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TENSOR PRODUCTS OF FINITE AND INFINITE DIMENSIONAL REPRESENTATIONS OF SEMISIMPLE LIE ALGEBRAS

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0. Introduction

0.0. Let $\mathfrak{g}$ be a semisimple Lie algebra, $U = U(\mathfrak{g})$ its enveloping algebra, $Z = Z(U)$ the ring of Laplace operators, i.e. the centre of U. Irreducible representations of $\mathfrak{g}$ are naturally distinguished by eigenvalues of Laplace operators, i.e. by characters of Z. It often happens that various properties of representations corresponding to different characters are the same. For example, it is well known that the decompositions of representations of the principal series of $SL(2, \mathbf{R})$

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at all integer points (i.e. points where a finite-dimensional subrepresentation exists) have similar structures.

Statements of this kind may be proved by the following trick: choose a finite dimensional $\mathfrak{g}$-module V and two characters θ, θ' of Z, i.e. two sets of eigenvalues of Laplace operators. For a $\mathfrak{g}$-module M with eigenvalue θ we construct a module M' equal to the θ'-component in the decomposition of the module $V \otimes M$ with respect to the eigenvalues of Laplace operators. It turns out that many properties of M' can be deduced from analogous properties of M. In particular, for some pairs (θ, θ') we obtain a one-to-one correspondence between two families of $\mathfrak{g}$-modules corresponding to θ and θ'.

This approach proved useful in various problems of the theory of infinite-dimensional representations of semisimple Lie algebras (see [1–8], [21–24]).

The aim of this article is to describe this method in the most general situation possible.

0.1. Brief contents of this paper. Choose a finite-dimensional $\mathfrak{g}$-module V. The map $M \mapsto V \otimes M$ defines a functor on the category of $\mathfrak{g}$-modules denoted by F_V. In §2 we study the relation between actions of Laplace operators on M and on $F_V(M)$. Namely, we define an action of the algebra $Z \otimes Z$ on $F_V(M) = V \otimes M$ via $(z_1 \otimes z_2)(v \otimes m) = z_1(v \otimes z_2 m)$ for z_1, $z_2 \in Z$, $v \in V$, $m \in M$. Theorem 2.5 describes the ideal of those elements from $Z \otimes Z$ that act trivially on all modules $F_V(M)$. This theorem generalizes the Kostant theorem from [8]; in fact, our proof repeats Kostant's.

The module M is called Z-finite if $\dim Zm < \infty$ for any $m \in M$. Theorem 2.5 implies that the functor F_V maps Z-finite modules into Z-finite ones.

0.2. Generally, the functor F_V is indecomposable. However, its restriction to the subcategory of Z-finite modules has a lot of direct summands. For example, denote the functor choosing the θ-component of every Z-finite module by $\mathrm{Pr}(\theta)$ for any character θ of the algebra Z. Then $F_V = \bigoplus_{\theta,\theta'} \mathrm{Pr}(\theta) \circ F_V \circ \mathrm{Pr}(\theta')$.

Usually infinite dimensional $\mathfrak{g}$-modules were investigated only by functors of the form $\mathrm{Pr}(\theta) \circ F_V \circ \mathrm{Pr}(\theta')$. However, these functors are often decomposable themselves. It is highly important that functors F_V corresponding to different modules V can have isomorphic direct summands. Section 3 deals with the study of direct summands of functors F_V on the category of Z-finite modules. We call these direct summands projective functors.

The main theorems of section 3 describe all projective functors (Theorem 3.3) and their morphisms (Theorem 3.5). The proof of these theorems is based on the following remarkable property of projective functors. Let θ be a character of Z and $\mathcal{M}(\theta)$ be the category of all $\mathfrak{g}$-modules with eigenvalue θ. Then the restriction of any projective functor F to $\mathcal{M}(\theta)$ is completely defined by the unique $\mathfrak{g}$-module $F(M_\chi)$, where M_χ is a Verma module with dominant weight χ (see [3]). In particular, direct summands of a functor F correspond to those of the module $F(M_\chi)$.

0.3. In the following sections of the paper we gather the dividends. In section 4 the following consequences of the properties of projective functors are derived.

(1) We prove that the functor $\Pr(\theta) \circ F_V \circ \Pr(\theta')$ defines an equivalence between the categories $\mathcal{M}(\theta)$ and $\mathcal{M}(\theta')$ for some pairs (θ, θ') (Theorem 4.1). This result generalizes the results of Zuckerman [1] (see also [22–24]).

(2) We establish a correspondence between two-sided ideals in the enveloping algebra $U(\mathfrak{g})$ and submodules in Verma modules (Theorem 4.3). (This has been obtained also by A. Joseph [26]). In particular, we reaffirm a result of Duflo [9] on primitive ideals in $U(\mathfrak{g})$ (Theorem 4.4).

(3) We study multiplicities in Jordan–Hölder series of Verma modules (see 4.5). In particular, we were able to compute these multiplicities for the algebra $\mathfrak{sl}(4)$ completely. (Jantzen [23, Sect. 5] obtained more general results.)

0.4. In the second chapter (sections 5, 6) of this paper projective functors are applied to the study of representations of complex semisimple Lie groups. The possibility of such application is based on the following. Let G_{C} be a complex Lie group whose Lie algebra (with a natural complex structure) coincides with $\mathfrak{g}$. To a Harish–Chandra module M over G_{C}, we assign the functor in the category of $\mathfrak{g}$-modules which is obtained by factoring a projective functor. Using the description of indecomposable projective functors, we give a classification of irreducible Harish–Chandra modules of the group G_{C} (Theorem 5.6). Note that the classification of irreducible Harish–Chandra modules of complex semisimple Lie groups was obtained by Zhelobenko [10] (see also [11]) by analytical methods (theory of intertwining operators). Our methods are purely algebraic.

Further, we give a more detailed description of Harish–Chandra modules. Namely, we define an equivalence between the category of

Harish–Chandra modules of the group $G_{\mathbb{C}}$ with given eigenvalues of Laplace operators and a category of modules that can be described in a sufficiently simple way (namely, a part of the category $\mathcal{O}$; see Theorem 5.9).

In section 6 we show that with respect to this equivalence representations of the principal series of the group $G_{\mathbb{C}}$ correspond to dual modules of Verma modules.

This implies that multiplicities in the Jordan–Hölder series of representations of the principal series of a complex group $G_{\mathbb{C}}$ coincide with multiplicities in the Jordan–Hölder series of Verma modules over $\mathfrak{g}$ (Theorem 6.7). Note that the relation between representations of complex groups and Verma modules was found by Duflo [9] and Duflo–Conze–Berline [12]. In particular, studying representations of the principal series we in fact reproduce results from [12], while Theorem 6.7 sharpens proposition 4 in [9] (see also [25, 26]).

0.5. In section 1 for the reader's convenience we have accumulated all necessary facts of algebraic nature. This includes some information on Abelian categories, on geometry of the weight space of semisimple Lie algebras $\mathfrak{g}$, and some general results on $\mathfrak{g}$-modules. In particular, we provide a detailed description of properties of the category $\mathcal{O}$ and of Verma modules. At the same time we introduce the necessary notations.

The paper has two appendices. The first gives the proof of one of the lemmas from section 1 that we could not find in the literature.

In appendix 2 we describe the relations between our notations for complex Lie groups and the standard ones.

1. Notations and preliminaries

1.1. *Algebras and modules*

In the sequel an algebraically closed field k of a characteristic 0 is fixed. All vector spaces, algebras and so on are defined over k; we shall usually write $\otimes$, dim instead of $\otimes_k$, $\dim_k$.

An algebra is an associative k-algebra with unit, a module is a left unitary module. Denote by A^0 the dual algebra of A.

Let M be an A-module, B a subalgebra of A. For an ideal J in B put $JM = \{\Sigma\, j_\alpha m_\alpha \mid j_\alpha \in J, m_\alpha \in M\}$ and $M/J = M/JM$.

An element of $m \in M$ is called B-finite if $\dim Bm < \infty$. The module M is called B-finite if all its elements are B-finite.

1.2. Category theory

We shall widely use the language of category theory, all the necessary facts and notions being contained in books by Bass [13, ch. I, II], or Mitchell [14], or MacLane [15].

We shall consider Abelian k-categories (see [13, II, 2]) and denote them by script capitals $\mathscr{A}$, $\mathscr{M}$, $\mathscr{O}$, $\mathscr{H}$. All functors are assumed to be k-linear and additive. For any algebra A denote by A-mod the category of A-modules. Denote by $\mathrm{Hom}(F, G)$ the space of morphisms of F into G for any two functors F, G.

Let $\mathscr{H} \subset \mathscr{A}$ be a complete subcategory. It is clear that $\mathscr{H}$ is Abelian, if $\mathscr{H}$ contains subquotients (i.e. if $\mathscr{H}$ contains with any object, all its subquotients). Otherwise this has to be proved separately.

Let $\mathscr{A}$ be category, $\mathscr{P}$ some class of objects in $\mathscr{A}$ closed with respect to finite direct sums. An object $A \in \mathscr{A}$ is called $\mathscr{P}$-generated (respectively $\mathscr{P}$-presentable) if there is an exact sequence $P \to A \to 0$ (respectively $P' \to P \to A \to 0$), where $P, P' \in \mathscr{P}$. Denote by $\mathscr{A}_{\mathscr{P}}$ a complete subcategory of $\mathscr{A}$ consisting of p-presentable objects.

We say that $\mathscr{A}$ contains enough projective (respectively enough injective) objects if any object of $\mathscr{A}$ is a quotient of a projective object (respectively a subobject of an injective object).

Suppose that the functor $F : \mathscr{A} \to \mathscr{B}$ is left adjoint to the functor $G : \mathscr{B} \to \mathscr{A}$, i.e. there is given a bifunctor isomorphism

$$\gamma : \mathrm{Hom}(FA, B) \xrightarrow{\sim} \mathrm{Hom}(A, GB)$$

(see [13, I, 7]). Then the following statements are valid:

(a) F is right exact and commutes with inductive limits and G is left exact and commutes with projective limits (see [13, I, 7]).

(b) If the functor G is exact, then F transforms projective objects into projective ones; conversely, if $\mathscr{A}$ has enough projective objects and F transforms them into projective ones, then the functor G is exact.

In fact, if P is a projective object of $\mathscr{A}$ and $B' \to B \to B''$ is an exact sequence in $\mathscr{B}$, then

$$\mathrm{Hom}_{\mathscr{A}}(P, G(B') \to G(B) \to G(B'')) \cong \mathrm{Hom}_{\mathscr{B}}(F(P), B' \to B \to B'')$$

are exact simultaneously (see [14, V, 7]).

(c) If F' is a direct summand of the functor F, then there is a direct summand G' of the functor G, such that F' is left adjoint to G'; any direct summand G' of a functor G is obtained by such a construction.

In fact, direct summands of F are described by idempotents in

$\operatorname{Hom}(F, F)$. Let $p: F \to F$ be an idempotent. Let us consider a bifunctor

$$H(A, B) = \operatorname{Hom}(FA, B): \mathscr{A}^0 \times \mathscr{B} \to k\text{-mod}$$

and define an idempotent $\bar{p}$ in the algebra $\operatorname{End} H(A, B)$ by the formula $\bar{p}(\varphi) = \varphi \circ p_A$, where φ, $\bar{p}(\varphi) \in \operatorname{Hom}(FA, B)$ and p_A is an idempotent in $\operatorname{Hom}(FA, FA)$. Let us prove that any idempotent $q \in \operatorname{End} H(A, B)$ is defined in this way. In fact, for a given A we obtain an idempotent in the algebra of endomorphisms on the functor $B \rightsquigarrow \operatorname{Hom}(FA, B)$. As is shown in [13, ch. II] to this idempotent there corresponds an idempotent $p_A: FA \to FA$. The set of idempotents $\{p_A, A \in \mathscr{A}\}$ defines the idempotent p.

Similarly, we prove that idempotents in $\operatorname{End} H$ correspond bijectively to idempotents in $\operatorname{End} G$.

Hence direct decompositions of the form $F = F' \oplus F''$ correspond bijectively to decompositions $G = G' \oplus G''$, and the functors F' and G' are conjugate.

1.3. *Bimodules and functors*

Let A, B be two algebras. Modules X over $A \otimes B^0$ will be called (A, B)-bimodule. We shall consider X as the left A-module and as the right B-module; if $a \in A$, $b \in B$, $x \in X$ then $(a \otimes b)x = axb$.

To each (A, B)-bimodule X assign a functor $h(X): B\text{-mod} \to A\text{-mod}$ via $h(X)(M) = X \otimes_B M$.

If the functor is right exact and commutes with inductive limits it will be called right continuous.

Proposition (see [13, II, 2]): (i) *The functor $h(X)$ is right continuous.*

(ii) *If Y is an (A, B)-bimodule, then*

$$\operatorname{Hom}_{A \otimes B^0}(X, Y) = \operatorname{Hom}_{\mathrm{Funct}}(h(X), h(Y)).$$

(iii) *Let $F: B\text{-mod} \to A\text{-mod}$ be a right continuous functor. Suppose $X = F(B) \in A\text{-mod}$ and define the right B-module structure on X via $r_X(b) = F(r_B(b))$, where $r_X(b)$ and $r_B(b)$ are the right multiplication morphisms by elements $b \in B$ in X and in B respectively. Then the functor F is naturally isomorphic to the functor $h(X)$.*

Usually the algebras A and B coincide. To emphasize their different roles we will write A^ℓ and A^r for A and B. Similarly the superscripts ℓ and r will be used for other objects, for example, if J is an ideal in A, then J^ℓ and J^r are the appropriate ideals in A^ℓ and A^r. The notion A^2 stands for the algebra $A \otimes A^0$.

1.4. *Lie algebras, weights and roots*

In the sequel we fix a semisimple Lie algebra $\mathfrak{g}$ and its Cartan subalgebra $\mathfrak{h}$.

We shall use the following notations (cf. [16], [17]):

$U = U(\mathfrak{g})$ – the enveloping algebra of $\mathfrak{g}$.

$\mathfrak{h}^*$ – the dual space to $\mathfrak{h}$; elements of $\mathfrak{h}^*$ are called weights.

$R \subset \mathfrak{h}^*$ – the system of roots in $\mathfrak{g}$ with respect to $\mathfrak{h}$.

If $\gamma \in R$, then $h_\gamma \in \mathfrak{h}$ is the dual root and σ_γ is the reflection corresponding to the root γ.

$$\sigma_\gamma(\chi) = \chi - \chi(h_\gamma)\gamma.$$

W – the Weyl group, i.e. the group generated by σ_γ.

$$\Lambda = \{\chi \in \mathfrak{h}^* \mid \chi(h_\gamma) \in \mathbf{Z} \quad \text{for } \gamma \in R\}$$

– the lattice of integer weights in $\mathfrak{h}^*$.

$\Gamma \subset \Lambda$ – the lattice generated by R.

For a weight χ, define a subset $R_\chi \subset R$ and subgroups W_χ, $W_{\chi+\Gamma}$ in W via

$$\begin{aligned} R_\chi &= \{\gamma \in R \mid \chi(h_\gamma) \in \mathbf{Z}\}, \\ W_\chi &= \{w \in W \mid w\chi = \chi\}, \\ W_{\chi+\Gamma} &= \{w \in W \mid w(\chi + \Gamma) = \chi + \Gamma\}. \end{aligned}$$

If $\chi - \psi \in \Lambda$, then $R_\chi = R_\psi$; $W_{\chi+\Gamma} = W_{\psi+\Gamma}$. If $W_\chi = \{e\}$, then the weight χ is called regular. Put

$$\Xi^0 = \{(\psi, \chi) \mid \psi, \chi \in \mathfrak{h}^*, \psi - \chi \in \Lambda\}.$$

Define a W-action on Ξ^0 by the formula $w(\psi, \chi) = (w\psi, w\chi)$. Denote by Ξ a quotient space with respect to this action.

1.5. *A partial ordering in* $\mathfrak{h}^*$

Fix a system of positive roots $R^+ \subset R$ and denote by $\mathfrak{n}^+$ the corresponding nilpotent subalgebra in $\mathfrak{g}$. Put

$\Gamma^+ \subset \Gamma$ – the semigroup generated by R^+.

$\rho = \frac{1}{2} \sum_{\gamma \in R^+} \gamma$ – the halfsum of positive roots.

If $\chi, \psi \in \mathfrak{h}^*$, $\gamma \in R^+$ we write $\psi \overset{\gamma}{<} \chi$, whenever $\psi = \sigma_\gamma \chi$, $\chi(h_\gamma) \in \mathbf{Z}$. We write $\psi < \chi$ whenever there exists sequences of weights $\psi_0, \psi_1, \ldots, \psi_n \in \mathfrak{h}^*$ and of roots $\gamma_1, \ldots, \gamma_n \in R^+$ such that $\psi = \psi_0 \overset{\gamma_1}{<} \psi_1 \overset{\gamma_2}{<} \cdots \overset{\gamma_n}{<} \psi_n = \chi$. The relation $<$ defines a partial ordering on $\mathfrak{h}$.

The weight χ is called dominant if it is maximal with respect to the ordering $<$, i.e. $\chi(h_\gamma) \notin \{-1, -2, \ldots\}$ for any $\gamma \in R^+$.

Call weights χ and ψ equivalent, and write $\chi \sim \psi$, if $W(\chi) = W(\psi)$ and $\chi + \Gamma = \psi + \Gamma$.

The conditions $\chi \sim \psi$ and $\psi \in W_{\chi+\Gamma}(\chi)$ are equivalent. It is clear that if $\psi < \chi$, then $\psi \sim \chi$. Conversely, the following lemma shows that if $\psi \sim \chi$, there is a weight φ such that $\chi < \varphi$ and $\psi < \varphi$.

LEMMA: *Let χ be a weight. Then*

(i) *R_χ is a root system, $W_{\chi+\Gamma}$ is its Weyl group. In particular, $W_{\chi+\Gamma}$ is generated by reflections.*

(ii) *There are weights $\chi_{\max}$ and $\chi_{\min}$ in the set $W_{\chi+\Gamma}(\chi)$ such that $\chi_{\min} < \psi < \chi_{\max}$ for any $\psi \in W_{\chi+\Gamma}(\chi)$.*

(iii) *Let χ be a dominant weight. Suppose that there are given weights $\psi = \chi + \lambda$, $\varphi = \chi + \mu$, where $\lambda, \mu \in \Lambda$ such that $\varphi < \psi$. Then $|\mu| \geq |\lambda|$; if $|\mu| = |\lambda|$ there is an element $w \in W_\chi$ such that $\mu = w\lambda$, $\varphi = w\psi$.*

(In (iii) $|\lambda|$ stands for the length of the weight λ with respect to a W-invariant inner product on Λ).

The proof of this Lemma is in Appendix 1.

1.6. *The Harish-Chandra theorem*

Let Z be the centre of the algebra U, Θ the set of characters of Z, i.e. the set of homomorphisms $\theta: Z \to k$. If $\theta \in \Theta$, then $J_\theta = \operatorname{Ker} \theta$ is a maximal ideal in Z.

Denote by $S(\mathfrak{h})$ the symmetric algebra of the space $\mathfrak{h}$. We shall consider elements $P \in S(\mathfrak{h})$ as polynomial functions on $\mathfrak{h}^*$ (if $P \in S(\mathfrak{h})$, $\chi \in \mathfrak{h}^*$ then $P(\chi)$ is the value of P at the point χ).

Denote by η^* the Harish-Chandra homomorphism $\eta^*: Z \to S(\mathfrak{h})$ (cf. [16, 7.4]). It is known that η^* defines an isomorphism of Z onto the subalgebra $S(\mathfrak{h})^W$ consisting of W-invariant elements ([16, 7.4.5]). Denote by η the corresponding map $\eta: \mathfrak{h}^* \to \Theta$, $\eta(\chi)(z) = \eta^*(z)(\chi)$ for $z \in Z$.

THEOREM: *η is an epimorphism and $\eta^{-1}\eta(\chi) = W(\chi)$ for any $\chi \in \mathfrak{h}^*$.*

The proof is in [16].

1.7. *The Kostant theorem*

Consider the adjoint action of the Lie algebra $\mathfrak{g}$ on U (defined via $\operatorname{ad} X(u) = Xu - uX$, for $X \in \mathfrak{g}$, $u \in U$) and denote by U^{ad} this $\mathfrak{g}$-module. For any finite dimensional $\mathfrak{g}$-module L we endow the space $\operatorname{Hom}_{\mathfrak{g}}(L, U^{\mathrm{ad}})$ with the Z-module structure via $(z\varphi)(\ell) = z\varphi(\ell)$, where $z \in Z$, $\varphi \in \operatorname{Hom}_{\mathfrak{g}}(L, U^{\mathrm{ad}})$, $\ell \in L$.

THEOREM (Kostant [18]): $\operatorname{Hom}_{\mathfrak{g}}(L, U^{\mathrm{ad}})$ *is a free Z-module, its rank equals the multiplicity of the zero weight in L.*

1.8. *Categories of U-modules*

In the sequel denote by $\mathcal{M}$ the category of all U-modules. Consider the following complete subcategories of $\mathcal{M}$.

$\mathcal{M}_f$ – the category of finitely generated modules.

$\mathcal{M}_{Zf}$ – the category of Z-finite modules.

Let $\theta \in \Theta$. Put

$$\begin{aligned} \mathcal{M}^n(\theta) &= \{M \in \mathcal{M} \mid J_\theta^n M = 0\} \\ \mathcal{M}(\theta) &= \mathcal{M}^1(\theta) \\ \mathcal{M}^\infty(\theta) &= \{M \in \mathcal{M} \mid \text{if } m \in M \text{ then } J_\theta^n(m) = 0 \\ &\qquad \text{for sufficiently large } n\}. \end{aligned}$$

It is clear that $\mathcal{M}(\theta) \subset \mathcal{M}^2(\theta) \subset \cdots \subset \mathcal{M}^\infty(\theta)$. All these subcategories are closed with respect to subquotients ($\mathcal{M}_f$ is closed because U is Noetherian, see [16]).

Put $U_\theta^n = U/J_\theta^n$, $U_\theta = U_\theta^1$. We shall often identity the category $\mathcal{M}^n(\theta)$ with U_θ^n-mod.

It is easy to check that each module $M \in M_{Zf}$ can be uniquely represented in the form $M = \bigoplus M_\theta$, where $M_\theta \in \mathcal{M}^\infty(\theta)$. Therefore $\mathcal{M}_{Zf}$ is a product of categories $\mathcal{M}^\infty(\theta)$, when θ ranges over Θ. Denote by $\operatorname{Pr}(\theta)$ the corresponding projection functor $\mathcal{M}_{Zf} \to \mathcal{M}^\infty(\theta)$, $M \rightsquigarrow M_\theta$.

1.9. *The category $\mathcal{O}$ and Verma modules*

A module $M \in \mathcal{M}$ is an $\mathcal{O}$-module, if it is finitely generated, $\mathfrak{h}$-diagonisable and $U(\mathfrak{n}^+)$-finite (see 1.1). Denote by $\mathcal{O}$ a complete subcategory in $\mathcal{M}$ consisting of $\mathcal{O}$-modules.

For any weight $\chi \in \mathfrak{h}^*$, define a Verma module $M_\chi \in \mathcal{M}$ via $M_\chi = U/U(I_{\chi-\rho} + \mathfrak{n})$, where $I_{\chi-\rho}$ is the ideal in $U(\mathfrak{h})$ generated by $h - (\chi - \rho)(h)$ for $h \in \mathfrak{h}$. We recall general properties of the category $\mathcal{O}$ and Verma modules (see [16], [19]).

(a) Let $M \in \mathcal{O}$, $\psi \in \mathfrak{h}^*$. Then the subspace $M^\psi \subset M$ consisting of vectors of weight ψ is finite-dimensional. Put $P(M) = \{\psi \in \mathfrak{h}^* \mid M^\psi \neq 0\}$. There are weights $\psi_1, \ldots, \psi_n \in \mathfrak{h}^*$ such that $P(M) \subset \cup_i (\psi_i - \Gamma^+)$. In particular, if $M \neq 0$, then there is a maximal weight χ in $P(M)$ (i.e. a weight χ such that $(\chi + \Gamma^+) \cap P(M) = \chi$).

(b) $\mathcal{O} \subset M_{Zf}$.

(c) $M_\chi \in \mathcal{O}$ for any weight $\psi \in \mathfrak{h}^*$. Besides, $\operatorname{End}_{\mathfrak{g}}(M_\chi) = k$ and the natural homomorphism $Z \to \operatorname{End}_{\mathfrak{g}}(M_\chi)$ coincides with the character $\theta = \eta(\chi)$. In particular, $M_\chi \in \mathcal{M}(\theta) = U_\theta$-mod.

(d) Each module M_χ has the unique simple quotient L_χ; $L_\psi \not\approx L_\chi$ for $\psi \neq \chi$ and the modules L_χ, $\chi \in \mathfrak{h}^*$ exhaust all simple modules in $\mathcal{O}$. Any simple $\mathfrak{g}$-module containing a vector f of the weight χ such that $\mathfrak{n}^+ f = 0$, is isomorphic to $L_{\chi+\rho}$. In particular, if $\lambda \in \Lambda$ is a dominant weight, then $L_{\lambda+\rho}$ is a finite dimensional $\mathfrak{g}$-module with the highest weight λ.

(e) If $M, M' \in \mathcal{O}$, then $\dim \operatorname{Hom}_U(M, M') < \infty$. Each module $M \in \mathcal{O}$ is of finite length; we denote by $JH(M)$ the multiset of factors of Jordan–Hölder series of a module M.

(f) Let V be a finite dimensional $\mathfrak{g}$-module, $\mu_1, \ldots, \mu_n$ a multiset of weights of the space V (i.e. there is a basis $v_1, \ldots, v_n$ of the module V such that the weight of the vector v_i equals μ_i). Then the module $V \otimes M_\chi$ has a filtration such that its quotients coincide with the multiset of modules $M_{\chi+\mu_i}$, $i = 1, 2, \ldots, n$.

1.10. *Involution in the category $\mathcal{O}$*

Fix an anti-automorphism $t: \mathfrak{g} \to \mathfrak{g}$, $X \mapsto {}^tX$, such that $t^2 = id$ and $t(h) = h$ for any $h \in \mathfrak{h}$.

Let $M \in \mathcal{O}$. Define a $\mathfrak{g}$-action on M^* by $(Xm^*)(m) = m^*({}^tXm)$, where $X \in \mathfrak{g}$, $m^* \in M^*$, $m \in M$. Denote by M^τ the $\mathfrak{g}$-submodule in M^* consisting of all $\mathfrak{h}$-finite vectors.

(a) $(M^\tau)^\psi = (M^\psi)^*$ (see 1.8.a)). In particular, $\dim (M^\tau)^\psi < \infty$, $(M^\tau)^\tau = M$ and τ is an exact (contravariant) functor.

(b) If $\chi \in \mathfrak{h}^*$, then $(L_\chi)^\tau \approx L_\chi$. In particular, if $M \in \mathcal{O}$ then $M^\tau \in \mathcal{O}$ and $JH(M^\tau) = JH(M)$.

Statement (b) easily follows from the irreducibility of the module L_χ^τ in view of (a) and $P(L_\chi^\tau) = P(L_\chi)$.

1.11. *Projective objects in the category $\mathcal{O}$ and multiplicities in Verma modules* (see [19])

(a) In the category $\mathcal{O}$, there are enough projective objects (see 1.2).

(b) For any weight χ there is a unique indecomposable projective object P_χ in $\mathcal{O}$ such that $\operatorname{Hom}(P_\chi, L_\chi) \neq 0$. We have

$\dim \mathrm{Hom}(P_\chi, M) = [M : L_\chi]$ for any $M \in \mathcal{O}$ (here $[M : L_\chi]$ is the multiplicity of L_χ in $JH(M)$).

(c) Put $d_{\chi,\psi} = [M_\chi : L_\psi] = \dim \mathrm{Hom}(P_\psi, M_\chi)$. Then $d_{\chi,\psi} > 0$ iff $\mathrm{Hom}(M_\psi, M_\chi) \neq 0$, iff $\chi > \psi$. Besides, $d_{\chi,\chi} = 1$.

(d) Each module P_ψ admits a filtration, the corresponding quotients being isomorphic to the modules M_χ. The multiplicity of M_χ in this filtration equals

$$[P_\psi : M_\chi] = d_{\chi,\psi} = [M_\chi : L_\psi].$$

Let S be an equivalence class of weights (see 1.5). Denote by $\mathcal{O}_S$ the complete subcategory in $\mathcal{O}$ consisting of modules M such that $P(M) \subset \cup_{\chi \in S} (\chi - \rho - \Gamma^+)$ and $M \in \mathcal{M}^\infty(\theta)$, where $\theta = \eta(\chi)$ for $\chi \in S$. From (a), (b), (c) we deduce that the decomposition $\mathfrak{h}^* = \cup S_\alpha$ into equivalence classes corresponds to a decomposition $\mathcal{O} = \bigoplus_\alpha \mathcal{O}_{S_\alpha}$, where $\mathcal{O}_{S_\alpha}$ are no longer decomposable.

If $S \subset \mathfrak{h}^*$ is the union of equivalence classes of weights, put $\mathcal{O}_S = \bigoplus_\alpha \mathcal{O}_{S_\alpha}$, $S_\alpha \subset S$.

1.12. *The Grothendieck group of the category $\mathcal{O}$*

Denote by $K(\mathcal{O})$ the Grothendieck group of the category $\mathcal{O}$. Let $M \in \mathcal{O}$; denote by $[M]$ the corresponding element of $K(\mathcal{O})$. We have $[M] = \Sigma_\psi [M : L_\psi][L_\psi]$.

Put $\delta_\chi = [M_\chi]$. Using 1.11(c), it is easy to check that elements δ_χ, $\chi \in \mathfrak{h}^*$ form a free basis of $K(\mathcal{O})$.

The following properties of δ_χ's hold

(a) If V is a finite dimensional $\mathfrak{g}$-module, $\mu_1, \ldots, \mu_n$ is a multiset of weights of V, then

$$[V \otimes M_\chi] = \sum_{i=1}^{n} \delta_{\chi+\mu_i} \quad \text{(see 1.9(f))}.$$

(b) $[P_\psi] = \Sigma_{\chi > \psi} d_{\chi,\psi} \delta_\chi$ (see 1.11(d)).

(c) Let us define an inner product $\{\cdot, \cdot\}$ on $K(\mathcal{O})$ by formulas $\{\delta_\chi, \delta_\psi\} = 0$ for $\chi \neq \psi$ and $\{\delta_\chi, \delta_\chi\} = 1$. If P is a projective object in $\mathcal{O}$ and $M \in \mathcal{O}$, then $\{[P], [M]\} = \dim \mathrm{Hom}(P, M)$.

In fact, it suffices to check this when $P = P_\psi$ and $M = M_\chi$; but then both the left-hand side and the right-hand side equal $d_{\chi,\psi}$ (see 1.11(b), (c), (d)).

Define a W-action on $K(\mathcal{O})$ by $w\delta_\chi = \delta_{w\chi}$ for $w \in W$, $\chi \in \mathfrak{h}^*$. This action preserves the inner product $\{\cdot, \cdot\}$.

(d) *The Weyl character formula.* If L_λ is a finite dimensional module, i.e. $\lambda-\rho\in\Lambda$ is a dominant weight, then

$$[L_\lambda]=\sum_{w\in W}(\det w)w\delta_\lambda.$$

1.13. *Theorems on annulators*

(a) Let $u\in U$, $u\neq 0$. Then there is a $\lambda\in\Lambda$, such that L_λ is a finite dimensional module and $uL_\lambda\neq 0$.

For any positive n one can choose λ so that $\lambda(h_\gamma)>n$ for any $\gamma\in R^+$ (see [20]).

(b) (The Duflo Theorem [9]). Let χ be a weight, $\theta=\eta(\chi)$, $u\in U_\theta$ (see 1.8). Suppose that $u\neq 0$. Then $uM_\chi\neq 0$.

1.14. Let V be finite dimensional $\mathfrak{g}$-module, $P(V)\subset\Lambda$ the set of its weights. This set is W-invariant.

For any irreducible module V with highest weight λ the weights belonging to the set $W\lambda$ are called extremal weights. If $\mu\in P(V)$, then $|\mu|\leq|\lambda|$. Moreover, if $|\mu|=|\lambda|$, then μ is an extremal weight. Besides, $P(V)\subset\bigcap_{w\in W}w(\lambda-\Gamma^+)$.

Denote by F_V the functor $F_V:\mathcal{M}\to\mathcal{M}$, $M\rightsquigarrow V\otimes M$ (see 2.1), denote by Φ_V the (U,U)-bimodule $V\otimes U$ (see 2.2).

I. PROJECTIVE FUNCTORS

2. The Kostant theorem

2.1. *The functors F_V*

Let V be a finite dimensional $\mathfrak{g}$-module. For any $\mathfrak{g}$-module M, define a $\mathfrak{g}$-module structure on $V\otimes M$ via $X(v\otimes m)=Xv\otimes m+v\otimes Xm$, for $v\in V$, $m\in M$. The correspondence $M\rightsquigarrow V\otimes M$ defines the functor $F_V: U\text{-mod}\to U\text{-mod}$. The main point of this paper is the description of such functors and their relations with Laplace operators (i.e. elements $z\in Z$).

Let us ennumerate the simplest properties of the functors F_V.

(a) The functor F_V is exact and commutes with (infinite) direct sums and products.

(b) To each $\mathfrak{g}$-morphism $\varphi: V_1\to V_2$ there corresponds the morphism of functors $F_{V_1}\to F_{V_2}$. If $V=k$ is the trivial one-dimensional $\mathfrak{g}$-module, then the functor F_V is naturally isomorphic to the identity functor $\mathrm{Id}_{\mathcal{M}}$ in $\mathcal{M}=U$-mod.

(c) The composition of functors $F_{V_1}\circ F_{V_2}$ is naturally isomorphic to the functor $F_{V_1\otimes V_2}$.

(d) If V^* is the $\mathfrak{g}$-module dual to V, then the functor F_{V^*} is both right and left adjoint to the functor F_V (to the morphism $\varphi : M \to V \otimes M$ corresponds the morphism $\varphi' : V^* \otimes M \to N$ defined by the formula $\varphi'(v^* \otimes m) = \Sigma \langle v^*, v_i \rangle n_i$ where $\varphi(m) = \Sigma\, v_i \otimes n_i$. The isomorphism $\mathrm{Hom}(V \otimes M, N) \cong \mathrm{Hom}(M, V^* \otimes N)$ is verified similarly, V being replaced by V^* and V^* by V.

2.2. Define a (U, U)-bimodule structure on the space $\Phi_V = V \otimes U$ by

$$X(v \otimes u) = Xv \otimes u + v \otimes Xu, \quad (v \otimes u)X = v \otimes uX$$

where $X \in \mathfrak{g}$, $v \in V$, $u \in U$. It is easy to see that the functor $h(\Phi_V) : \mathcal{M} \to \mathcal{M}$ ($M \rightsquigarrow \Phi_V \otimes_U M$, see 1.3) is naturally isomorphic to the functor F_V.

For any (U, U)-bimodule Y define the adjoint action, ad, of the Lie algebra $\mathfrak{g}$ on Y by the formula $\operatorname{ad} X(y) = Xy - yX$, $X \in \mathfrak{g}$, $y \in Y$. Denote by Y^{ad} this $\mathfrak{g}$-module.

LEMMA: (i) *There is a natural isomorphism*

$$\mathrm{Hom}_{U^2}(\Phi_V, Y) \cong \mathrm{Hom}_{\mathfrak{g}}(V, Y^{\mathrm{ad}}).$$

(ii) *If $V = V \otimes 1 \subset \Phi_V$, then $UV = VU = \Phi_V$.*

PROOF: (i) To each U^2-module morphism (see 1.3) $\varphi : \Phi_V \to Y$ there corresponds the morphism $\psi : V \to Y^{\mathrm{ad}}$, $\psi(v) = \varphi(v \otimes 1)$. Conversely, from ψ we recover the morphism φ via $\varphi(v \otimes u) = \psi(v)u$. Clearly this correspondence defines the required isomorphism.

(ii) Since $VU = \Phi_V$, it suffices to verify that UV is invariant with respect to the right action of $\mathfrak{g}$. This follows from the formula

$$\begin{aligned} u(v \otimes 1)X &= u(v \otimes X) = u(X(v \otimes 1) - Xv \otimes 1) \\ &= uX(v \otimes 1) - u(Xv \otimes 1), \quad \text{where } X \in \mathfrak{g}, u \in U, v \in V. \end{aligned}$$

2.3. COROLLARY: (i) $F_V(\mathcal{M}_f) \subset \mathcal{M}_f$, $F_V(\mathcal{O}) \subset \mathcal{O}$.

(ii) *In the categories $\mathcal{M}$, $\mathcal{M}_f$ and $\mathcal{O}$, the functors F_V transforms projective objects into projective ones.*

PROOF: (i) Lemma 2.2(ii) implies that the functor F_V is exact and $F_V(U) = \Phi_V \in \mathcal{M}_f$ so $F_V(\mathcal{M}_f) \subset \mathcal{M}_f$.

If $M \in \mathcal{O}$ then $F_V(M) = V \otimes M$ is $\mathfrak{h}$-diagonisable and $U(\mathfrak{n}^+)$-finite because $V \in \mathcal{O}$; also, as $F_V(M) \in \mathcal{M}_f$, we have $F_V(M) \in \mathcal{O}$.

(iii) follows from (i), 2.1(a), (b) and 1.2.

2.4. *Action of Laplace operators on the functors F_V*

Let $M \in \mathcal{M}$. There is a relation between the Z-action on $F_V(M)$ (see 1.6) and the Z-action on M. The Kostant theorem describes this relation. Define a Z^2-action on the functor F_V (i.e. a homomorphism $Z^2 \to \mathrm{End}_{\mathrm{Funct}}(F_V)$). For $z = \Sigma\, a_i^\ell \otimes b_i^r \in Z^2$ and $a_i^\ell, b_i^r \in Z$ define the Z^2-action on $F_V(M)$, when $M \in \mathcal{M}$ by putting $z(v \otimes m) = \Sigma\, a_i(v_i \otimes b_i m)$, where $v \in V$, $m \in M$.

The functor F_V was identified with the (U, U)-bimodule Φ_V (see 1.3 and 2.2). The Z^2-action described above coincides with the natural action of the subalgebra $Z^2 \subset U^2$ on U^2-module Φ_V. Denote by I_V the kernel of this action, $I_V = \{z \in Z^2 \mid z(V \otimes M) = 0 \text{ for any } M \in \mathcal{M}\}$.

For an explicit description of the ideal $I_V \subset Z^2$, let us consider Z^2 as a subalgebra of $S(\mathfrak{h} \oplus \mathfrak{h})$, polynomial functions depending on the pair $\psi, \chi \in \mathfrak{h}^*$.

Consider the embedding

$$\eta^* \otimes \eta^* : Z^2 = Z \otimes Z \to S(\mathfrak{h}) \otimes S(\mathfrak{h}) = S(\mathfrak{h} \oplus \mathfrak{h}).$$

It is clear that the image of Z^2 consists exactly of polynomial functions $Q(\psi, \chi)$ that are W-invariant in each variable (see 1.6).

2.5. THEOREM: *Let* $z \in Z^2$, $Q(z) = (\eta^* \otimes \eta^*)(z) \in S(\mathfrak{h} \otimes \mathfrak{h})$. *Then the following conditions are equivalent*:

(i) $z \in I_V$.

(ii) $Q(\chi + \mu, \chi) \equiv 0$ *for any weight* $\mu \in P(V)$ (*see* 1.14).

This theorem is a refinement of the Kostant theorem [8]. Our proof, in fact, repeats that of Kostant.

PROOF: (a) A weight $\lambda \in \Lambda$ is called n-dominant, $n \in \mathbf{N}$, if $\lambda(h_\gamma) > n$ for any $\gamma \in R^+$. One can choose a positive integer $n = n(V)$, such that for any n-dominant weight λ

$$V \otimes L_\lambda = \bigoplus L_{\lambda + \mu_i},$$

where $\mu_1, \ldots, \mu_k$ is the multiset of weights of V.

In fact, since the functor F_V is exact and preserves the category $\mathcal{O}$, the map $[M] \to [F_V(M)]$ can be extended to a homomorphism $K(\mathcal{O}) \to K(\mathcal{O})$. In $K(\mathcal{O})$ the following equality, due to 1.12, holds:

$$[V\otimes L_\lambda]=\sum_i\sum_{w\in W}\det w\delta_{w\lambda+\mu_i}=\sum_i\sum_{w\in W}\det w\delta_{w(\lambda+\mu_i)}=[\bigoplus L_{\lambda+\mu_i}]$$

if all weights $\lambda+\mu_i-\rho$ are dominant.

Since both modules are finite dimensional, hence completely reducible, they are isomorphic.

(b) Consider the action of an element $z\in Z^2$ on $V\otimes L_\lambda=\bigoplus L_{\lambda+\mu_i}$. The definition of the action of z and 1.9(c), (d) imply that z multiplies each component $L_{\lambda+\mu_i}$ by a scalar equal to $Q(\lambda+\mu_i,\lambda)$.

(c) Let $z\in I_V$. Then $z(V\otimes L_\lambda)=0$, i.e. $Q(\lambda+\mu,\lambda)=0$, for any $\mu\in P(V)$. Thus, the polynomial function $Q_\mu(\chi)=Q(\chi+\mu,\chi)$ vanishes on every n-dominant weight λ. Since such weights are dense in $\mathfrak{h}^*$ in the Zariski topology, we have $Q_\mu(\chi)\equiv 0$.

(d) (ii)$\Rightarrow$(i). Let Q satisfy condition (ii). Then $Q(\lambda+\mu,\lambda)=0$ for any $\mu\in P(V)$, hence $z(V\otimes L_\lambda)=0$ for all n-dominant weights λ.

Let $v_1,\dots,v_n$ be a basis of V. Consider the action of z on $V\otimes U$ and define elements $u_{ij}\in U$ by $z(v_j\otimes 1)=\sum_{i=1}^n v_i\otimes u_{ij}$. Evidently, the action of z on $V\otimes M$ for any $\mathfrak{g}$-module M is defined by

$$z(v_j\otimes m)=\sum v_i\otimes u_{ij}m.$$

Therefore the condition $z(V\otimes L_\lambda)=0$ means that $u_{ij}L_\lambda=0$ for any i,j. Since this is true for any n-dominant weight, 1.13(a) implies that $u_{ij}=0$ for any i and j.

Thus $z(V\otimes M)=0$ for any $\mathfrak{g}$-module M, i.e. $z\in I_V$.

2.6. Corollary: (i) *Z^2/I_V is finitely generated Z^r-module.*
(ii) *$F_V(\mathcal{M}_{Zf})\subset\mathcal{M}_{Zf}$ (see* 1.8).

Proof: (i) Put $A=S(\mathfrak{h})$, $B=S(\mathfrak{h})^W$,

$$A^2=A\otimes A=S(\mathfrak{h}\otimes\mathfrak{h}),\quad B^2=B\otimes B\subset A^2,$$
$$J=\{Q\in A^2\mid Q(\chi+\mu,\chi)\equiv 0 \text{ for any } \mu\in P(V)\}.$$

J is an ideal in A^2, $J_V=J\cap B^2$ is an ideal in B^2. It is necessary to verify that B^2/J_V is a finitely generated B^r-module.

Since W is a finite group, A is a finitely generated B-module. Define a B^r-module homomorphism $i:B^2/J_V\to\bigoplus_{\mu\in P(V)}A$ via $i(Q)=i_\mu(Q)$, where $i_\mu(Q)(\chi)=Q(\chi+\mu,\chi)$. It is easy to check that i is well-defined and is an embedding. Since $\bigoplus_{\mu\in P(V)}A$ is a finitely generated B-module and B is Nöetherian, the algebra B^2/J_V is finitely generated over B^r.

(ii) If J is an ideal in Z of finite codimension then by (i) the algebra $Z^2/(J_V+Z^\ell \otimes J)$ is finite dimensional, hence the ideal $J'=\{z\in Z \mid z\otimes 1 \in (J_V+Z^\ell\otimes J)\}$ has finite codimension in Z. If M is a $\mathfrak{g}$-module such that $JM=0$, it is clear that $J'(V\otimes M)=0$, i.e. $V\otimes M\in \mathcal{M}_{Zf}$. Since the functor F_V commutes with inductive limits, $F_V(\mathcal{M}_{Zf})\subset \mathcal{M}_{Zf}$.

Using Theorem 2.5, it is not difficult to prove the following statement that strengthens Corollary 2.6(ii).

Let χ be a weight, $\theta=\eta(\chi)$, $\theta_\mu=\eta(\chi+\mu)$. Then $F_V(\mathcal{M}^\infty(\theta))\subset \Pi_{\mu\in P(V)}\mathcal{M}^\infty(\theta_\mu)$ (see 1.8).

We omit the proof of this statement here, since as in section 3 we shall obtain stronger results.

3. Decomposition of the functors F_V

3.1. *Projective functors*

Let V be a finite dimensional $\mathfrak{g}$-module. Then the functor F_V preserves the subcategory $\mathcal{M}_{Zf}\subset\mathcal{M}$ consisting of Z-finite modules (See 2.6). Denote by $F_V|_{\mathcal{M}_{Zf}}$ the restriction of F_V to $\mathcal{M}_{Zf}$.

The functor $F_V|_{\mathcal{M}_{Zf}}$ has a lot of direct summands. An example is an identity functor $\mathrm{Id}:\mathcal{M}\to\mathcal{M}$ (corresponding to $V=k$) which is indecomposable, while its restriction to $\mathcal{M}_{Zf}$ decomposes into the direct sum $\bigoplus_{\theta\in\Theta}\mathrm{Pr}(\theta)$ (see 1.8). The aim of this section is to give sufficiently explicit description of indecomposable components of functors $F_V|_{\mathcal{M}_{Zf}}$.

DEFINITION: *The functor $F:\mathcal{M}_{Zf}\to\mathcal{M}_{Zf}$ is called projective if it is isomorphic to a direct summand of the functor $F_V|_{\mathcal{M}_{Zf}}$ for a finite dimensional $\mathfrak{g}$-module V.*

The meaning of the term projective will be clear from section 5.

We show that each projective functor decomposes into direct sum of indecomposable functors and describe all indecomposable projective functors (Theorem 3.3).

3.2. Let us list the simplest properties of projective functors.

LEMMA: *Let F, G be projective functors. Then*

(i) *The functor F is exact and preserves direct sums and products. Direct summands of F are projective.*

(ii) *Functors $F\oplus G$ and $F\circ G$ are projective.*

(iii) *We have a natural isomorphism of functors $F=$*

$\bigoplus_{\theta,\theta'} \Pr(\theta') \circ F \circ \Pr(\theta)$ *where* $\theta, \theta' \in \Theta$ *(see* 1.8) *and all the functors* $\Pr(\theta') \circ F \circ \Pr(\theta)$ *are projective.*

(iv) *The functor* F *preserves* $\mathcal{O}$ *and transforms projective objects of* $\mathcal{O}$ *into projective ones.*

(v) *There is a projective functor* F' *left adjoint to the functor* F. *(Similarly there is a projective functor* F'' *right adjoint to the functor* F).

Proof follows from 2.1, 2.2, 1.8, and 1.3(c).

3.3. *Basic theorems on projective functors*

Section 3 deals with the classification of projective functors. In subsections 3.3–3.5 we give formulations of the main theorems. The other subsections deal with the proofs of these theorems.

Put $\Xi^0 = \{(\psi, \chi) \mid \psi, \chi \in \mathfrak{h}^*, \psi - \chi \in \Lambda\}$. Define a W-action on Ξ^0 via $w(\psi, \chi) = (w\psi, w\chi)$ and denote by Ξ the quotient set with respect to this action. Each element $\xi \in \Xi$ can be written as a pair (ψ, χ). The pair (ψ, χ) is called a proper notation for ξ if χ is a dominant weight and $\psi < W_\chi(\psi)$, i.e. $\psi < w\psi$ for any $w \in W_\chi$. It is clear that each element $\xi \in \Xi$ can be written properly.

Put $\eta^r(\psi, \chi) = \eta(\chi)$ for any $(\psi, \chi) \in \Xi^0$. Evidently η^r is the map $\eta^r : \Xi \to \Theta$.

THEOREM: (i) *Each projective functor decomposes into a direct sum of indecomposable projective functors.*

(ii) *For each element* $\xi \in \Xi$ *there is indecomposable projective functor* F_ξ, *a unique up to isomorphism, such that*

(a) $F_\xi(M_\varphi) = 0$ if $\eta^r(\xi) \neq \eta(\varphi)$, $\varphi \in \mathfrak{h}^*$.

(b) *If* ξ *is written properly,* $\xi = (\psi, \chi)$, *then* $F_\xi(M_\chi) = P_\psi$ *(see* 1.11).

The map $\xi \rightsquigarrow F_\xi$ *defines a bijection of* Ξ *with the set of isomorphism classes of indecomposable projective functors.*

3.4. Assign to each projective functor F an endomorphism F^K of the group $K(\mathcal{O})$ (see 1.12) via $F^K([M]) = [FM]$, $M \in \mathcal{O}$. The F^K are well-defined, because $F(\mathcal{O}) \subset \mathcal{O}$ and F is exact (see 3.1).

For example, $(F_V)^K(\delta_\chi) = \Sigma\, \delta_{\chi+\mu_i}$, where $\mu_1, \ldots, \mu_n$ is the multiset of weights of V (see 1.9f); $\Pr(\theta)^K(\delta_\chi) = \delta_\chi$ for $\eta(\chi) = \theta$ and $(\Pr(\theta))^K(\delta_\chi) = 0$ for $\eta(\chi) \neq \theta$ (see 1.9(c)).

THEOREM: *Let* F, G *be projective functors. Then*

(i) *If* $F^K = G^K$, *then* F *is isomorphic to* G.

(ii) $(F \oplus G)^K = F^K \oplus G^K$, $(F \circ G)^K = F^K \circ G^K$.

(iii) *If the functor F is left adjoint to the functor G, then the operators F^K and G^K are conjugate with respect to the inner product $\{\cdot,\cdot\}$ in $K(\mathcal{O})$ (see* 1.12).

(iv) *The operator F^K commutes with W-action (see* 1.12).

If $\xi \in \Xi$, then by this theorem we can explicitly define the operator F_ξ^K corresponding to the indecomposable functor F_ξ. Namely, suppose $\xi = (\psi, \chi)$ is written properly. Then $F_\xi^K(\delta_\varphi) = 0$ if $\varphi \notin W(\chi)$ and $F_\xi^K(\delta_{w\chi}) = \Sigma_{\varphi > \psi}\, d_{\varphi,\psi}\delta_{w\varphi}$ (see 1.12).

Thus, all constants $d_{\varphi,\psi}$ being known, we can define the decomposition of the functor F into indecomposable functors, using F^K. In particular, we can write out the multiplication table (in terms of $d_{\varphi,\psi}$):

$$F_\xi \circ F_{\xi'} = \sum C_{\xi,\xi'}^{\xi''} F_{\xi''}.$$

3.5. Choose a character $\theta \in \Theta$. To prove Theorems 3.3, 3.4, we consider the restriction of projective functors to the subcategory $\mathcal{M}(\theta)$ consisting of the modules M which satisfy $J_\theta M = 0$ (see 1.8). For a projective functor F, denote by $F(\theta)$ its restriction to the subcategory $\mathcal{M}(\theta)$.

DEFINITION: The functor $F: \mathcal{M}(\theta) \to \mathcal{M}$ is called a projective θ-functor, if it is isomorphic to a direct summand of the functor $F_V(\theta)$ for a finite dimensional $\mathfrak{g}$-module V.

As we shall see below, each projective θ-functor is of the form $F(\theta)$ for some projective functor F.

The crucial point in the description of projective functors is the following theorem, describing the space $\mathrm{Hom}(F, G)$ for projective θ-functors F, G (here $\mathrm{Hom}(F, G)$ stands for the space of morphisms of the functor F into the functor G).

THEOREM: *Let F, G be projective θ-functors, $\chi \in \eta^{-1}(\theta)$ a weight. Define the homomorphism*

$$i_\chi: \mathrm{Hom}(F, G) \to \mathrm{Hom}(FM_\chi, GM_\chi)$$

via $i_\chi(\varphi) = \varphi_{M_\chi}$ (here $\varphi_{M_\chi}: FM_\chi \to GM_\chi$ is the value of the functor morphism $\varphi: F \to G$ on the module $M_\chi \in \mathcal{M}(\theta)$). Then i_χ is a monomorphism. If χ is a dominant weight, then i_χ is an isomorphism.

This theorem reduces the study of projective functors to the study of projective objects of category $\mathcal{O}$.

Proofs of the theorems on projective functors

The remainder of this section deals with the proof of Theorems 3.3, 3.4, 3.5. In subsection 3.6 we prove Theorem 3.5; in subsection 3.7 we discuss how to extend projective θ-functors to projective functors. In subsections 3.8–3.10 we prove Theorems 3.4 and 3.3.

3.6. Proof of the Theorem 3.5: (a) Let F be projective θ-functor. By definition, there is a decomposition $F_V(\theta) = F \oplus F^{\perp}$, where V is a finite dimensional $\mathfrak{g}$-module. Hence there is the decomposition

$$\mathrm{Hom}(F_V(\theta), G) = \mathrm{Hom}(F, G) \oplus \mathrm{Hom}(F^{\perp}, G).$$

This decomposition implies that it suffices to prove the theorem only for $F = F_V(\theta)$. Similarly, we may assume that $G = F_L(\theta)$, where L is a finite dimensional $\mathfrak{g}$-module.

(b) We prove that i_χ is injective. Choose a basis $v_1, \ldots, v_n$ in V and $\ell_1, \ldots, \ell_m$ in L. Consider a morphism $\varphi_{U_\theta} : F(U_\theta) \to G(U_\theta)$ for any morphism $\varphi \in \mathrm{Hom}(F, G)$ (i.e. $\varphi_{U_\theta} : V \otimes U_\theta \to L \otimes U_\theta$), and define elements $u_{ij} \in U_\theta$ by formula $\varphi_{U_\theta}(v_j \otimes 1) = \Sigma\, \ell_i \otimes u_{ij}$, $i = 1, 2, \ldots, m$; $j = 1, 2, \ldots, n$.

For $M \in \mathcal{M}(\theta)$, $m \in M$ the map $u \rightsquigarrow um$ is a $\mathfrak{g}$-module morphism $U_\theta \to M$. This immediately implies that the morphism $\varphi_M : V \otimes M \to L \otimes M$ is given by the formula $\varphi_M(v_j \otimes m) = \Sigma\, \ell_i \otimes u_{ij}m$. Therefore, if $i_\chi(\varphi) = \varphi_{M_\chi} = 0$, we have $u_{ij}M_\chi = 0$ for any i, j. It follows from 1.13 that $i_\chi(\varphi) = 0$ implies $u_{ij} = 0$, i.e. $\varphi = 0$. Thus i_χ is injective.

(c) Prove that if χ is a dominant weight, then $\dim \mathrm{Hom}(F, G) \geq \dim \mathrm{Hom}_{\mathcal{O}}(FM_\chi, GM_\chi)$. For this, let us estimate both dimensions separately.

Consider the (U, U_θ)-bimodules $\Phi_V/J_\theta^r = V \otimes U_\theta$ and $\Phi_L/J_\theta^r = L \otimes U_\theta$ (see 2.2). Since $\mathcal{M}(\theta) = U_\theta$-mod 1.3 implies that

$$\mathrm{Hom}(F, G) = \mathrm{Hom}_{(U, U_\theta)}(\Phi_V/J_\theta^r, \Phi_L/J_\theta^r)$$

$$= \mathrm{Hom}_{(U, U)}(\Phi_V/J_\theta^r, \Phi_L/J_\theta^r) = \mathrm{Hom}_{(U, U)}(\Phi_V, \Phi_L/J_\theta^r).$$

By Lemma 2.2(i) the latter space coincides with

$$\mathrm{Hom}_{\mathfrak{g}}(V, (\Phi_L/J_\theta^r)^{\mathrm{ad}}) = \mathrm{Hom}_{\mathfrak{g}}(V, (L \otimes U_\theta)^{\mathrm{ad}}) = \mathrm{Hom}_{\mathfrak{g}}(V \otimes L^*, U_\theta^{\mathrm{ad}}).$$

Since the representation of $\mathfrak{g}$ in U^{ad} is completely reducible, this space coincides with $(\mathrm{Hom}_{\mathfrak{g}}(V\otimes L^*, U^{\mathrm{ad}}))/J_\theta$. The dimension of this space is equal to $\dim(V\otimes L^*)^0$, i.e. to the multiplicity of the weight 0 in $V\otimes L^*$ (see 1.7). Thus

$$\dim \mathrm{Hom}(F, G) = \dim(V\otimes L^*)^0.$$

Let us estimate the dimension of the second space. We have

$$\begin{aligned}\mathrm{Hom}_{\mathcal{O}}(FM_\chi, GM_\chi) &= \mathrm{Hom}_{\mathcal{O}}(V\otimes M_\chi, L\otimes M_\chi)\\ &= \mathrm{Hom}_{\mathcal{O}}(M_\chi, (V^*\otimes L)\otimes M_\chi).\end{aligned}$$

The space $(V^*\otimes L)\otimes M_\chi$ admits a filtration, the corresponding quotients being isomorphic to $M_{\chi+\lambda_i}$, where $\lambda_1, \dots, \lambda_n$ is a multiset of weights of $V^*\otimes L$. Since $\mathrm{Hom}_{\mathcal{O}}(M_\chi, M_\psi) = 0$ for a dominant weight χ with $\psi \neq \chi$ (see 1.11(c)) and $\mathrm{Hom}_{\mathcal{O}}(M_\chi, M_\chi) = k$, we see that $\dim \mathrm{Hom}_{\mathcal{O}}(FM_\chi, GM_\chi)$ does not exceed the multiplicity of the weight 0 in $V^*\otimes L$. Thus

$$\begin{aligned}\dim \mathrm{Hom}_{\mathcal{O}}(FM_\chi, GM_\chi) &\le \dim(V^*\otimes L)^0 = \dim(V\otimes L^*)^0\\ &= \dim \mathrm{Hom}(F, G).\end{aligned}$$

(d) Theorem 3.5 follows immediately from (b) and (c).

REMARK: It can be proved similarly that if χ is an antidominant weight (i.e. χ is minimal with respect to the ordering $<$), then i_χ is an isomorphism. In fact,

$$\mathrm{Hom}(V\otimes M_\chi, L\otimes M_\chi) = \mathrm{Hom}((L^*\otimes V)\otimes M_\chi, M_\chi)$$

and since $\mathrm{Hom}_{\mathcal{O}}(M_\psi, M_\chi) = 0$, when $\psi \neq \chi$ (see 1.11(c)), we see that $\dim \mathrm{Hom}_{\mathcal{O}}(V\otimes M_\chi, L\otimes M_\chi)$ does not exceed the multiplicity of the weight 0 in $L^*\otimes V$.

We do not know if the corresponding statement is true for an arbitrary weight χ. It seems not.

3.7. Let F be a projective functor. We want to show that its restriction to the subcategory $\mathcal{M}^\infty(\theta)$ is thoroughly defined by its restriction to the subcategory $\mathcal{M}(\theta)$.

Denote the restriction of the functor F onto $\mathcal{M}^n(\theta)$, $n = 1, 2, \dots, \infty$ by $F^n(\theta)$. Note that all categories $\mathcal{M}_{Zf}$, $\mathcal{M}^n(\theta)$ are Z-categories, so that

Z acts on all functors F, $F^n(\theta)$ (to the morphism $z: M \to M$, $z \in Z$ corresponds the morphism $F(z): FM \to FM$). In particular, if F, G are projective functors, all spaces $\mathrm{Hom}(F^n(\theta), G^n(\theta))$ are Z-modules.

PROPOSITION: *Let F, G be projective functors. Then each morphism $\varphi: F(\theta) \to G(\theta)$ could be continued up to a morphism $\hat{\varphi}: F^\infty(\theta) \to G^\infty(\theta)$. If φ is an isomorphism, then so is $\hat{\varphi}$. If $F = G$ and φ is idempotent, then for $\hat{\varphi}$ we can choose an idempotent.*

PROOF OF THE PROPOSITION: Put $H^n = \mathrm{Hom}(F^n(\theta), G^n(\theta))$, $n = 1, 2, \ldots, \infty$. Denote by $r_{nm}: H^n \to H^m$ (for $m \leq n$) the natural map that assigns to each functor morphism its restriction to the subcategory $\mathcal{M}^m(\theta) \subset \mathcal{M}^n(\theta)$.

(a) $H^\infty = \varprojlim H^n$. This follows from the fact that the functor F commutes with inductive limits and each module $M \in \mathcal{M}^\infty(\theta)$ is the inductive limit of the modules $M^n \in \mathcal{M}^n(\theta)$.

(b) Prove that $H^n = H^\infty / J_\theta^n$. As in 3.6, step (a) we assume that $F = F_V$, $G = F_L$. Further, as in 3.6, step (c) we prove that $H^n = (\mathrm{Hom}_{\mathfrak{g}}(L^* \otimes V, U^{\mathrm{ad}}))/J_\theta^n$. This and (a) imply the desired statement.

(c) Thus, we have shown that $H^\infty = \varprojlim H^\infty / J_\theta^n$, i.e. the space H^∞ is complete in the J_θ-adic topology.

Since $\mathrm{Hom}(F(\theta), G(\theta)) = H^1 = H^\infty / J_\theta$, each element $\varphi \in H^1$ can be lifted to an element $\hat{\varphi} \in H^\infty$.

Let φ be isomorphism, $\psi = \varphi^{-1}$. In order to prove that $\hat{\varphi}$ is an isomorphism, it suffices to verify that morphisms $\hat{\varphi}\hat{\psi}$ and $\hat{\psi}\hat{\varphi}$ are invertible. Therefore, we may assume that $F = G$ and $\varphi = 1$. But $\hat{\varphi} = 1 - \alpha$, where $\alpha \in J_\theta H^\infty$, so as $\hat{\varphi}^{-1} = 1 + \alpha + \alpha^2 + \cdots$ and this series converges because of completeness of H^∞ in J_θ-adic topology.

If $F = G$ and $\varphi \in \mathrm{End}\, F(\theta)$ is an idempotent, then $\hat{\varphi}$ can be chosen as idempotent, too, due to [13, III, 2.10].

3.8. The following corollary follows from Theorem 3.5 and Proposition 3.7.

COROLLARY: *Let F, G be projective functors, χ be a dominant weight, $\theta = \eta(\chi)$. Then each isomorphism $FM_\chi \cong GM_\chi$ can be continued up to an isomorphism of functors $F^\infty(\theta) \to G^\infty(\theta)$ and each decomposition $FM_\chi = \bigoplus M_i$ can be continued up to a decomposition of functors $F^\infty(\theta) = \bigoplus F_i$ with $F_i M_\chi = M_i$.*

Let F be projective functor. Then $F = \bigoplus_{\theta \in \Theta} F \circ \mathrm{Pr}(\theta)$. Each of the functors $F \circ \mathrm{Pr}(\theta)$ is defined by its restriction to the subcategory

$\mathcal{M}^{\infty}(\theta)$. Hence, this corollary implies that $F \circ \Pr(\theta)$ decomposes into a direct sum of a finite number of indecomposable projective functors. This proves Theorem 3.3(i).

If F is indecomposable projective functor, then $F = F \circ \Pr(\theta)$ for a $\theta \in \Theta$. So, if $\eta(\chi) \neq \theta$, then $FM_\chi = 0$. If $\eta(\chi) = \theta$ and χ is a dominant weight, then by Corollary 3.8 and 3.2(iv), we see that FM_χ is indecomposable projective object of $\mathcal{O}$, i.e. $FM_\chi = P_\psi$ for a $\psi \in \mathfrak{h}^*$.

3.9. Proof of Theorem 3.4: (a) Statement (ii) is evident. To prove (i), it suffices to show that $F^{\infty}(\theta) = G^{\infty}(\theta)$ for any $\theta \in \Theta$. By 3.8 this is equivalent to $FM_\chi \approx GM_\chi$, where $\chi \in \eta^{-1}(\theta)$ is a dominant weight. Since $F^K = G^K$, we have $[FM_\chi] = [GM_\chi]$. But FM_χ and GM_χ are projective objects in $\mathcal{O}$ and by 1.12(b) their isomorphism class is defined by their image in $K(\mathcal{O})$.

(b) Let us prove (iii). We must show that $\{F^K x, y\} = \{x, G^K y\}$, $x, y \in K(\mathcal{O})$. By 1.12(b), we can assume that $x = [P]$, where P is projective object in $\mathcal{O}$ and $y = [M]$, $M \in \mathcal{O}$. Since FP is also a projective object, we have $\{F^K x, y\} = \{[FP], [M]\} = \dim \operatorname{Hom}(FP, M)$ and $\{x, G^K y\} = \{[P], [GM]\} = \dim \operatorname{Hom}(P, GM)$ (see 1.12(c)).

The desired equality follows from the fact that the functor F is left adjoint to the functor G.

(c) To prove (iv), let us show that the operator $F^K \in \operatorname{End} K(\mathcal{O})$ commutes with the W-action. The decomposition $F = \bigoplus \Pr(\theta') \circ F \circ \Pr(\theta)$ allows us to take

$$(*) \qquad F = \Pr(\theta') \circ F \circ \Pr(\theta),$$

where $\theta, \theta' \in \Theta$ are fixed characters.

We carry out the proof in 2 steps.

(d) Let $\chi \in \eta^{-1}(\theta)$ be dominant weight. Put $S = \eta^{-1}(\theta') \cap (\chi + \Lambda)$. We say that the weight χ dominates the character θ', if $\chi - \psi$ is a dominant weight for any $\psi \in S$. Let us prove then, that the operator F^K commutes with W-action. (We assume that condition (*) holds).

Put $G_V = \Pr(\theta') \circ F_V \circ \Pr(\theta)$. We prove that the operator F^K is a linear combination of the operators G_V^K for some finite dimensional $\mathfrak{g}$-modules V. Due to explicit formulas (see 3.4), G_V^K commutes with W. This implies that F^K commutes with W.

It suffices to consider the case of indecomposable F. Then $FM_\chi = P_\psi$, where $\psi \in S$ (see 3.8).

Choose $h \in \mathfrak{h}$, so that $\gamma(h) \in \mathbf{Z}^+$ for any $\gamma \in R^+$ (e.g. $h = \Sigma\, h_\gamma$, $\gamma \in R^+$) and prove by induction in $i(\psi) = (\chi - \psi)(h)$.

Let $\lambda = \chi - \psi$, L be the indecomposable module with highest weight λ and L^* its dual. Let us decompose the functor G_{L^*} into indecomposable ones. By 3.8 such a decomposition is of the form $G_{L^*} = \bigoplus F_\varphi$, where $F_\varphi M_\chi = P_\varphi$ and all φ's belong to S. On the other hand, $[G_{L^*}(M_\chi)] = \Sigma\, a_\varphi \delta_\varphi$, where $a_\psi = 1$ and $a_\varphi > 0$ only when

$$\varphi \in \chi + P(L^*) = \chi - P(L) = \psi + (\lambda - P(L)) \subset \psi + \Gamma^+.$$

Hence, the decomposition of G_{L^*} is of the form $F \bigoplus (\bigoplus F_{\varphi_i})$, where $\varphi_i \in S$, $\varphi_i \neq \psi$ and $\varphi_i \in \psi + \Gamma^+$, so that $i(\varphi_i) > i(\psi)$. Induction on $i(\psi)$ immediately implies that the operator F^K is a linear combination of operators G_V^K.

At the same time, we have proved that in the considered case there exists an indecomposable projective functor F such that $FM_\chi = P_\psi$.

(e) Now consider the general case. Let χ be a maximal weight in $\eta^{-1}(\theta)$. Choose an integer n sufficiently large and put $\varphi = \chi + n\rho \in \mathfrak{h}^*$, $\theta'' = \eta(\varphi)$ (see 1.5). For n sufficiently large, φ dominates θ and θ'.

By (d), there is an indecomposable projective functor G satisfying $GM_\varphi \approx P_\chi = M_\chi$, so that $G^K(\delta_\varphi) = \delta_\chi$.

Let $F = \Pr(\theta') \circ F \circ \Pr(\theta)$ be a projective functor. It follows from (iv) that operators G^K and $(F \circ G)^K = F^K \circ G^K$ commute with W-action. To prove that F^K commutes with W-action it suffices to verify that $wF^K(\delta_\chi) = F^K(\delta_{w\chi})$ for any $w \in W$. But this follows from

$$wF^K(\delta_\chi) = wF^K \circ G^K(\delta_\varphi) = F^K \circ G^K(w\delta_\varphi) = F^K \circ wG^K(\delta_\varphi) = F^K(w\delta_\chi)$$

since $F^K \circ G^K$ and G^K commute with W.

Theorem 3.4 is entirely proved.

3.10. It remains to prove Theorem 3.3(ii).

(a) For a projective functor F, put $a_F(\psi, \chi) = \{\delta_\psi, F^K\delta_\chi\}$, where $\psi, \chi \in \mathfrak{h}^*$. If χ is a dominant weight, then FM_χ is a projective module so that $a_F(\psi, \chi) \geq 0$ for any ψ (see 1.12(c)). Since operator F^K commutes with W-action we have that $a_F(\psi, \chi) \geq 0$ for any ψ, χ. Put $S(F) = \{(\psi, \chi) \mid a_F(\psi, \chi) > 0\}$, $S^m(F) = \{(\psi, \chi) \in S(F) \mid |\chi - \psi| \text{ is maximal in } S(F)\}$. The non-negativity of the coefficients $a_F(\psi, \chi)$ implies that if $F = \bigoplus F_\alpha$, then $S(F) = \cup\, S(F_\alpha)$, hence $S^m(F) \subset \cup\, S^m(F_\alpha)$. In particular, since $S(F_V) \subset \Xi^0$, we have $S(F) \subset \Xi^0$. The operator F^K commutes with W-action, therefore $S(F)$ and $S^m(F)$ are W-invariant.

(b) Let F be an indecomposable functor. Then $S^m(F)/W$ consists only of one element.

In fact, $F = F \circ \mathrm{Pr}(\theta)$ for a $\theta \in \Theta$. If $\chi \in \eta^{-1}(\theta)$ is a dominant weight, then by 3.8 $FM_\chi = P_\psi$ for some weight ψ. We prove that $S^m(F) = W(\psi, \chi)$. It suffices to verify that $(\varphi, \chi) \in S^m(F)$ implies $\varphi \in W_\chi(\psi)$. Since $a_F(\varphi, \chi) > 0$ only if $\varphi < \psi$ (see 1.12(b)), this statement follows from Lemma 1.5(c).

(c) We have assigned to each indecomposable projective functor F an element $\xi \in S^m(F)/W \subset \Xi$. It follows from (b) that if $\xi = (\psi, \chi)$ is written properly, then $FM_\chi \cong P_\psi$. It only remains to show that each element $\xi \in \Xi$ is obtained in this way.

Let $\xi = W(\psi, \chi)$ and let V be the finite dimensional $\mathfrak{g}$-module with an extremal weight $\psi - \chi$. It is clear that (ψ, χ) belongs to $S^m(F_V)$. The functor F_V being decomposed into a sum of indecomposable summands $F_V = \bigoplus F_\alpha$ we see (cf. (a)) that $(\psi, \chi) \in S^m(F_\alpha)$ for some α.

Q.E.D.

Theorem 3.3 is entirely proved.

4. Consequences of the properties of projective functors

The equivalence of categories $\mathcal{M}^\infty(\theta)$

4.1. Let $\theta, \theta' \in \Theta$ be characters of Z and let V be a finite dimensional $\mathfrak{g}$-module. Denote by $F_{\theta,V,\theta'}$ the functor $\mathrm{Pr}(\theta') \circ F_V \circ \mathrm{Pr}(\theta)$, which we will consider as a functor from $\mathcal{M}^\infty(\theta)$ into $\mathcal{M}^\infty(\theta')$.

THEOREM: *Let $\theta, \theta' \in \Theta$. Suppose $\chi \in \eta^{-1}(\theta)$, $\psi \in \eta^{-1}(\theta')$ satisfy*

(a) $\psi - \chi \in \Lambda$.

(b) *χ and ψ are dominant.*

(c) $W_\chi = W_\psi$.

Then the categories $\mathcal{M}^\infty(\theta)$ and $\mathcal{M}^\infty(\theta')$ are equivalent. Equivalence is defined by functors $F_{\theta',V,\theta}: \mathcal{M}^\infty(\theta) \to \mathcal{M}^\infty(\theta')$ and $F_{\theta,V^*,\theta'}: \mathcal{M}^\infty(\theta') \to \mathcal{M}^\infty(\theta)$, where V, V^* are finite dimensional $\mathfrak{g}$-modules with extremal weights $\lambda = \psi - \chi$ and $-\lambda$ respectively.

PROOF: By Theorem 3.3, there are indecomposable projective functors F_1 and F_2 such that $F_1M_\chi = P_\psi$, $F_2M_\psi = P_\chi$. Since ψ and χ are dominant, $P_\psi = M_\psi$ and $P_\chi = M_\chi$. Therefore $F_2F_1M_\chi \cong M_\chi$, and Theorem 3.3 implies that the functor F_2F_1 is isomorphic to $\mathrm{Pr}(\theta)$. Similarly $F_1F_2 \approx \mathrm{Pr}(\theta')$. By restricting F_1 and F_2 to subcategories $\mathcal{M}^\infty(\theta)$ and $\mathcal{M}^\infty(\theta')$, we obtain the desired equivalency of categories.

It remains only to verify that $F_1 = F_{\theta',V,\theta}$, i.e. that $F_{\theta',V,\theta}M_\chi = M_\psi$. It is equivalent to inequality $\eta(\chi + \mu) \neq \theta'$ for any $\mu \in P(V)$, $\mu \neq \lambda$ which easily follows from Lemma 1.5(b), (c).

4.2. REMARKS: (a) Let $\mathscr{H}$ be a complete subcategory of $\mathscr{M}$ preserved by all the F_V. For example, either $\mathscr{H}$ is category of Harish–Chandra modules with respect to a reductive subalgebra $\mathfrak{g}_0 \subset \mathfrak{g}$ (see [16]) or $\mathscr{H} = \mathscr{O}$. Put $\mathscr{H}_\theta = \mathscr{H} \cap \mathscr{M}^\infty(\theta)$. Then, under the assumptions of Theorem 4.1, the functors $F_{\theta',V,\theta}$ and $F_{\theta,V^*,\theta'}$ define the equivalence of the categories $\mathscr{H}_\theta$ and $\mathscr{H}_{\theta'}$.

(b) Suppose that condition (c) of the Theorem is replaced by the weaker condition.

(c) $W_\psi \subset W_\chi$.

Then the functor $F_{\theta,V^*,\theta'} \circ F_{\theta',V,\theta}$ is isomorphic to a direct sum of $|W_\chi/W_\psi|$ copies of the identity functor

$$\mathrm{Id}: \mathscr{M}^\infty(\theta) \to \mathscr{M}^\infty(\theta).$$

It suffices to verify this statement for the module M_χ, which is straightforward, if we use 1.5 and 3.4.

(c) The statements of remarks (a) and (b) refine results of Zuckerman [1] on the behaviour of the functors $F_{\theta',V,\theta}$ on irreducible G. Harish–Chandra modules. (D. Vogan [22] and T. Enright [24] have also refined these results, see also [23].)

Two-sided ideals and submodules of Verma modules

4.3. Let $\theta \in \Theta$, χ be a dominant weight in $\eta^{-1}(\theta)$. Denote by Ω_θ the lattice of two-sided ideals in the algebra U_θ, U_θ included. Evidently, this lattice is equivalent to the lattice of two-sided ideals in the algebra U containing the ideal $J_\theta \subset Z$. Denote by Ω_χ the sub-module lattice of M_χ.

Define a map $\nu: \Omega_\theta \to \Omega_\chi$ by assigning to each two-sided ideal $J \subset U_\theta$ the submodule $\nu(J) = J(M_\chi) \subset M_\chi$.

It is clear that $\nu(J_1 + J_2) = \nu(J_1) + \nu(J_2)$, $\nu(J_1 J_2) = J_1 \nu(J_2)$, it follows from $J_1 \subset J_2$ that $\nu(J_1) \subset \nu(J_2)$.

THEOREM: *Let χ be dominant weight, $\theta = \eta(\chi)$,*

(i) *if χ is a regular weight (i.e. $W_\chi = \{e\}$), then the map $\nu: \Omega_\theta \to \Omega_\chi$ is a lattice isomorphism.*

(ii) *In general, ν is an embedding.*

Denote by $\mathscr{P}$ the class of modules isomorphic to direct sums of P_ψ, for $\psi < \chi$ and $\psi < W_\chi(\psi)$. Then the image of ν consists exactly of all $\mathscr{P}$-generated submodules in M_χ (see 1.2). (A. Joseph [26] has obtained this theorem by different methods.)

PROOF: (i) is essentially a special case of (ii). In fact, (ii) implies that if χ is a regular weight, then ν is a bijection. Now $a \subset b$ iff $a+b=b$ and $a \cap b = \sup\{x \mid x \subset a, x \subset b\}$. Since ν preserves sums, it preserves inclusions and intersections.

Prove statement (ii). Let (φ, F) be a pair such that $F : \mathcal{M}(\theta) \to \mathcal{M}$ is a projective θ-functor, $\varphi : F \to \mathrm{Id}(\theta)$ is a functor morphism. For such a pair put

$$J(\varphi, F) = \mathrm{Im}(\varphi_{U_\theta} : FU_\theta \to U_\theta),$$
$$M(\varphi, F) = \mathrm{Im}(\varphi_{M_\chi} : FM_\chi \to M_\chi).$$

(a) *$J(\varphi, F)$ is a two-sided ideal in U_θ and $\nu(J(\varphi, F)) = M(\varphi, F)$.*

In fact, the functor F corresponds to a (U, U_θ)-bimodule X and φ corresponds to a bimodule morphism $\tilde{\varphi} : X \to U_\theta$. It is clear that $J(\varphi, F) = \tilde{\varphi}(X)$ is a submodule, i.e. the two-sided ideal in U_θ, and $M(\varphi, F) = \mathrm{Im}(X \otimes_{U_\theta} M_\chi \to U_\theta \otimes_{U_\theta} M_\chi) = J(\varphi, F)M_\chi = \nu(J(\varphi, F))$.

(b) *Each two-sided ideal J in U_θ is of the form $J(\varphi, F)$ for some F and φ.*

Since U_θ is Nöetherian, J has a finite number of generators $u_1, \ldots, u_n$. These generators belong to a finite dimensional space $V \subset J$ which is invariant with respect to the adjoint action ad (see 2.2). The morphism $V \to U^{\mathrm{ad}}$ may be continued to a bimodule morphism $\hat{\varphi} : V \otimes U_\theta \to U_\theta$ (see 2.2), i.e. to a functor morphism $\varphi : F_V(\theta) \to \mathrm{Id}\,\theta$. It is clear that $J = J(\varphi, F_V)$.

(c) *If $M(\varphi, F) \subset M(\varphi', F')$, then $J(\varphi, F) \subset J(\varphi', F')$. In particular, if $M(\varphi, F) = M(\varphi', F')$, then $J(\varphi, F) = J(\varphi', F')$.*

Since $\mathrm{Im}\,\varphi_{M_\chi}(FM_\chi) \subset \mathrm{Im}\,\varphi'_{M_\chi}(F'M_\chi)$ and FM_χ is a projective object in $\mathcal{O}$ there is a morphism $\tilde{\alpha} : FM_\chi \to F'M_\chi$ such that $\varphi'_{M_\chi} \circ \tilde{\alpha} = \varphi_{M_\chi}$. By Theorem 3.5 the morphism $\tilde{\alpha}$ is continued to a morphism of functors $\alpha : F \to F'$ and $\varphi' \circ \alpha = \varphi$. This immediately implies that $J(\varphi, F) = \mathrm{Im}\,\varphi_{U_\theta} \subset \mathrm{Im}\,\varphi'_{U_\theta} = J(\varphi', M')$.

(d) It follows from (a)–(c) that ν is an embedding. It follows from (a), (b) that $\nu(\Omega_\theta)$ consists exactly of those submodules $M \subset M_\chi$ that are of the form $M = M(\varphi, F)$. By Theorem 3.5 they are exactly submodules of the form $\varphi(FM_\chi)$, where F is a projective θ-functor, $\varphi \in \mathrm{Hom}_{\mathcal{O}}(FM_\chi, M_\chi)$. The statement of the theorem follows from 3.3, 3.8 and 1.11(c).

4.4. *The Duflo theorem*

Theorem 4.3 implies a simple proof of the following remarkable result of Duflo (see also [26]).

THEOREM [9]: *Let $\theta \in \Theta$, J a prime two-sided ideal in U_θ. Then a weight $\psi \in \eta^{-1}(\theta)$ exists such that $J = \mathrm{Ann}\, L_\psi$.*

In fact, consider Jordan–Hölder series of the module $M = M_\chi/\nu(J)$. Let $L_1, L_2, \ldots, L_n$ be its consequent compositional factors and $I_i = \mathrm{Ann}_{U_\theta}(L_i)$. Since $JM = 0$ we have $J \subset I_i$ for any i. Besides, ideal $I = I_1 I_2 \ldots I_n$ annuls M and by 4.3 $I \subset J$. Since J is a prime ideal, we have that $J = I_i$, for some i. By 1.11(c) $L_i = L_\psi$, where $\psi < \chi$, so that $J = \mathrm{Ann}_{U_\theta}(L_\psi)$.

Note that in general the weight ψ in the Duflo theorem cannot be chosen uniquely. Using theorems 3.3, 3.4 and 4.3 it is possible to refine a choice of ψ.

4.5. *Multiplicities in Verma modules*

Now apply the results of section 3 to the study of multiplicities $d_{\varphi\psi} = [M_\varphi : L_\psi]$. Choose a dominant weight χ and put $\theta = \eta(\chi)$. Let us restrict ourselves to the case of the regular weight χ.

Let $\bar{W} = W_{\chi+\Gamma}$, $S = \bar{W}(\chi)$, K_S a subgroup in $K(\mathcal{O})$ generated by δ_φ, $\varphi \in S$. Let us identify K_S with the group ring of the group $\bar{W}$, assigning to an element $w \in \bar{W}$ the element $\delta_{w\chi} \in K_S$.

On $\bar{W}$, consider an ordering such that $w < w'$ iff $w\chi < w'\chi$. It is possible to show (see [9]) that this ordering is defined by a system of positive roots $R_\chi^+ = R_\chi \cap R^+$ in R_χ. Namely, let us consider a system of simple roots $\Pi_\chi \subset R_\chi^+$, and put $\sigma_\alpha w \overset{\alpha}{<} w$ for any $\alpha \in R_\chi^+$, if $\ell(\sigma_\alpha w) > \ell(w)$ (ℓ is a length function on $\bar{W}$ corresponding to Π_χ). It is transitive closure of all relations $\overset{\alpha}{<}$ that is the relation of order (in [9] the opposite relation $>$ is considered).

For any $w \in \bar{W}$, consider an indecomposable projective functor $F_w = F_{(w\chi,\chi)}$ such that $F_w M_\chi = P_{w\chi}$ (see 3.3). Then operator F_w^K preserves a subgroup K_S. Since F_w^K commutes with $\bar{W}$ action, we have in $\mathbf{Z}[\bar{W}] = K_S$, that F_w^K is an operator of right multiplication by $f_w = F_w^K(\delta_\chi)$.

Let us realize elements of $\mathbf{Z}[\bar{W}]$ as functions on $\bar{W}$. By definition of F_w, we have $f_w(s) = \{[P_{w\chi}], \delta_{s\chi}\} = d_{s\chi, w\chi} = [M_{s\chi} : L_{w\chi}]$ (see 1.1(c), (d)). We wish to compute multiplicities $d_{\varphi,\psi}$, i.e. functions f_w. Describe their properties.

(1) $f_w(s) \in \mathbf{Z}^+$ and $f_w(s) \geq f_w(s')$ for $s \geq s'$; besides $f_w(s) > 0$ iff $s > w$ and $f_w(w) = 1$ (see 1.11(c), (d)).

(2) A semigroup (with respect to addition) of $\mathbf{Z}[\bar{W}]$ generated by f_w, $w \in \bar{W}$ is invariant with respect to convolution, i.e. $f_{w_1} * f_{w_2} = \Sigma\, C^w_{w_1,w_2} f_w$, where $C^w_{w_1,w_2} \in \mathbf{Z}^+$.

Since $F_{w_1} \circ F_{w_2}$ is projective functor and hence it may be decomposed into the sum of indecomposable functors, we have (2). Derive from (1) and (2) some consequences.

(3) Let $\sigma \in \bar{W}$ be a simple reflection. Then $f_\sigma(\sigma) = f_\sigma(e) = 1$, $f_\sigma(s) = 0$ for $s \neq \sigma, e$ (e is the identity of $\bar{W}$).

In fact, it follows from (1) that $f_\sigma = \sigma + ne$ where $n > 0$. Therefore, $f_\sigma * f_\sigma = 2n\sigma + (n^2+1)e = 2nf_\sigma + (1-n^2)f_e$. Since elements f_w are linearly independent in $\mathbf{Z}(\bar{W}]$, (2) implies that $1-n^2 \geq 0$, hence $n = 1$. In particular, $f_\sigma * f_\sigma = 2f_\sigma$.

(4) Let $s > w$ and $\ell(w) = \ell(s)+1$. Then $f_w(s) = 1$.

The proof is by induction on $\ell(w)$. If $\ell(w) = 1$ our statement is contained in (3). Choose a simple reflection σ in $\bar{W}$ so that $w = \sigma w'$, where $w' > w$. Then it is clear that $f_\sigma * f_{w'} = f_w + \Sigma\, C_t f_t$, where $\ell(t) \leq \ell(w)$. Since $f_{w'}(s) = 0$ and $s \neq w'$ and $f_{w'}(\sigma s) \leq 1$ by inductive hypothesis and $f_{w'}(\sigma w') = 0$, $f_{w'}(w') = 1$, we have $(f_\sigma * f_{w'})(s) \leq 1$. Hence $f_w(s) \leq 1$. Q.E.D.

(5) Let σ be simple reflection in $\bar{W}$ and $\sigma w > w$. Then $f_w(\sigma s) = f_w(s)$ for any s. Similarly, if $w\sigma > w$, then $f_w(s\sigma) = f_w(s)$. In particular, if $w_0 \in \bar{W}$ is the element of maximal length, then $f_{w_0}(s) = 1$ for any s.

Consider a function $f = f_\sigma * f_w$. It is clear that $f(s) = 0$ if $s < w$ and $f(w) = 2$. Hence, the decomposition of f is of the form $f = 2f_w + \cdots$. Comparing sums of coefficients we see that $f = 2f_w$, hence $f_w(\sigma s) = f_w(s)$. We similarly prove the second statement.

(6) $f_w(s) = f_{w^{-1}}(s^{-1})$ for any w, $s \in \bar{W}$. It easily follows from 3.4(c).

Example: Using properties (1)–(6) it is possible to find out all functions f_w for the group $G = SL_4$ (i.e. $\bar{W}$ is the Weyl group of the root system A_3). Provide the answer.

(a) $f_w(s) = 0$, for $s \not> w$; $f_w(s) = 1, 2$ for $s > w$.

(b) $f_w(s) = 2$ in the following cases:

(i) $w = (3412)$, $s = (1324)$ and (1234)

(ii) $w = (4231)$, $s = (2143)$, (2134), (1234) and (1243).

We depict elements of S_4 by substitutions. For simple reflections we take elementary transpositions.

We have to note that the results of (4)–(6) and some more general results about $f_w(s)$ are contained in Jantzen thesis [23, sect. 5].

II. REPRESENTATIONS OF COMPLEX GROUPS

5. Equivalent subcategories in the category $\mathcal{O}$ and in the category of Harish–Chandra modules

In section 4 we had proved that sometimes the two-sided ideal lattice in U_θ coincides with the submodule lattice in M_χ. On the other hand Duflo [9] had realized this lattice as submodule lattice of a

Harish–Chandra module of the algebra $U^2 = U \otimes U^0$. In this section we will connect these two approaches.

5.1. *Harish–Chandra modules*

We will use the following notations:

$\mathfrak{g}^2 = \mathfrak{g} \oplus \mathfrak{g}^0$, where $\mathfrak{g}^0$ is the opposite Lie algebra.

$\mathfrak{k} = \{(X, -X),\ X \in \mathfrak{g}\} \subset \mathfrak{g}^2$ the diagonal subalgebra naturally isomorphic to $\mathfrak{g}$.

$U^2 = U(\mathfrak{g}^2) = U(\mathfrak{g}) \otimes U(\mathfrak{g}^0)$.

$Z^2 = Z \otimes Z \subset U^2$ – the centre of algebra U^2.

We shall often realize U^2-module Y as (U, U) bimodule.

For abuse of notation we denote by $Y/\mathfrak{k}$ both the restriction of the representation of $\mathfrak{g}^2$ in Y on a Lie subalgebra $\mathfrak{k} = \mathfrak{g}$ and the space of representation itself. It is evident that $Y/\mathfrak{k}$ coincides with $\mathfrak{g}$-module Y^{ad} constructed via (U, U)-bimodule Y, see 2.2.

For any finite dimensional $\mathfrak{g}$-module V denote by Φ_V and U^2-module $V \otimes U$ constructed in 2.2.

5.2. Definition: *The U^2-module M is $\mathfrak{k}$-algebraic* if the module $M/\mathfrak{k}$ can be decomposed into the direct sum of finite-dimensional irreducible $\mathfrak{k}$-modules. Denote by $\mathscr{H}$ a complete subcategory in U^2-mod, consisting of algebraic modules. Denote by $\mathscr{H}_f$ a complete subcategory of $\mathscr{H}$ consisting of finitely generated modules.

Lemma: (i) *Categories $\mathscr{H}$ and $\mathscr{H}_f$ are closed with respect to subquotients.*

(ii) *Modules Φ_V belong to $\mathscr{H}_f$; any module $Y \in \mathscr{H}_f$ is isomorphic to a quotient of a module Φ_V.*

(iii) *Modules Φ_V are projective objects in categories $\mathscr{H}$ and $\mathscr{H}_f$.*

Proof: (i) holds because U^2 is Nöetherian.

(ii) By 2.2, $\Phi_V \in \mathscr{H}_f$. Let $Y \in \mathscr{H}_f$ and let $y_1, \dots, y_n$ be its generators. It follows from the definition of an algebraicness that there is a finite dimensional $\mathfrak{k}$-invariant subspace $V \subset Y$ containing $y_1, \dots, y_n$. Inclusion $V \hookrightarrow Y$ defines by 2.2 a morphism $\Phi_V \to Y$; since $\operatorname{Im} \Phi_V$ contains generators y_i, we have $\operatorname{Im} \Phi_V = Y$.

(iii) Let $Y' \to Y \to Y''$ be an exact sequence in $\mathscr{H}$. Then by 2.2

$$\operatorname{Hom}_{\mathscr{H}}(\Phi_V, Y' \to Y \to Y'') = \operatorname{Hom}_{\mathfrak{k}}(V, Y' \to Y \to Y'').$$

But this sequence is exact, because Y', Y, Y'' are completely reducible $\mathfrak{k}$-modules.

5.3. PROPOSITION: *Let* $Y \in \mathcal{H}_f$. *The following statements are equivalent.*

(i) *For any finite-dimensional* $\mathfrak{g}$*-module* V *the space* $\mathrm{Hom}_{\mathfrak{g}}(V, Y/\mathfrak{k}) = \mathrm{Hom}_{\mathcal{H}}(\Phi_V, Y)$ *is finite-dimensional.*

(ii) *Any irreducible representation of* $\mathfrak{k}$ *enters in the restriction* $Y/\mathfrak{k}$ *with a finite multiplicity.*

(iii) *The ideal* $\mathrm{Ann}_{Z^2}(Y) = \{z \in Z^2 \mid zY = 0\}$ *has a finite codimension in* Z^2.

(iv) *The module* Y *is* Z^r *finite, (here* $Z^2 = Z \otimes Z \subset U \otimes U^0$, $Z^r = 1 \otimes Z \subset Z^2$, *see* 2.4).

The module Y satisfying these conditions we call a Harish–Chandra module (with respect to $\mathfrak{k}$). Denote by $\mathcal{HCK}$ the category of such modules. It is a complete subcategory in $\mathcal{H}_f$ closed with respect to subquotients.

PROOF: Clearly (i)$\Rightarrow$(ii)$\Rightarrow$(iii)$\Rightarrow$(iv).

(iv)$\Rightarrow$(i). Let Y be Z^r-finite module. Let y_i, $i = 1, \ldots, n$ be generators of Y, $J_i = \{z \in Z^r \mid zy_i = 0\}$, $J = \cap J_i$. Then $\dim(Z^r/J) < \infty$ and $JY = 0$. By 5.2(ii), Y is a quotient of the module Φ_L, hence the module Φ_L/J so we may assume that $Y = \Phi_L/J$. Then for any finite-dimensional $\mathfrak{k}$-module V we have

$$\begin{aligned}\mathrm{Hom}_{\mathfrak{k}}(V, Y) = \mathrm{Hom}_{\mathfrak{k}}(V, \Phi_L)/J &= \mathrm{Hom}_{\mathfrak{g}}(V, L \otimes U^{\mathrm{ad}})/J \\ &= \mathrm{Hom}_{\mathfrak{g}}(L^* \otimes V, U^{\mathrm{ad}})/J.\end{aligned}$$

(the first equality uses the complete irreducibility of the $\mathfrak{k}$-module Φ_L). The latter space is finite-dimensional in view of 1.7.

5.4. *Classification of irreducible Harish–Chandra modules*

As in 1.8, we can decompose the category $\mathcal{HCK}$ into the sum of categories corresponding to different characters of Z^2. Each character of Z^2 is determined by a pair (θ_1, θ_2), where θ_1 is a character of Z^ℓ and θ_2 is a character of Z^r. Accordingly, put $\mathcal{H}_f^m(\theta)^r = \{Y \in \mathcal{H}_f \mid (J_\theta^r)^m Y = 0\}$, $\mathcal{H}_f^\infty(\theta)^r = \cup_m \mathcal{H}_f^m(\theta)^r$. Similarly define $\mathcal{H}_f^m(\theta)^\ell$.

Our nearest aim is to study the category $\mathcal{H}_f(\theta)^r = \mathcal{H}_f^1(\theta)^r$, where $\theta \in \Theta$ is a fixed character of Z. This category is a complete subcategory in the category of (U, U_θ)-bimodules. For any finite-dimensional $\mathfrak{g}$-module V put $\Phi_V(\theta) = \Phi_V/J_\theta^r \in \mathcal{H}_f(\theta)^r$.

Recall that we denote by Ξ the set of equivalence classes of pairs (ψ, χ), ψ, $\chi \in \mathfrak{h}^*$, $\psi - \chi \in \Lambda$. where $(w\psi, w\chi) \sim (\psi, \chi)$ for $w \in W$. We put $\eta^r(\psi, \chi) = \eta(\chi) \in \Theta$. A pair (ψ, χ) is called proper, if χ is a dominant weight and $\psi < W_\chi \psi$.

PROPOSITION: (i) *The modules $\Phi_V(\theta)$ are projective objects in $\mathcal{H}_f(\theta)^r$. Each module $Y \in \mathcal{H}_f(\theta)^r$ is a quotient of a module $\Phi_V(\theta)$.*

(ii) *If $Y, Y' \in \mathcal{H}_f(\theta)$, then* $\dim \mathrm{Hom}_{U^2}(Y, Y') < \infty$.

(iii) *Each projective object in $\mathcal{H}_f(\theta)^r$ can be decomposed into the finite direct sum of indecomposable projective objects.*

(iv) *To each element $\xi \in \Xi$, such that $\eta^r(\xi) = \theta$ there corresponds an indecomposable projective object $P_\xi \in \mathcal{H}_f(\theta)$ which is determined up to isomorphism by the following condition:*

If ξ is written properly, $\xi = (\psi, \chi)$, then $P_\xi \otimes_U M_\chi \cong P_\psi$.

Each indecomposable projective object $P \in \mathcal{H}_f(\theta)^r$ is isomorphic to one of the objects P_ξ.

PROOF: (i) It is clear that each morphism $\Phi_V \to Y$, $Y \in \mathcal{H}_f(\theta)^r$, factors through a morphism $\Phi_V(\theta) \to Y$. Hence (i) follows from 5.2(ii).

(iii) If Y is a quotient of a module $\Phi_V(\theta)$ then $\mathrm{Hom}_{U^2}(Y, Y') \subset \mathrm{Hom}_{U^2}(\Phi_V(\theta), Y') = \mathrm{Hom}_{U^2}(\Phi_V, Y') = \mathrm{Hom}_{\mathfrak{f}}(V, Y')$. This space is finite-dimensional by 5.3.

(iii) follows from (ii).

(iv) Let P be a projective object in $\mathcal{H}_f(\theta)^r$. By (i) we have that P is a quotient of the module $\Phi_V(\theta)$ and since P is a projective object, it is a direct summand of $\Phi_V(\theta)$.

The module $\Phi_V(\theta)$ can be realized as a projective θ-functor $F_V(\theta): \mathcal{M}(\theta) \to \mathcal{M}$ (see 1.3 and 3.5). To a decomposition of the module $\Phi_V(\theta)$ into direct summands there corresponds a decomposition of the functor $F_V(\theta)$ into direct sum of subfunctors. Therefore, statement (iv) follows from Theorems 3.3–3.5.

5.5. LEMMA: *Each projective module P_ξ has the unique simple quotient L_ξ and* $\dim \mathrm{Hom}(P_\xi, L_\xi) = 1$ *while* $\dim \mathrm{Hom}(P_\xi, L_{\xi'}) = 0$ *for $\xi' \neq \xi$.*

PROOF: (a) There exists an irreducible quotient due to the Zorn lemma.

(b) Since End P_ξ is a finite-dimensional ring and it does not have and idempotents different from 0 and 1, it is local (see [13], Chapter III) and its maximal ideal is nilpotent.

(c) If P_1, P_2 are proper submodules of P_ξ then $P_1 + P_2 \neq P_\xi$. Otherwise by the projectivity of P_ξ the map $\mathrm{Hom}(P_\xi, P_1) \oplus \mathrm{Hom}(P_\xi, P_2) \to \mathrm{Hom}(P_\xi, P_\xi)$ would be an epimorphism, i.e. in End P_ξ we would have $1 = \varphi_1 + \varphi_2$, where $\mathrm{Im}\, \varphi_i \subset P_i$. Since End P_ξ is a local ring, one of morphisms φ_1 or φ_2 would be invertible, i.e. one of submodules P_1 or P_2 would coincide with P_ξ.

Hence, we have proved that the moduule P_ξ has the unique irreducible quotient L_ξ.

(d) By (c) $\mathrm{Hom}(P_\xi, L_\xi) = \mathrm{Hom}(L_\xi, L_\xi) = k$. If $\mathrm{Hom}(P_\xi, L_{\xi'}) \neq 0$, then $L_\xi \cong L_{\xi'}$. Let us lift the non-zero morphisms $\varphi' : P_\xi \to L_{\xi'}$, and $\psi' : P_{\xi'} \to L_\xi$ to morphisms $\varphi : P_\xi \to P_{\xi'}$ and $\psi : P_{\xi'} \to P_\xi$. It is clear that $\varphi\psi$ and $\psi\varphi$ are not nilpotent. This implies that $\varphi\psi$ and $\psi\varphi$ are invertible, i.e. φ and ψ are isomorphisms. Hence, $P_\xi \cong P_{\xi'}$ and by 5.4(iv) we have that $\xi = \xi'$.

5.6. THEOREM: *Assign to each element $\xi \in \Xi$ the irreducible quotient L_ξ of the module P_ξ. This map defines a one-to-one correspondence between Ξ and the set of equivalence classes of simple Harish-Chandra modules.*

This theorem follows from 5.5 because each simple module of $\mathscr{H}$ belongs to the category $\mathscr{H}_f(\theta)^r$ for some θ and by 5.4 is a quotient of the module P_ξ.

5.7. COROLLARY: (i) *If $J \subset Z^2$ is an ideal of finite codimension, we have a finite number of simple Harish-Chandra modules L, such that $JL = 0$.*

(ii) *Let $Y \in \mathscr{H}$ be a module satisfying*:

(a) *Y is annihilated by an ideal $J \subset Z^2$ of a finite codimension.*

(b) $\dim \mathrm{Hom}_{\mathfrak{k}}(V, Y) < \infty$ *for any finite dimensional $\mathfrak{g}$-module V. Then Y is of a finite length*; *in particular, each Harish-Chandra module has a finite length.*

PROOF: (i) follows straightforward from Theorem 5.6.

(ii) Let $L_1, \ldots, L_n$ be a set of simple modules such that $JL_i = 0$ (see (i)). Then, there is a finite dimensional $\mathfrak{g}$-module V (not necessarily irreducible) such that $\mathrm{Hom}_{\mathfrak{k}}(V, L_i) \neq 0$ for any i. Note that on $\mathscr{H}$, a functor $X \mapsto \mathrm{Hom}_{\mathfrak{k}}(V, X)$ is exact. If X is a non-zero subquotient of Y, then X has an irreducible subquotient L. Since $JL = 0$, $L = L_i$ so that $\mathrm{Hom}_{\mathfrak{k}}(V, L) \neq 0$, and hence $\mathrm{Hom}_{\mathfrak{k}}(V, X) \neq 0$. This immediately implies that the length of Y does not exceed $\dim \mathrm{Hom}_{\mathfrak{k}}(V, Y)$.

5.8. *The functor T_χ*

We may consider each module $Y \in \mathscr{H}$ as (U, U)-bimodule. Therefore, the bifunctors $\mathscr{H} \otimes \mathscr{M} \to \mathscr{M} : (Y, M) \mapsto Y \otimes_U M$ and $\mathscr{H} \otimes \mathscr{H} \to \mathscr{H} : (Y, Y') Y \otimes_U Y'$ are defined. These bifunctors are right exact with respect to each variable and there is a natural isomorphism $(Y \otimes_U Y') \otimes_U M = Y \otimes_U (Y' \otimes_U M)$. The functors $\otimes_U \Phi_V$, $\Phi_V \otimes_U : \mathscr{H} \to \mathscr{H}$ are exact and the functor $\Phi_V \otimes_U : \mathscr{M} \to \mathscr{M}$ coincides with the functor F_V considered in §§2, 3.

Since each module $Y \in \mathcal{H}_f$ is a quotient of the module Φ_V, we have that for $Y \in \mathcal{H}_f$ the functor $Y \otimes_U : \mathcal{M} \to \mathcal{M}$ preserves subcategories $\mathcal{M}_{Zf}$, $\mathcal{M}_{zf}$, $\mathcal{O}$ while the functor $Y \otimes_U : \mathcal{H} \to \mathcal{H}$ preserves subcategories $\mathcal{H}_f$ and $\mathcal{HCh}$.

In §§2–4, we had fixed the module $Y \in \mathcal{H}_f$ and had considered the functor $Y \otimes_U : \mathcal{M} \to \mathcal{M}$. Now we do conversely, we choose a module $M \in \mathcal{M}$ and study the functor $T_M = \otimes_U M : \mathcal{H} \to \mathcal{M}$.

It is clear that the functor T_M preserves the left action of Z and multiplication from the left by $Y \in \mathcal{H}$ (i.e. $T_M(z) = z$ for $z \in Z^\ell$ and $T_M(Y \otimes_U Y') = Y \otimes_U T_M(Y')$.

5.9. Choose a character $\theta \in \Theta$ and let us show, using functors T_M, that the category $\mathcal{H}_f(\theta)^r$ is equivalent to a complete subcategory of a category $\mathcal{O}$. Choose a weight $\chi \in \eta^{-1}(\theta)$ and consider the functor $T_\chi = T_{M_\chi}$. It is clear that the functor T_χ is defined by its values on the subcategory $\mathcal{H}_f(\theta)^r$ and $T_\chi(\mathcal{H}_f) \subset \mathcal{O}$. Therefore, we shall consider T_χ as the functor $T_\chi : \mathcal{H}_f(\theta)^r \to \mathcal{O}$.

Let χ be a dominant weight. Denote by $\mathcal{P}(\chi)$ the class of projective objects of the category $\mathcal{O}$ consisting of direct sums of modules P_ψ, where $\psi \in \chi + \Lambda$, $\psi < W_\chi(\psi)$. Let $\mathcal{O}_{\mathcal{P}(\chi)}$ be the complete subcategory of $\mathcal{O}$ consisting of $\mathcal{P}(\chi)$-presentable modules (see 1.2).

THEOREM: *Let χ be a dominant weight, $\theta = \eta(\chi)$.*

(i) *If χ is a regular weight, then the functor T_χ defines the equivalence between $\mathcal{H}_f(\theta)^r$ and $\mathcal{O}_{\chi+\Lambda}$ (this category consists of modules $M \in \mathcal{O}$ such that $M^\psi = 0$ for $\psi \notin \chi + \Lambda$; see* 1.11).

(ii) *In general case the functor T_χ defines the equivalence between $\mathcal{H}_f(\theta)^r$ and the complete subcategory $\mathcal{O}_{\mathcal{P}(\chi)}$ of the category $\mathcal{O}$.*

Note that the category $\mathcal{O}_{\chi+\Lambda}$ is closed with respect to subquotients so that for a regular dominant weight χ the functor $T_\chi : \mathcal{H}_f(\theta)^r \to \mathcal{O}$ is exact. Generally, it is not so. For example if $\mathfrak{g} = \mathfrak{sl}(4)$ and α, β, γ are simple roots (α is orthogonal to γ) and χ is a weight such that $\chi(h_\alpha) = \chi(h_\gamma) = 0$, $\chi(h_\beta) = 1$, then it is easy to see that the functor T_χ is not exact.

5.10. PROOF OF THEOREM 5.9: Statement (i) is a special case of the statement (ii), because for a regular weight χ the class $\mathcal{P}(\chi)$ contains all projective objects of $\mathcal{O}_{\chi+\Lambda}$ so that $\mathcal{O}_{\mathcal{P}(\chi)} = \mathcal{O}_{\chi+\Lambda}$. To prove (ii) we use the following general fact.

PROPOSITION: *Let $\mathcal{A}$, $\mathcal{B}$ be Abelian categories, $\mathcal{P}$ the class of* *projective objects of $\mathscr{A}$ and $T:\mathscr{A}\to\mathscr{B}$ right exact functor. Suppose that*

(i) *$\mathscr{A}=\mathscr{A}_{\mathscr{P}}$, i.e. all objects of $\mathscr{A}$ are $\mathscr{P}$-presentable.*

(ii) *$T(P)$ is a projective object in $\mathscr{B}$ for any $P\in\mathscr{P}$. Denote by $T(\mathscr{P})$ the class of such objects.*

(iii) *If $P, P'\in\mathscr{P}$ then the map*

$$F_{P,P'}:\mathrm{Hom}_{\mathscr{A}}(P,P')\to\mathrm{Hom}_{\mathscr{B}}(T(P),T(P'))$$

is isomorphism.

Then functor T is fully faithful and defines the equivalence of the category $\mathscr{A}$ with the complete subcategory $\mathscr{B}_{T(\mathscr{P})}$ of $\mathscr{B}$ consisting of $T(\mathscr{P})$-presentable objects.

PROOF: (a) If $A\in\mathscr{A}$, then object $T(A)\in\mathscr{B}$ is $T(\mathscr{P})$-presentable. In fact, by (i) there is an exact sequence $P'\to P\to A\to 0$, and since functor T is right exact, so is the sequence $T(P')\to T(P)\to T(A)\to 0$. Let us prove that each object $B\in\mathscr{B}_{T(\mathscr{P})}$ is isomorphic to an object of the form $T(A)$, $A\in\mathscr{A}$. In fact, let $T(P')\xrightarrow{\alpha}T(P)\to B\to 0$ be its $T(\mathscr{P})$-presentation. By (iii) there is a morphism $\beta:P'\to P$ such that $T(\beta)=\alpha$. Put $A=\mathrm{Coker}\,\beta$. Functor T transforms the exact sequence $P'\xrightarrow{\beta}P\to A\to 0$ into the exact sequence $T(P')\xrightarrow{\alpha}T(P)\to T(A)\to 0$, which immediately implies that B is isomorphic to $T(A)$.

(b) Let $A\in\mathscr{A}$, $Q\in\mathscr{P}$. Then $T_{Q,A}:\mathrm{Hom}_{\mathscr{A}}(Q,A)\to\mathrm{Hom}_{\mathscr{B}}(T(Q),T(A))$ is isomorphism.

In fact, consider the $\mathscr{P}$-presentation $P'\to P\to A\to 0$ and its T image $T(P')\to T(P)\to T(A)\to 0$. Since Q and $T(Q)$ are projective objects, the following sequences are exact

$$\begin{array}{ccccccc}
\mathrm{Hom}(Q,P') & \longrightarrow & \mathrm{Hom}(Q,P) & \longrightarrow & \mathrm{Hom}(Q,A) & \longrightarrow & 0\\
\downarrow{\scriptstyle T_{Q,P'}} & & \downarrow{\scriptstyle T_{Q,P}} & & \downarrow{\scriptstyle T_{Q,A}} & & \\
\mathrm{Hom}(T(Q),T(P')) & \to & \mathrm{Hom}(T(Q),T(P)) & \to & \mathrm{Hom}(T(Q),T(A)) & \to & 0.
\end{array}$$

Since vertical arrows $T_{Q,P'}$ and $T_{Q,P}$ are isomorphisms due to condition (iii) so is $T_{Q,A}$.

(c) Let $A, C\in\mathscr{A}$. Then $T_{A,C}:\mathrm{Hom}_{\mathscr{A}}(A,C)\to\mathrm{Hom}_{\mathscr{B}}(T(A),T(C))$ is isomorphism.

Consider again exact sequences $P'\to P\to A\to 0$ and $T(P')\to T(P)\to T(A)\to 0$.

Using the exactness of the functor Hom, we obtain

$$\begin{array}{ccccccc}
0\to & \mathrm{Hom}(A,C) & \longrightarrow & \mathrm{Hom}(P,C) & \longrightarrow & \mathrm{Hom}(P',C) & \\
 & \downarrow{\scriptstyle T_{A,C}} & & \downarrow{\scriptstyle T_{P,C}} & & \downarrow{\scriptstyle T_{P',C}} & \\
0\to & \mathrm{Hom}(T(A),T(C)) & \to & \mathrm{Hom}(T(P),T(C)) & \to & \mathrm{Hom}(T(P'),T(C)). &
\end{array}$$

As it is shown in (b), vertical arrows $T_{P',C}$ and $T_{P,C}$ are isomorphisms. Hence, so is $T_{A,C}$. Proposition is proved.

Now apply this proposition to $\mathcal{A} = \mathcal{H}_f(\theta)^r$, $\mathcal{B} = \mathcal{O}$, $T = T_\chi$ and let $\mathcal{P}$ be the class of projective objects in $\mathcal{H}_f(\theta)^r$. Let us verify conditions (i)–(iii) of the previous proposition.

Condition (i) follows from Proposition 5.4(i). To check (ii) and (iii) we consider projective objects of $\mathcal{H}_f(\theta)^r$ as projective θ-functors $\mathcal{M}(\theta) \to \mathcal{M}$; it may be done because they are direct summands of modules $\Phi_V(\theta)$ considered as functors $F_V(\theta)$. The validity of conditions (ii)–(iii) now follows from 3.2 and Theorem 3.5.

To prove Theorem 5.9 it only remains to verify that the class of objects $T(\mathcal{P}) = T_\chi(\mathcal{P})$ coincides with the class of objects $\mathcal{P}(\chi)$. It immediately follows from 5.4(iii)–(iv).

5.11. Let χ be a dominant weight $\theta = \eta(\chi)$, $Y \in \mathcal{H}_f(\theta)^r$, $M = T_\chi(Y)$. If χ is a regular weight, then Theorem 5.9 implies that the submodule lattices of M and Y are isomorphic. Generally, it is not so for an arbitrary χ.

Denote by Ω_Y and Ω_M the submodule lattices of Y and M and define a map $\nu : \Omega_Y \to \Omega_M$ in the following way.

If $Y' \subset Y$, i.e. when an embedding $\varphi : Y' \to Y$ is given, put $\nu(Y') = \mathrm{Im}(T_\chi(\varphi) : T_\chi(Y') \to T_\chi(Y))$.

THEOREM: *The map $\nu : \Omega_Y \to \Omega_M$ is an embedding. Its image consists exactly of all $\mathcal{P}_\chi$-generated submodules of M (see 1.2).*

This theorem is proved exactly as Theorem 4.3.

5.12. From Theorem 5.9 we deduce an interesting corollary.

PROPOSITION: *Let χ, ψ be weights that belong to the same orbit of the Weyl group W. Then categories $\mathcal{O}_{\chi+\Lambda}$ and $\mathcal{O}_{\psi+\Lambda}$ are equivalent, if considered as Z-categories.*

PROOF: Let $\psi = w\chi$. Replacing χ by $\chi + \lambda$ and ψ by $\psi + w\lambda$, where $\lambda \in \Lambda$, we may assume that χ and ψ are regular. Further, replacing χ and ψ by dominant elements of $W_{\chi+\Gamma}(\chi)$ and $W_{\psi+\Gamma}(\psi)$, we may assume that χ and ψ are dominant.

Suppose $\theta = \eta(\chi) = \eta(\psi)$. By Theorem 5.9 categories $\mathcal{O}_{\chi+\Lambda}$ and $\mathcal{O}_{\psi+\Lambda}$ are both equivalent to the same category $\mathcal{H}_f(\theta)^r$. Hence they are equivalent.

6. Multiplicities in representations of principal series and in Verma modules

6.1. *The functor H*

Theorem 5.9 shows that the functor T_χ defines an equivalency of categories $\mathscr{H}_f(\theta)^r$ and $\mathcal{O}_{\chi+\Lambda}$ for a regular dominant weight χ. In particular, it has the inverse functor. We should like to describe this functor explicitly. For this, we use a method borrowed from [12]. Below, we apply this explicit description for computation of multiplicities of representations of principal series.

Let $M, N \in \mathcal{M}$. Endow the space $\mathrm{Hom}_k(M, N)$ with U^2-module structure via $((u \otimes u^0)\varphi)(m) = u\varphi(u^0 m)$, where $m \in M$, $\varphi \in \mathrm{Hom}_k(M, N)$, $u \otimes u^0 \in U^2$. Denote by $H(M, N)$ the space of $\mathfrak{k}$-finite vectors in $\mathrm{Hom}_k(M, N)$. It is easy to verify that $H(M, N)$ is U^2-submodule of $\mathrm{Hom}_k(M, N)$. Evidently $H(M, N) \in \mathscr{H}$.

Therefore, we have obtained a bifunctor $H : \mathcal{M} \times \mathcal{M} \to \mathscr{H}$ contravariant in the first variable and covariant in the second one.

For each module $M \in \mathcal{M}$ define a functor $H_M : \mathcal{M} \to \mathscr{H}$ via $H_M(N) = H(M, N)$.

LEMMA: (i) *The functor $H(M, N)$ is left exact in each variable.*

(ii) *The functor H_M is right adjoint to the functor T_M, i.e.*

$$\mathrm{Hom}(T_M(Y), N) = \mathrm{Hom}(Y, H_M(N)) \quad \textit{for } N \in \mathcal{M},\ Y \in \mathscr{H}.$$

(iii) *The functor $N \mapsto H(M, N)$ is Z^ℓ-linear and the functor $M \mapsto H(M, N)$ is Z^r-linear. If V is a finite dimensional $\mathfrak{g}$-module then*

$$H(M, V \otimes N) = \Phi_V \otimes_U H(M, N),$$

$$H(V \otimes M, N) = H(M, N) \otimes_U \Phi_{V^*}.$$

PROOF: (i) The functor $(M, N) \mapsto \mathrm{Hom}_k(M, N)$ is exact and the functor of passage to $\mathfrak{k}$-finite elements is left exact.

(ii) $\mathrm{Hom}_U(T_M(Y), N) = \mathrm{Hom}_U(Y \otimes_U M, N)$

$$= \mathrm{Hom}_{U^2}(Y, \mathrm{Hom}_k(M, N)).$$

Since $Y \in \mathscr{H}$, in the latter equation $\mathrm{Hom}_k(M, N)$ may be replaced by $H(M, N)$.

(iii) is quite straightforward.

6.2. We shall be mainly interested in the functor $H(M, N)$ for $M, N \in \mathcal{O}$, i.e. we shall consider the functor $H : \mathcal{O} \times \mathcal{O} \to \mathscr{H}$.

LEMMA: (i) *If* $M, N \in \mathcal{O}$, *then* $H(M, N) \in \mathcal{HCh}$.

(ii) *If M is projective object in* $\mathcal{O}$, *then the functor* $H_M : \mathcal{O} \to \mathcal{HCh}$ *is exact.*

PROOF: (i) Let $J_1 = \mathrm{Ann}_Z(N)$, $J_2 = \mathrm{Ann}_Z(M)$. Then by 6.1(iii), $H(M, N)$ is annihilated by the ideal $J = J_1^{\ell} \otimes J_2^{\ell} \subset Z^2$ of a finite codimension. Therefore by proposition 5.3 it suffices to verify that $\dim \mathrm{Hom}_{\mathfrak{k}}(V, H(M, N)) < \infty$ for any finite dimensional $\mathfrak{g}$-module V.

By 6.1 we have

$$\begin{aligned}\mathrm{Hom}_{\mathfrak{k}}(V, H(M, N)) &= \mathrm{Hom}_{U^2}(\Phi_V, H(M, N))\\ &= \mathrm{Hom}_U(\Phi_V \otimes_U M, N).\end{aligned}$$

The latter space is finite-dimensional since both $\Phi_V \otimes_U M$ and N belong to $\mathcal{O}$.

Note that similarly, if M is a module of a finite length and N is finitely generated, then $H(M, N) \in \mathcal{HCh}$.

(ii) follows from 6.1(ii) and 1.2.

6.3. Let χ be a weight, $\theta = \eta(\chi)$. Put $H_\chi = H_{M_\chi} : \mathcal{O} \to \mathcal{H}_f(\theta)^r$. The functor H_χ is right adjoint to the functor T_χ. Therefore, Theorem 5.9 immediately implies the following statement.

PROPOSITION: *Let* χ *be a dominant weight. Then*

(i) *If* χ *is a regular weight, then* H_χ *defines equivalence between* $\mathcal{O}_{\chi+\Lambda}$ *and* $\mathcal{H}_f(\theta)^r$, *inverse to the functor* T_χ.

(ii) *In general,* H_χ *defines the equivalence between* $\mathcal{O}_{\mathcal{P}(\chi)}$ *and* $\mathcal{H}_f(\theta)^r$ *inverse to* T_χ. *In particular, the functor* $H_\chi \circ T_\chi$ *is isomorphic to the identity functor.*

6.4. Now we may describe the relations between representations of principal series of Lie algebra $\mathfrak{g}^2$ and modules $M_\psi \in \mathcal{O}$. First, give the definition of representations of principal series.

Let $M, N \in \mathcal{O}$. On $N \otimes M$ define a $\mathfrak{g}^2$-module structure via $(x, y)(n \otimes m) = (-x^t n) \otimes m - (n \otimes ym)$, where t is the anti-automorphism of $\mathfrak{g}$ considered in 1.10. The conjugate module $(N \otimes M)^*$ is naturally identified with the space of bilinear forms $B : N \times M \to k$ and $\mathfrak{g}^2$-action is defined via $[(x, y)B](n, m) = B(x^t n, m) + B(n, xm)$. Denote by $\mathrm{Dual}(N, M)$ the representation of $\mathfrak{g}^2$ in the space of $\mathfrak{k}$-finite vectors in $(N \otimes M)^*$.

For each pair of weights $\psi, \chi \in \mathfrak{h}^*$ define $X(\psi, \chi)$ to be the representation of principal series by setting $X(\psi, \chi) = \mathrm{Dual}(M_\psi, M_\chi)$.

This representation differs somewhat from the traditional definition (see [10], [11]). As it is shown in Appendix 2, $X(\psi, \chi)$ may be identified with $\mathscr{L}(-\psi, -\chi)$, where $\mathscr{L}$ is a representation of principal series in the sense of Duflo [11].

6.5. To find out how representations of principal series are transformed by the functor T_χ, we must describe the relation between functors Dual and $H(\cdot, \cdot)$. For this purpose we use a contravariant functor $\tau: \mathcal{O} \to \mathcal{O}$, $M \mapsto M^\tau$, constructed in 1.10.

PROPOSITION: *Let $M, N \in \mathcal{O}$. Then* $\mathrm{Dual}(N, M) = H(M, N^\tau)$.

PROOF: Define a morphism $i: H(M, N^\tau) \to \mathrm{Dual}(N, M)$ via $i(\varphi)(n, m) = \langle \varphi(m), n\rangle$ where $\varphi: \mathrm{Hom}(M, N^\tau)$, $i(\varphi) \in \mathrm{Dual}(N, M) = (N \otimes M)^*$ and $\langle \cdot, \cdot \rangle$ is a natural pairing of N^τ and N. It is easy to verify, that i is a U^2-module homomorphism and i is injective.

Define a morphism $j: \mathrm{Dual}(N, M) \to \mathrm{Hom}_k(M, N^*)$. If $B \in \mathrm{Dual}(N, M) \subset (N \otimes M)^*$, $m \in M$, then define the linear functional $j(B)(m)$ on N via $[j(B)(m)](n) = B(n, m)$. It is easy to see that j is a U^2-module homomorphism if N^* is considered together with the U-action described in 1.10.

Prove that if $\varphi \in \mathrm{Hom}_k(M, N^*)$ is a $\mathfrak{k}$-finite homomorphism, then it belongs to the subspace $H(M, N^\tau)$. It suffices to verify that $\varphi(M) \subset N^\tau$. Let A be finite-dimensional $\mathfrak{k}$-invariant subspace in $\mathrm{Hom}_k(M, N^*)$ and $B = U(\mathfrak{h})m$. Then the subspace $AB \subset N^*$ is finite-dimensional $\mathfrak{h}$-invariant because $h\varphi(m) = (h\varphi - \varphi h)(m) + \varphi(hm)$ for $h \in \mathfrak{h}$. Hence $\varphi(M) \subset N^\tau$.

Thus we have constructed a morphism $j: \mathrm{Dual}(N, M) \to H(M, N^\tau)$. It is easy to verify that $ij = 1$, $ji = 1$. The proposition is proved.

6.6. COROLLARY: *If $\chi, \psi \in \mathfrak{h}^*$, then the representation $X(\psi, \chi)$ of principal series is isomorphic to $H(M_\chi, M_\psi^\tau) = H_\chi(M_\psi^\tau)$.*

6.7. Now we can describe the Jordan–Hölder series of representations of principal series.

THEOREM: *Let $\chi, \psi \in \mathfrak{h}^*$ and $\xi \in \Xi$ (see 5.4). Suppose χ is a dominant weight. If ξ can not be written properly $\xi = (\varphi, \chi)$, then $[X(\psi, \chi): L_\xi] = 0$. If ξ can be written properly, $\xi = (\varphi, \chi)$, where $\varphi < W_\chi(\varphi)$, then $[X(\psi, \chi): L_\xi] = [M_\psi : L_\varphi]$.*

PROOF: The multiplicity of L_ξ in $X(\psi, \chi)$ coincides with

$\dim \mathrm{Hom}_{\mathscr{H}}(P_\xi, X(\psi, \chi))$. We have

$$\begin{aligned}[X(\psi, \chi): L_\xi] &= \dim \mathrm{Hom}_{\mathscr{H}}(P_\xi, X(\psi, \chi)) \\ &= \dim \mathrm{Hom}_{\mathscr{H}}(P_\xi, H_\chi(M_\psi^\tau)) = \dim \mathrm{Hom}_U(T_\chi(P_\xi), M_\psi^\tau) \\ &= \dim \mathrm{Hom}_U(P_\varphi, M_\psi^\tau) = [M_\psi^\tau : L_\varphi] = [M_\psi : L_\varphi].\end{aligned}$$

(1.11(b), 6.5, 1.4, 1.11(d) are used.)

REMARKS: (a) $X(w\psi, w\chi)$ and $X(\psi, \chi)$ are known to have the same Jordan–Hölder series (see e.g. Duflo [11]). Therefore, our theorem gives the description of multiplicities in representations of principal series in terms of multiplicities of Verma modules.

(b) The similar theorem is valid for representations induced from parabolic subgroups.

(c) A. Joseph [26] and T. Enright [25] have also proved the similar theorem.

Appendix 1

In this Appendix we prove Lemma 1.5.

Let $\chi_1, \ldots, \chi_r \in \mathfrak{h}^*$. Any equation L of the form $\Sigma_{i=1}^r n_i w_i(\chi_i) = \lambda$, where $n_i \in \mathbf{Z}$, $w_i \in W$, $\lambda \in \Lambda$ is called a relation.

We shall use the following translation principle.

The translation principle

Let $\mathfrak{h}_{\mathbf{R}}^* = \Lambda \otimes_{\mathbf{Z}} \mathbf{R}$. *If* $\chi_1, \ldots, \chi_r \in \mathfrak{h}^*$, *then weights* $\chi_1', \ldots, \chi_r' \in \mathfrak{h}_{\mathbf{R}}^*$ *exist such that satisfy exactly the same relations as* $\chi_1, \ldots, \chi_r$ *do.*

Using the principle, all questions of geometric nature put in the weight space $\mathfrak{h}^*$ we can solve in the space $\mathfrak{h}_{\mathbf{R}}^*$.

To prove this translation principle consider $H = (\mathfrak{h}^*)^r$ and $H_{\mathbf{R}} = (\mathfrak{h}_{\mathbf{R}}^*)^r$. Each set $\mathscr{L} = \{L_\alpha\}$ of relations defines affine subspaces $H^{\mathscr{L}} \subset H$ and $H_{\mathbf{R}}^{\alpha} \subset H_{\mathbf{R}}$ where $H^{\mathscr{L}} = \{(\chi_1, \ldots, \chi_r) \in H \mid L_\alpha(\chi_1, \ldots, \chi_r) = 0$ for any $\alpha\}$ and $H_{\mathbf{R}}^{\mathscr{L}}$ is defined similarly. Since all these subspaces are defined over $\mathbf{Z}$, we have that $H^{\mathscr{L}_1} \subset H^{\mathscr{L}_2}$ iff $H_{\mathbf{R}}^{\mathscr{L}_1} \subset H_{\mathbf{R}}^{\mathscr{L}_2}$.

Let $\mathscr{L}$ be a set of all relations that are satisfied by weights $\chi_1, \ldots, \chi_r$. Then for each relation $L \notin \mathscr{L}$ we have $H^{\mathscr{L} \cup L} \subsetneq H^{\mathscr{L}}$, so that $H_{\mathbf{R}}^{\mathscr{L} \cup L} \subsetneq H_{\mathbf{R}}$. Since there is only a countable number of relations L, we have that the set $\cup_{L \bar{\in} \mathscr{L}} H_{\mathbf{R}}^{\mathscr{L} \cup L}$ does not cover $H_{\mathbf{R}}^{\mathscr{L}}$ and we can choose a point $(\chi_1', \ldots, \chi_r') \in H_{\mathbf{R}}^{\mathscr{L}}$ that does not belong to this set. Weights $\chi_1', \ldots, \chi_r'$ are the desired ones.

PROOF OF LEMMA 1.5: Using the translation principle it is possible to assume that we are in $\mathfrak{h}_{\mathbf{R}}^*$. Then (i) follows from [17, Ex. I, 227].

(ii) R being replaced by R_χ, we may assume that $W_{\chi+\Gamma} = W$, i.e. $\chi \in \Lambda$. Then statement (ii) is contained in [17].

(iii) It suffices to consider the case $\varphi \overset{\gamma}{<} \psi$, where $\gamma \in R_\chi^+ = R_\chi \cap R^+$. Then $\chi(h_\gamma) \geq 0$ and $\psi(h_\gamma) \geq 0$, so that χ and ψ are to the one side of the hyperplane π_γ corresponding to γ. Then $\varphi = \sigma_\gamma \psi$, where σ_γ is the reflection in the hyperplane π_γ. This immediately implies that $|\varphi - \chi| \geq |\psi - \chi|$ and the equality is possible only if either $\psi \in \pi_\gamma$ or $\chi \in \pi_\gamma$. In the former case $\varphi = \psi$, while in the latter case $\varphi \in W_\chi(\psi)$.

Appendix 2

Usually (see e.g. [9], [10] or [11]) Lie algebra of a complex group is identified with $\mathfrak{a} = \mathfrak{g} \oplus \mathfrak{g}$ and a subalgebra $\mathfrak{k}_\mathfrak{a} = \{(x, -x^t) \in \mathfrak{a} \mid x \in \mathfrak{a}\}$ is considered. It had been more convenient for us to identify it with the Lie algebra $\mathfrak{g}^2 = \mathfrak{g} \otimes \mathfrak{g}^0$. In this Appendix we provide the dictionary for translation from one language into another.

Put $\mathfrak{h}_\mathfrak{a} = \mathfrak{h} \oplus \mathfrak{h} \subset \mathfrak{a}$, $\mathfrak{n}_\mathfrak{a} = \mathfrak{n} \oplus \mathfrak{n} \subset \mathfrak{a}$. A weight of the algebra $\mathfrak{h}_\mathfrak{a}$ is defined by a pair (p, q) where $p, q \in \mathfrak{h}^*$. The Verma module $M(p, q)$ of the algebra $\mathfrak{a}$ with respect to subalgebras $\mathfrak{h}_\mathfrak{a}$, $\mathfrak{n}_\mathfrak{a}$ is naturally identified with $M_p \otimes M_q$. The representation of principal series $\mathscr{L}(p, q)$ is defined as the representation of $\mathfrak{a}$ in the space of $\mathfrak{k}_\mathfrak{a}$-finite vectors of a module $M(-p, -q)^*$ (see [9], [11]).

Let us identify Lie algebras $\mathfrak{a}$ and $\mathfrak{g}^2$ via $i: \mathfrak{a} \to \mathfrak{g}^2$, $i(x, y) = (-x^t, -y)$. It is clear that $i(\mathfrak{k}_\mathfrak{a}) = \mathfrak{k}$. It is a straightforward verification that these isomorphisms transform an $\mathfrak{a}$-module $M(-p) \otimes M(-q)$ into a $\mathfrak{g}^2$-module $M_{-p} \otimes M_{-q}$ with the action defined in 6.4. Hence, $\mathfrak{a}$-module $\mathscr{L}(p, q)$ is identified with $\mathfrak{g}^2$-module $X(-p, -q)$. This implies that the irreducible $\mathfrak{a}$-module $\mathscr{V}(p, q)$ considered by Duflo in [9, 1.1] transforms into a $\mathfrak{g}^2$-module $L_{(-p,-q)}$. Thus, in terms of [9], Theorem 6.7 means that if $-q$ is a dominant weight and $p' \geq W_q(p')$, then $[\mathscr{L}_{p,q} : \mathscr{V}_{p',q}] = [M_{-p} : L_{-p'}]$.

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REFERENCES

[1] C. ZUCKERMAN: Tensor products of finite and infinite dimensional representations of semisimple Lie groups. *Ann. of Math.* *106* (1977) 295–308.
[2] J.N. BERNSTEIN, I.M. GELFAND and S.I. GELFAND: The structure of representations generated by vectors of highest weight, *Funk. Anal. Appl.* *51* (1971), N1, 1–9 (Russian).
[3] W. BORHO and J.C. JANTZEN: Uber primitive Ideale in der Einhullenden einer halbeinfachen Lie-Algebra. *Inventiones Math.* *39* (1977) 1–53.
[4] N.R. WALLACH: On the Enright–Varadarajan modules, *Ann. Sci. Ecole Norm. Sup.* *9* (1976) 81–101.
[5] A.W. KNAPP and N.R. WALLACH: Szegö kernels associated with discrete series, *Inventiones Math.* *34* (1976) 163–200.
[6] W. SCHMIDT: L^2-cohomology and the discrete series, *Ann. of Math.* *103* (1976) 375–394.
[7] M.F. ATIYAH and W. SCHMIDT: A geometric construction of the discrete series for semisimple Lie groups, *Inventiones Math.* *42* (1977) 1–62.
[8] B. KOSTANT: On tensor product of a finite and infinite dimensional representation. *Jour. Func. Anal. v. 20 N 4* (1975) 257–285.
[9] M. DUFLO: Sur la classification des ideaux primitifs dans l'algèbre envelloppante d'une algèbre de Lie semi-simple. *Ann. of Math.* *105* (1977) 107–120.
[10] D.P. ZHELOBENKO: *Harmonic analysis on semisimple complex Lie groups*, Moscow, "Nauka", 1974 (Russian).
[11] M. DUFLO: Représentations irréductibles des groupes semisimples complexes, *Lecture Notes in Math.* *497* (1975) 26–88.
[12] M. DUFLO and N. CONZE-BERLINE: Sur les représentations induites des groupes semi-simple complexes, *Compositio Math.* *34* (1977) 307–336.
[13] H. BASS: *Algebraic K-theory*, Benjamin, 1968.
[14] B. MITCHELL: *Theory of categories*, Academic Press, N.Y., 1965.
[15] S. MACLANE: *Categories for the Working Mathematician*, Springer, New York–Berlin–Heidelberg, 1971.
[16] J. DIXMIER: *Algèbres envelloppantes.* Paris, Gauthier-Villars, 1974.
[17] N. BOURBAKI: *Groupes et algèbres de Lie, Ch. 4, 5, 6.* Paris, Hermann, 1968.
[18] B. KOSTANT: Lie group representations on polynomial rings, *Amer. J. Math.* *81* (1959) 937–1032.
[19] J.N. BERNSTEIN, I.M. GELFAND and S.I. GELFAND: On a category of G-modules. *Funk. Anal. Appl.* *10* (1976) N2, 1–8 (Russian).
[20] B. KOSTANT: A formula for the multiplicity of a weight, *Trans. Amer. Math. Soc.* *93* (1959) 53–73.
[21] J. LEPOWSKY and N.R. WALLACH: Finite- and infinite-dimensional representations of linear semisimple groups, *Trans. Amer. Math. Soc.* *184* (1973) 223–246.
[22] D. VOGAN: Irreducible Characters of Semisimple Lie Groups I, *Duke Math. J.* *46* (1979) 61–00.
[23] J.C. JANTZEN: Moduln mit einem höchsten Gewicht, *Habilitationsschrift*, Universität Bonn, 1977.
[24] T. ENRIGHT: On the Irreducibility of the Fundamental Series of a Real semisimple Lie Algebra, *Ann. of Math.* *110* (1979) 1–82.
[25] T. ENRIGHT: The Representations of Complex Semisimple Lie Groups (preprint).
[26] A. JOSEPH: Dixmier's Problem for Verma and Principal Series Submodules (preprint), December, 1978.

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