i_2}(x_1,x_2) = \det\big(Q_{i_j}(x_k)\big).$$

Then

$$\int\int_{a<x_1<x_2} Q_{i_1,i_2}(x_1,x_2)Q_{j_1,j_2}(x_1,x_2)\,d\psi(x_1)\,d\psi(x_2) = \pi_{i_1}\pi_{i_2}\delta_{i_1,j_1}\delta_{i_2,j_2}.$$

The Harish-Chandra C-function of $\mathrm{SU}(2,2)$ is

$$C(\underline{\lambda}) = (\lambda_1^2-\lambda_2^2)^{-1}\frac{2^{i\lambda_1}\Gamma(i\lambda_1)}{\Gamma(\frac{i\lambda_1+1}{2})\,\Gamma(\frac{i\lambda_1+1}{2})}\frac{2^{i\lambda_2}\Gamma(i\lambda_2)}{\Gamma(\frac{i\lambda_2+1}{2})\,\Gamma(\frac{i\lambda_2+1}{2})}$$

and C_0 is a positive constant. We let

$$B_\nu(\underline{\lambda}) = \prod_{j=1}^{2}\left|\Gamma\left(\nu-\frac{3}{2}+i\frac{\lambda_j}{2}\right)\right|^2$$

(B_ν is the symbol of the transform on the domain D, see [UU].) Note that

$$C(\underline{\lambda}) = (\lambda_1^2-\lambda_2^2)^{-1}c(\lambda_1)c(\lambda_2), \quad B_\nu = b_{\nu-1}(\lambda_1)b_{\nu-1}(\lambda_2)$$

where b_κ and c are the corresponding symbol function and Harish-Chandra c-function for $\mathrm{SU}(1,1)$ in Section 1.

Using Lemma 2.10 (with $x_1 = \lambda_1^2$ and $x_2 = \lambda_2^2$) we then get

PROPOSITION 2.11.

(1) If $\nu \ge \frac{3}{2}$ *then we have the following orthogonality relation*

$$C_0\int_{\lambda_1>\lambda_2>0} \widetilde{e_{\underline{\mathbf{m}},\nu}}(\underline{\lambda})\overline{\widetilde{e_{\underline{\mathbf{m}}',\nu}}(\underline{\lambda})}B_\nu(\underline{\lambda})\left|C(\underline{\lambda})\right|^{-2}d\underline{\lambda} = \delta_{\underline{\mathbf{m}},\underline{\mathbf{m}}'},$$

where C_0 *is a positive constant.*

(2) If $1 < \nu < \frac{3}{2}$ *then we have the following orthogonality relation*

$$\begin{aligned} &C_1\int_{\lambda_1>\lambda_2>0} \widetilde{e_{\underline{\mathbf{m}},\nu}}(\underline{\lambda})\overline{\widetilde{e_{\underline{\mathbf{m}}',\nu}}(\underline{\lambda})}B_\nu(\underline{\lambda})|C(\underline{\lambda})|^{-2}\,d\underline{\lambda}\\ &\quad + C_2\int_{\underline{\lambda}=i(2\nu-3)f_1+yf_2,y>0} \widetilde{e_{\underline{\mathbf{m}},\nu}}(\underline{\lambda})\overline{\widetilde{e_{\underline{\mathbf{m}}',\nu}}(\underline{\lambda})}c_\nu(y)\,dy = \delta_{\underline{\mathbf{m}},\underline{\mathbf{m}}'},\end{aligned}$$

where C_1 *and* C_2 *are positive constants and*

$$c_\nu(y) = \left|\big((2\nu-3)^2+y^2\big)\frac{\Gamma(\frac{2\kappa-3+iy}{2})\Gamma(\frac{iy+1}{2})\,\Gamma(\frac{iy+1}{2})}{\Gamma(iy)}\right|^2.$$

Propositions 2.9 and 2.11 now give us

PROPOSITION 2.12.

(1) *If $\nu \geq \frac{3}{2}$ then the operator pair $(\Box_1, \Box_2)$ on $(H_\nu \otimes \overline{H_\nu})_0$ is unitarily equivalent to the operator pair (n_1, n_2) of multiplication operators by $n_1(\underline{\lambda})$ and $n_2(\underline{\lambda})$ on $L^2\big(\mathfrak{a}_+^*, B_\nu(\underline{\lambda})|C(\underline{\lambda})|^{-2}\,d\underline{\lambda}\big)$ and thus the spectrum of $\big(\sigma(\Box_1, \Box_2), (H_\nu \otimes \overline{H_\nu})_0\big)$ is $\big\{\big(n_1(\underline{\lambda}), n_2(\underline{\lambda})\big); \underline{\lambda} \in \mathfrak{a}_+^*\big\}$. Moreover each point $\big(n_1(\underline{\lambda}), n_2(\underline{\lambda})\big)$ in the spectrum is of multiplicity one.*

(2) *If $1 < \nu < \frac{3}{2}$ then (1) is true with $L^2\big(\mathfrak{a}_+^*, B_\nu(\underline{\lambda})|C(\underline{\lambda})|^{-2}d\underline{\lambda}\big)$ replaced by*

$$L^2\big(\mathfrak{a}_+^*, B_\nu(\underline{\lambda})|C(\underline{\lambda})|^{-2}\,d\underline{\lambda}\big) \oplus L^2\big(i(2\nu-3)f_1 + \mathbb{R}^+ f_2, c_\nu(y)\,dy\big),$$

where $i(2\nu-3)f_1 + \mathbb{R}^+ f_2$ stands for the subset $\{i(2\nu-3)f_1 + yf_2; y \in \mathbb{R}^+\}$.

PROOF. We need only to prove that the map $e_{\underline{\mathbf{m}}} \longmapsto \widetilde{e_{\underline{\mathbf{m}},\nu}}$ from $(H_\nu \otimes \overline{H_\nu})_0$ into the corresponding spaces of $\underline{\lambda}$, is *onto*. We assume $\nu \geq \frac{3}{2}$. The remaining case is similar. This is equivalent to that the linear span of the polynomials $\widetilde{e_{\underline{\mathbf{m}},\nu}}$ are dense in the corresponding Hilbert space. From Lemma 1.1 (see also the remark thereafter) we see that the the linear span of the functions $e_{m_1}(\lambda_1)e_{m_2}(\lambda_2)$ are dense in the space $L^2(\mathbb{R}^+ \times \mathbb{R}^+, b_{\nu-1}(\lambda_1)b_{\nu-1}(\lambda_2)|c(\lambda_1)c(\lambda_2)|^{-2}d\lambda_1 d\lambda_2)$. Now the map

$$f(\lambda_1, \lambda_2) \longmapsto f(\lambda_1, \lambda_2) - f(\lambda_2, \lambda_1)$$

is bounded from $L^2(\mathbb{R}^+ \times \mathbb{R}^+, b_{\nu-1}(\lambda_1)b_{\nu-1}(\lambda_2)\big|c(\lambda_1)c(\lambda_2)\big|^{-2}\,d\lambda_1\,d\lambda_2)$ into

$$L^2\big(\mathfrak{a}_+^*, b_{\nu-1}(\lambda_1)b_{\nu-1}(\lambda_2)|c(\lambda_1)c(\lambda_2)|^{-2}\,d\lambda_1\,d\lambda_2\big);$$

moreover it is easily seen that the map is onto. Under this map the functions $e_{m_1}(\lambda_1)e_{m_2}(\lambda_2)$ are mapped into $e_{m_1}(\lambda_1)e_{m_2}(\lambda_2) - e_{m_1}(\lambda_2)e_{m_2}(\lambda_1)$. Thus they are dense in the later space. Moreover they are skew-symmetric in m_1 and m_2 and if $m_1 = m_2 = 0$ they are identically zero. Thus the functions $e_{m_1+1}(\lambda_1)e_{m_2}(\lambda_2) - e_{m_1+1}(\lambda_2)e_{m_2}(\lambda_1)$ with $m_1 \geq m_2$ are dense in the space. This is equivalent to that $\widetilde{e_{\underline{\mathbf{m}},\nu}}(\underline{\lambda})$ are dense in $L^2\big(\mathfrak{a}_+^*, B_\nu(\underline{\lambda})|C(\underline{\lambda})|^{-2}\,d\underline{\lambda}\big)$. ∎

Recall the representation $\mathcal{H}(\underline{\lambda})$ defined in Section 1. Combining Lemma 2.4 and Proposition 2.12 we now get

THEOREM 2.13. *Suppose $\nu \geq \frac{3}{2}$. We have the following Plancherel formula and the decomposition of the representation*

$$\pi_\nu \otimes \overline{\pi_\nu} \cong \int_{\lambda_1 > \lambda_2 > 0}^{\oplus} \mathcal{H}(\underline{\lambda}) B_\nu(\underline{\lambda})|C(\underline{\lambda})|^{-2}\,d\underline{\lambda}.$$

3. The tensor products $\pi_\nu \otimes \overline{\pi_\nu}$ for $1 < \nu < \frac{3}{2}$.

Let $1 < \nu < \frac{3}{2}$. It follows from Lemma 2.4 and Proposition 2.12 that for a. e. $\underline{\lambda} = i(2\nu-3)f_1 + yf_2$ with $y \in \mathbb{R}^+$ there is a representation $\mathcal{H}\big(i(2\nu-3)f_1 + yf_2\big)$ of G appearing in the decomposition of $\pi_\nu \otimes \overline{\pi_\nu}$ so that it has K-fixed vector and $(\Box_1, \Box_2)$ acts on it as a pair of scalar $\big(n_1(\underline{\lambda}), n_2(\underline{\lambda})\big)$, $\underline{\lambda} = i(2\nu-3)f_1 + yf_2$. Thus $\mathcal{H}\big(i(2\nu-3)f_1 + yf_2\big)$ can be identified as functions on D generated by the spherical function $\phi_{i(2\nu-3)f_1+yf_2}$ and moreover $\phi_{i(2\nu-3)f_1+yf_2}$ is positive definite.

THEOREM 3.1. *We have the following Plancherel formula and the decomposition of the representation, for* $1 < \nu < \frac{3}{2}$,

$$\pi_\nu \otimes \overline{\pi_\nu} \cong \int_{\lambda_1 > \lambda_2 > 0}^{\oplus} H\,(\underline{\lambda}) B_\nu(\underline{\lambda}) |C(\underline{\lambda})|^{-2}\, d\underline{\lambda} \oplus \int_{\mathbb{R}^+}^{\oplus} H\left(i(2\nu-3)f_1 + yf_2\right) c_\nu(y)\, dy.$$

We note that for the group SU$(1,1)$ the symbol of the Berezin transform is $b_\nu(\lambda) = \Gamma(\nu - \frac{1}{2} + \frac{i\lambda}{2})\Gamma(\nu - \frac{1}{2} - \frac{i\lambda}{2})$, and it has a simple pole at the point $i(\frac{1}{2} - \nu)$ on $i\mathbb{R}^+$ when $\nu < \frac{1}{2}$. This gives the discrete part in the decomposition of the tensor product of the analytic continuation of the holomophic discrete series $\pi_\nu \otimes \overline{\pi_\nu}$ of SU$(1,1)$ ([OZ]). However for SU$(2,2)$ the symbol of the Berezin transform, $B_\nu(\underline{\lambda})$, has pole on a line when $\nu < \frac{3}{2}$. This gives us the above continuous decomposition.

We know from the Knapp-Speh classification of unitary representations of SU$(2,2)$ ([KS, Main Theorem, (iii)-(c)]) that $H\left(i(2\nu-3)f_1 + \lambda_2 f_2\right)$ is a unitary submodule of the (non-unitary) induced representation $\mathrm{Ind}_P\Big(1 \otimes i\big(i(2\nu-3)f_1 + \lambda_2 f_2\big) \otimes 1\Big)$ and the whole induced representation is reducible.

4. **The tensor products** $\pi_\nu \otimes \overline{\pi_\nu}$ **for** $\nu = 1$**.** We now consider the case $\nu = 1$. The K-types appearing in the representation π_1 are of the form $\underline{\mathbf{m}} = (m_1, 0)$ with $m_1 \geq 0$. The corresponding polynomials $e_{\underline{\mathbf{m}},1}$ are now

$$P_{\underline{\mathbf{m}},1}(\underline{\lambda}) = \frac{p_{m_1+1,0}(\lambda_1) - p_{m_1+1,0}(\lambda_2)}{\lambda_2^2 - \lambda_1^2}$$

and if $\underline{\lambda} = (\lambda_1, \lambda_2) = (-i, y)$ then

$$p_{m_1+1,0}(-i) = 0$$

and

$$p_{m_1+1,0}(y) = -\frac{1}{(m_1+1)!\,(m_1+1)!} \frac{1+y^2}{2^2} S_{m_1}\left(-\left(\frac{y}{2}\right)^2; \frac{3}{2}, \frac{1}{2}, \frac{1}{2}\right)$$

where S_{m_1} is the continuous Hahn polynomials introduced in Section 1. Thus

$$P_{\underline{\mathbf{m}},1}(\underline{\lambda}) = 2^{-2} \frac{1}{(m_1+1)!\,(m_1+1)!} S_{m_1}\left(-\left(\frac{y}{2}\right)^2; \frac{3}{2}, \frac{1}{2}, \frac{1}{2}\right)$$

where ${}_3F_2$ is the hypergeometric function and S_n is the continuous dual Hahn polynomials as in Section 1.

Lemma 2.8 when $\nu = 1$ takes the following form.

LEMMA 4.1. *We have the following orthogonality relation*

$$c_0 \int_0^\infty \widetilde{e_{\underline{\mathbf{m}},1}}(-i,y)\overline{\widetilde{e_{\underline{\mathbf{m}}',1}}(-i,y)} \left| \frac{\Gamma(\frac{3+iy}{2})\Gamma(\frac{1+iy}{2})\Gamma(\frac{1+iy}{2})}{\Gamma(iy)} \right|^2 dy = \delta_{\underline{\mathbf{m}},\underline{\mathbf{m}}'}$$

for $\underline{\mathbf{m}} = (m_1, 0)$ *and* $\underline{\mathbf{m}}' = (m_1', 0)$ *with* $m_1 \geq 0$ *and* $m_1' \geq 0$. *Here* c_0 *is a positive constant.*

PROOF. This is just the Wilson's orthogonality relation, see (4.4) on p. 697, [W]. ∎

Thus the map $e_{\underline{\mathbf{m}}} \mapsto \widetilde{e_{\underline{\mathbf{m}},1}}(-i, y)$ induces a unitary operator from $(\pi_1 \otimes \overline{\pi_1})_K$ onto the L^2-space on $(0, \infty)$ with the corresponding measure given in Lemma 4.1. That the map is isometric follows from the Lemma. The *onto* claim can be proved as Lemma 1.1. Consequently this map induces a unitary equivalence between the pair of operators $(\Box_1, \Box_2)$ on $(H_1 \otimes \overline{H_1})_0$ and $\left(n_1(\underline{\lambda}), n_2(\underline{\lambda})\right)$ on the above L^2-space of functions of $y \in (0, \infty)$ with $\underline{\lambda} = (-i, y)$. Note that both $\Box_2$ and $n_2(\underline{\lambda})$ are identically zero on the corresponding spaces. Thus we have the following decomposition for the tensor product $\pi_1 \otimes \overline{\pi_1}$. Denote $H(-if_1 + yf_2)$ the representation generated by the spherical function $\phi_{-if_1+yf_2}$.

THEOREM 4.2. *We have the following Plancherel formula and the decomposition of the representation*

$$\pi_1 \otimes \overline{\pi_1} \cong \int_{\mathbb{R}^+}^{\oplus} H(-if_1 + yf_2) \left| \frac{\Gamma(\frac{3+iy}{2})\Gamma(\frac{1+iy}{2})\Gamma(\frac{1+iy}{2})}{\Gamma(iy)} \right|^2 dy.$$

5. **The tensor products of holomorphic discrete series of semisimple Lie group.** We note that much of our calculations above can be generalized to a general bounded symmetric domain and their holomorphic discrete series. We follow the notation in [FK]. Let $D = G/K$ be an irreducible bounded symmetric domain of rank r. Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be the Cartan decomposition. Let $\gamma_1 > \gamma_2 > \cdots > \gamma_r$ be the Harish-Chandra strongly orthogonal roots and let $2f_1, 2f_2, \ldots, 2f_r$ be their Cayley transform on a maximal abelian subspace $\mathfrak{a}$ of $\mathfrak{p}$. Denote a and b the root multiplicity of $f_j \pm f_k$ and f_j respectively, and $p = a(r-1) + 2 + b$ be the genus of the symmetric domain. We let π_ν be the analytic continuation of holomorphic discrete series of a general bounded symmetric domain as defined in [FK]. An orthogonal expansion of the reproducing kernel in terms of $K_{\underline{\mathbf{m}}}$ has been found in that paper, which then gives an generalization of our Lemma 2.6. However we are unable to find a similar expansion as in Lemma 2.7 for a general bounded symmetric domain; even for $\nu = 0$, namely the expansion of the spherical function in terms of $K_{\underline{\mathbf{m}}}$ (which are up to constants the Jack symmetric polynomials), is still an open question [Op]. Nevertheless using the intertwining operator R defined in Section 2, the result of Faraut-Koranyi [FK] and the result of Unterberger-Upmeier [UU] on the symbol of the Berezin transform RR^* we can derive the following

THEOREM 5.1. *Let* $\nu > \frac{p-1}{2}$ *and* $H(\underline{\lambda})$ *be the principal series representation of* G *with parameter* $\lambda \in \mathfrak{a}^*$. *We have the following decomposition formula for the tensor product* $\pi_\nu \otimes \overline{\pi_\nu}$*:*

$$\pi_\nu \otimes \overline{\pi_\nu} \cong \int_{\mathfrak{a}^*/W}^{\oplus} H(\underline{\lambda}) d\underline{\lambda}$$

It would be interesting to find out the decomposition of $\pi_\nu \otimes \overline{\pi_\nu}$ for other (unitary) values of ν. We plan to pursue this question in the future.

REFERENCES

[AW] R. Askey and J. Wilson, *Some basic hypergeometric orthogonal polynomials that generalize Jacobi polynomials*. Mem. Amer. Math. Soc., (319) **54**(1985).

[B] F. A. Berezin, *General concept of quantization*. Commun. Math. Phys. **40**(1975) 153–174.

[EHW] T. J. Enright, R. Howe and N. R. Wallach, *A classification of unitary highest weight modules*. Progr. Math. **40**(1983), 97–143.

[FK] J. Faraut and A. Koranyi, *Function spaces and reproducing kernels on bounded symmetric domains*. J. Funct. Anal. **89**(1990), 64–89.

[He] S. Helgason, *Groups and geometric analysis*. Academic Press, London, 1984.

[Hoo] B. Hoogenboom, *Spherical functions and differential operators on complex Grassmann manifolds*. Ark. Mat. **20**(1982), 69–85.

[How] R. Howe, *On Some Results of Strichartz and of Rallis and Schiffman*. J. Funct. Anal. **32**(1979), 297–303.

[Hu] L. K. Hua, *Harmonic analysis of functions of several complex variables in the classical domains*. Amer. Math. Soc., Providence, Rhode Island, 1963.

[KM] S. Karlin and J. McGregor, *Some properties of orthogonal polynomials.* In: Theory and Application of Special Functions (Ed. R. Askey), Academic Press, 1975, 521–550.

[Kn] A. W. Knapp, *Representation Theory of Semisimple Groups*. Princeton University Press, Princeton, New Jersey, 1986.

[KS] A. W. Knapp and B. Speh, *Irreducible Unitary Representations of* $\mathrm{SU}(2,2)$. J. Funct. Anal. **45**(1982), 41–73.

[Ma] W. Magnus and F. Oberhettinger, *Formulas and theorems for the functions of mathematical physics*. Chelsea, New York, 1949.

[M] F. I. Mautner, *Unitary representations of locally compact groups. II*. Ann. of Math. **52**(1950), 528–556.

[Op] E. Opdam, Private communication.

[Or] B. Ørsted, *Composition series for analytic continuations of holomorphic discrete series representations of* $\mathrm{SU}(n,n)$. Trans. Amer. Math. Soc. **260**(1980), 563–573.

[OZ] B. Ørsted and G. Zhang, *L^2-versions of the Howe Correspondence I*. Math. Scand., to appear.

[PZ] J. Peetre and G. Zhang, *A weighted Plancherel formula III. The case of a hyperbolic matrix ball*. Collect. Math. **43**(1992), 273–301.

[Pu] L. Pukánszky, *On the Kronecker products of irreducible representations of the* 2×2 *real unimodular group. Part I*. Trans. Amer. Math. Soc. **100**(1961), 116–152.

[Re1] J. Repka, *Tensor products of unitary representations of* $\mathrm{SL}_2(R)$. Amer. J. Math. **100**(1978), 930–932.

[Re2] J. Repka, *Tensor products of holomorphic discrete series*. Can. J. Math. **31**(1979), 836–844.

[IES] I. E. Segal, *An extension of Plancherel's formula to separable unimodular groups*. Ann. of Math. **52** (1950), 272–292.

[UU] A. Unterberger and H. Upmeier, *Berezin transform and invariant differential operators*. Preprint.

[Wa] N. R. Wallach, *The analytic continuation of the discrete series I, II*. Trans. Amer. Math. Soc. **251**(1979), 1–17, 19–37.

[W] J. Wilson, *Some hypergeometric orthogonal polynomials.* SIAM. J. Math. Anal. **11**(1980), 690–701.

[Z] G. Zhang, *Tensor products of weighted Bergman spaces and invariant Ha-plitz operators*. Math. Scand. **71**(1992), 85–95.

Institut for Matematik og Datalogi
Odense Universitet
DK-5230 Odense M
Denmark

Department of Mathematics
University of Karlstad
S-651 88 Karlstad
Sweden