Tensor products of spherical and equivariant immersions

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In [C1, C2, C3, C4], B.-Y. Chen introduced the tensor product of two immersions of a given Riemannian manifold; he proved that the set of all immersions of the given manifold, provided with direct sum and tensor product, defines a commutative semiring.

In [DDVV] we introduced $\mathcal{I}$, the commutative semiring of all transversal immersions of all differentiable manifolds in Euclidean spaces, provided with the binary operations direct sum and tensor product. In this paper we further investigate which immersions define a subsemiring or a multiplicative subsemigroup ; in particular, we fix our attention on spherical immersions of differentiable manifolds, isometric and equivariant immersions of Riemannian manifolds and immersions of finite type.

Denote by $\mathbb{E}^n$ the n-dimensional Euclidean space with Euclidean metric $\langle\ ,\ \rangle$. The n-dimensional sphere with radius r is denoted by $S^n(r)$. Let $f : M \to \mathbb{E}^m$ be an immersion of a differentiable manifold in a Euclidean space. Then f is said to be *transversal* in a point $p \in M$ if and only if the position vector $f(p)$ is not tangent to M at p, i.e. $f(p) \notin f_*(T_pM)$. If f is transversal in every point of M, then f shortly is called transversal. Consider two differentiable manifolds M and N of dimensions r resp. s and assume that $f : M \to \mathbb{E}^m$ and $h : N \to \mathbb{E}^n$ are two transversal immersions. Then the direct sum map $f \oplus h : M \times N \to \mathbb{E}^{m+n} : (p, q) \mapsto (f(p), h(q))$ and the tensor product map $f \otimes h : M \times N \to \mathbb{E}^{mn} : (p, q) \mapsto f(p) \otimes h(q)$ are again two transversal immersions. We define a symmetric relation $\sim$ as follows : if $f : M \to \mathbb{E}^m$ is an immersion and $i : \mathbb{E}^m \subset \mathbb{E}^n$ is a linear isometric immersion, then

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Received by the editors November 1993
Communicated by A. Warrinier
AMS Mathematics Subject Classification : 53C40, 53B25, 58G25

$f \sim i(f)$; if $f : M \to \mathbb{E}^m$ is an immersion and $d : N \to M$ is a diffeomorphism, then $f \sim f \circ d$. Then $\sim$ is an equivalence relation on the set of all transversal immersions Z and we put $\mathcal{I} = \mathcal{Z}/\sim$. It is proved in [DDVV] that $\mathcal{I}, \oplus, \otimes$ is a commutative semiring, with $e : \{0\} \to \mathbb{E}^1 : 0 \mapsto 1$ as identity element.

Theorem 1.

- *The set of all spherical immersions of all differentiable manifolds induces a subsemiring $\mathcal{I}_\int$ of $\mathcal{I}, \oplus, \otimes$;*
- *the tensor product of two spherical immersions f and h of Riemannian manifolds M and N is an isometric immersion of the Riemannian product $M \times N$ if and only if $f(M)$ is contained in a sphere of radius r, $h(N)$ is contained in a sphere of radius r', f is a homothetical immersion with factor $1/r'$ and h is a homothetical immersion with factor $1/r$*
- *the tensor product of two spherical isometric immersions f and h of Riemannian manifolds M and N such that both $f(M)$ and $h(N)$ are contained in a sphere of radius 1, is an isometric immersion of the Riemannian product $M \times N$*
- *if f and g are two homothetical spherical immersions in a sphere of radius 1 with the same factor λ, then $f \otimes g$ is again a homothetical immersion of the Riemannian product with factor λ.*

Corollary. The set of all spherical immersions of all differentiable manifolds into a unit sphere induces a subsemigroup $\mathcal{I}_{\sqcap\int}, \otimes$ of $\mathcal{I}, \otimes$.

Proof Let $f : M \to \mathbb{E}^m$ and $h : N \to \mathbb{E}^n$ be immersions such that $f(M) \subseteq S^{m-1}(r)$ and $h(N) \subseteq S^{n-1}(s)$. Then, obviously, f and h are transversal immersions such that $f \otimes h$ is also an immersion. It easily follows that

$$(f \oplus h)(M \times N) \subseteq S^{m+n-1}(\sqrt{r^2 + s^2})$$

and that

$$(f \otimes h)(M \times N) \subseteq S^{mn-1}(rs).$$

In order to prove (2), we assume that (M, g) and (N, g') are Riemannian manifolds. Take $p \in M$ with $v_p, v'_p \in T_pM$ and $q \in N$ with $w_q, w'_q \in T_qN$. Then

$$\begin{aligned}
\langle (f \otimes h)_*(v_p, w_q), (f \otimes h)_*(v'_p, w'_q)\rangle &= \langle f(p) \otimes h_*(w_q) + f_*(v_p) \otimes h(q), \\
&\qquad f(p) \otimes h_*(w'_q) + f_*(v'_p) \otimes h(q)\rangle \\
&=\langle f(p) \otimes h_*(w_q), f(p) \otimes h_*(w'_q)\rangle + \langle f(p) \otimes h_*(w_q), f_*(v'_p) \otimes h(q)\rangle \\
&\quad + \langle f_*(v_p) \otimes h(q), f(p) \otimes h_*(w'_q)\rangle + \langle f_*(v_p) \otimes h(q), f_*(v'_p) \otimes h(q)\rangle \\
&=\langle f(p), f(p)\rangle\langle h_*(w_q), h_*(w'_q)\rangle + \langle f(p), f_*(v'_p)\rangle\langle h_*(w_q), h(q)\rangle + \\
&\quad \langle f_*(v_p), f(p)\rangle\langle h(q), h_*(w'_q)\rangle + \langle f_*(v_p), f_*(v'_p)\rangle\langle h(q), h(q)\rangle
\end{aligned} \tag{*}$$

If $\langle f(p), f(p)\rangle = r^2$ and $\langle h(q), h(q)\rangle = {r'}^2$, then

$$\begin{aligned}
\langle (f \otimes h)_*(v_p, w_q), (f \otimes h)_*(v'_p, w'_q)\rangle &=\langle f_*(v_p), f_*(v'_p)\rangle {r'}^2 + \langle h_*(w_q), h_*(w'_q)\rangle r^2 \\
&=g(v_p, v'_p) + g'(w_q, w'_q),
\end{aligned}$$

and thus $f \otimes h$ is an isometric immersions. The converse can be proved in the same way, based on $(*)$. (3) is an immediate consequence and (4) can be proved similarly as (2). $\square$

If (M, g) is a compact Riemannian homogeneous manifold, denote by G the identity component of the group of all isometries of M. Then G is a compact Lie group which acts transitively on M. Thus $M = G/K$, where K is the isotropy subgroup of G at a point $x \in M$.

Definition. An (isometric) immersion $f : (M, g) \to \mathbb{E}^m$ is said to be *equivariant* if and only if there exists a Lie homomorphism $\psi : G \to SO(m)$ such that $f(q(p)) = \psi(q)(f(p)), \forall q \in G$ and $p \in M$.

Theorem 2. *The set $\mathcal{E}$ of all isometric equivariant transversal immersions of all compact homogeneous Riemannian manifolds induces a subsemiring of $\mathcal{I}, \oplus, \otimes$.*

Proof. Let $f : (M, g) \to \mathbb{E}^m$ and $h : (M', g') \to \mathbb{E}^{m'}$ two isometric equivariant transversal immersions of the compact homogeneous Riemannian manifolds (M, g) and (M', g'). Denote by G (resp. G') the identity component of the group of all isometries of M (resp. M').

Then there exists $\psi : G \to SO(m)$ (resp. $\psi' : G' \to SO(m')$) such that

$$\begin{cases} f(q(\alpha)) = \psi(q)(f(\alpha)), & \forall \alpha \in M, q \in G \\ h(q'(\beta)) = \psi'(q')(h(\beta)), & \forall \beta \in M', q' \in G'. \end{cases}$$

Define

$$\begin{cases} \psi_\oplus : G \times G' \to SO(m + m') \\ \psi_\otimes : G \times G' \to SO(mm') \end{cases}$$

by

$$\begin{cases} \psi_\oplus(q, q') = (\; \psi(q) \quad 00 \quad \psi'(q') \;) \\ \psi_\otimes(q, q') = \psi(q) \otimes \psi(q'). \end{cases}$$

Then we have

$$\begin{aligned} (f \oplus h)((q, q')(\alpha, \beta)) &= (f(q(\alpha)), h(q'(\beta))) \\ &= (\psi(q)(f(\alpha)), \psi'(q')(h(\beta)) \\ &= \psi_\oplus(q, q')(f(\alpha), h(\beta)) \\ &= \psi_\oplus(q, q')[(f \oplus h)(\alpha, \beta)] \end{aligned}$$

$$\begin{aligned} (f \otimes h)((q, q')(\alpha, \beta)) &= f(q(\alpha)) \otimes h(q'(\beta)) \\ &= \psi(q)(f(\alpha)) \otimes \psi'(q')(h(\beta)) \\ &= \psi_\otimes(q, q')[(f \otimes h)(\alpha, \beta)] \end{aligned}$$

Thus, $f \oplus h$ and $f \otimes h$ are isometric equivariant immersions. $\square$

We now concentrate on the tensor product of spherical isometric immersions (with spherical it is meant that the image is contained in a sphere of radius 1).

Lemma A. *Assume that $f : (M, g) \to \mathbb{E}^m$ and $h : (N, g') \to \mathbb{E}^n$ are two spherical isometric immersions with $\mathrm{Im}(f) \subseteq S^{m-1}(R)$ and $\mathrm{Im}(h) \subseteq S^{n-1}(R')$. Let $p \in M$ and $q \in N$. Suppose $\dim M = r$ with basis of T_pM given by $e_1, \ldots, e_r$ and let a basis of the normal bundle of $f(M)$ in $S^{m-1}(R)$ at p be given by $\xi_1, \xi_2, \ldots, \xi_{m-r-1}$; suppose $\dim N = s$ with basis of T_qN given by $f_1, \ldots, f_s$ and let a basis of the normal bundle of $h(N)$ in $S^{n-1}(R')$ at q be given by $\zeta_1, \zeta_2, \ldots, \zeta_{n-s-1}$. Then a basis of $(f \otimes h)_*(T_{(p,q)}(M \times N))$ is given by*

$$\{f(p) \otimes h_*(f_j), f_*(e_i) \otimes h(q) | 1 \le i \le r, 1 \le j \le s\}.$$

A basis of the normal space of $(f \otimes h)(M \times N)$ in $S^{mn-1}(RR')$ at (p, q) is given by

$$\begin{aligned}
\{f_*(e_i) \otimes h_*(f_j) &, 1 \le i \le r, 1 \le j \le s;\\
f_*(e_i) \otimes \zeta_j &, 1 \le i \le r, 1 \le j \le n-s-1;\\
\xi_i \otimes h_*(f_j) &, 1 \le i \le m-r-1, 1 \le j \le s;\\
\xi_i \otimes \zeta_j &, 1 \le i \le m-r-1, 1 \le j \le n-s-1;\\
\xi_i \otimes h(q) &, 1 \le i \le m-r-1;\\
f(p) \otimes \zeta_j &, 1 \le j \le n-s-1\}.
\end{aligned}$$

Proof. The first part follows from [DDVV], Lemma 2. The second part is then obvious. □

Lemma B. *With the same notations as in Lemma A, put $R = R' = 1$, $X = (X_1, X_2)$, $Y = (Y_1, Y_2) \in T_{p,q}(M \times N)$. If the second fundamental forms of $f : M \to S^{m-1}(1)$ and $h : N \to S^{n-1}(1)$ are given by σ_f and σ_h, then the second fundamental form of $f \otimes h : M \times N \to S^{mn-1}$ is given by*

$$\sigma_{f\otimes h}(X, Y) = f_*(X_1) \otimes h_*(Y_2) + f_*(Y_1) \otimes h_*(X_2) + f(p) \otimes \sigma_h(X_2, Y_2) + \sigma_f(X_1, Y_1) \otimes h(q).$$

Proof. Let D denote the affine connection of Euclidean space, and let ∇^1 and ∇^2 denote the Levi Civita connections of M and N. Then

$$\begin{aligned}
D_X(f \otimes h)_*(Y) &= D_X(f(p) \otimes h_*(Y_2)) + D_X(f_*(Y_1) \otimes h(q))\\
&= D_{X_1} f(p) \otimes h_*(Y_2) + f(p) \otimes D_{X_2} h_*(Y_2)\\
&+ D_{X_1} f_*(Y_1) \otimes h(q) + f_*(Y_1) \otimes D_{X_2} h(q)\\
&= f_*(X_1) \otimes h_*(Y_2) + f(p) \otimes h_*(\nabla^2_{X_2} Y_2) + f(p) \otimes \sigma_h(X_2, Y_2)\\
&- \langle Y_1, Y_2\rangle f(p) \otimes h(q) + f_*(\nabla^1_{X_1} Y_1) \otimes h(q) + \sigma_f(X_1, Y_1) \otimes h(q)\\
&- \langle X_1, X_2\rangle f(p) \otimes h(q) + f_*(Y_1) \otimes h_*(X_2).
\end{aligned}$$

The result now follows from Lemma A.

In the following theorem we use the same notations as in Lemma A.

Theorem 3.

- *The set* $\mathcal{I}_{\Updownarrow\sqcap\int}$ *of all minimal spherical isometric immersions of all Riemannian manifolds into a unit sphere is a subsemigroup of* $\mathcal{I}_{\sqcap\int}, \otimes$*;*
- *The set* $\mathcal{I}_{\{\sqcup\sqcap\int}$ *of all finite type spherical isometric mmersions of all Riemannian manifolds into a unit sphere is a subsemigroup of* $\mathcal{I}_{\sqcap\int}, \otimes$*;*
- *The set* $\mathcal{I}_{\sqcup\nabla\int}$ *of all spherical isometric immersions of all manifolds into a sphere* S^{2m-1} *which are totally real with respect to at least one complex structure on* $\mathbb{E}^{2m}$ *is an ideal of* $\mathcal{I}_{\int}, \oplus, \otimes$*;*

Proof. Assume f and h are minimal spherical isometric immersions; then it is easy to see that the mean curvature vector of the tensor product immersion

$$H_{f\otimes h}(p,q) = \frac{1}{r+s}\left(\sum_{j=1}^{s}\sigma_{f\otimes h}((0,f_j),(0,f_j)) + \sum_{i=1}^{r}\sigma_{f\otimes h}((e_i,0),(e_i,0))\right).$$

Therefore, by Lemma B,

$$H_{f\otimes h}(p,q) = \frac{1}{r+s}\left(\sum_{j=1}^{s} f(p)\otimes\sigma_h(f_j,f_j) + \sum_{i=1}^{r}\sigma_f(e_i,e_i)\otimes h(q)\right)$$

and thus

$$H_{f\otimes h} = \frac{1}{r+s}(sf\otimes H_h + rH_f\otimes h).$$

It is now clear that if f and h are minimal, then $f\otimes h$ is minimal too.

In order to prove (2), assume that $f = f_1 + \ldots + f_t$ with $\Delta_f f_i = \lambda_i f_i (1 \le i \le t)$ and $h = h_1 + \ldots + h_u$ with $\Delta_h h_j = \mu_j h_j (1 \le j \le u)$. Then $f\otimes h = \sum_{i,j} f_i \otimes h_j$ with $\Delta(f_i\otimes h_j) = (\lambda_i + \mu_j)(f_i\otimes h_j)$ and thus $f\otimes h$ is of finite type $\le tu$.

For proving (3), we first remark that the direct sum of any two totally real immersions is again totally real. Next let $f : M \to S^{2m-1}$ be a totally real spherical immersion with respect to a complex structure J on $\mathbb{E}^{2m}$, and let $h : N \to S^{n-1}$ be a spherical immersion. We prove that $f\otimes h$ is totally real with respect to the complex structure $J\otimes I$ on $\mathbb{E}^{2mn}$. This follows immediately from

$$\begin{aligned}&\langle (J\otimes I)(f\otimes h)_*(v_p,w_q),(f\otimes h)_*(v'_p,w'_q)\rangle\\ &\quad=\langle Jf(p),f(p)\rangle\langle h_*(w_q),h_*(w'_q)\rangle + \langle Jf(p),f_*(v'_p)\rangle\langle h_*(w_q),h(q)\rangle\\ &\quad+\langle Jf_*(v_p),f(p)\rangle\langle h(q),h_*(w'_q)\rangle + \langle Jf_*(v_p),f_*(v'_p)\rangle\langle h(q),h(q)\rangle = 0.\square\end{aligned}$$

Examples. The tensor product of two circles of radius 1 is a flat surface of type 1 in a 3-dimensional unit sphere. In particular, it is the Clifford torus. The tensor product of a circle of radius 1 and a small circle on a 2-dimensional unit sphere is a flat surface of type 2. The tensor product of two small circles on a 2-dimensional unit sphere is a flat surface of type 3. In general, the tensor product of two curves of finite type on a unit sphere (cf. [CDDVV]) is a flat torus of finite type.

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