ILLINOIS JOURNAL OF MATHEMATICS
Volume 31, Number 1, Spring 1987

TENSOR PRODUCTS OF TSIRELSON'S SPACE

BY

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Tsirelson's space T has attracted considerable interest during the past few years, somewhat eclipsing the original space T^* discovered in 1973 by B. S. Tsirelson [12]. However, in [1], the first two authors and Dineen showed that T^* held the greater interest, from the point of view of holomorphic functions. Specifically, the main result of [1] is that for all positive integers n, $P(^nT^*)$ is reflexive. As a consequence, it is shown that the space $(H(T^*), \tau_\omega)$ of complex-valued holomorphic functions on T^*, endowed with the Nachbin ported topology, is reflexive. Here, we continue our study of multilinear properties of T^* by showing that $P(^nT^*)$ is "Tsirelson-like", in the sense that it is reflexive, with (not unconditional) basis, and contains no l_p space for $1 < p < \infty$. In fact, our method of proof enables us to prove that $(H(T^*, l_p), \tau_\omega)$ and $P(^nT^*, l_p)$ are reflexive for all $n = 1, 2, \ldots$ and all p, $1 < p < \infty$.

Our notation and terminology will follow the earlier paper [1]. Given Banach spaces X and Y, $L(^nX, Y)$ is the Banach space of continuous n-linear mappings $A\colon X \times \cdots \times X \to Y$, with norm

$$\|A\| = \sup\left\{\|A(x_1, \ldots, x_n)\|\colon x_j \in X, \|x_j\| \leq 1, 1 \leq j \leq n\right\}.$$

$L(^nX)$ denotes $L(^nX, K)$ where $K = R$ or C. An important observation for us will be the fact that $L(^nX, Y)$ is isometrically isomorphic to the space $L(\hat{\otimes}_\pi^n X, Y)$ of linear mappings between the n-fold completed projective tensor product of X with itself and Y. Similarly the space $L_s(^nX, Y)$ of symmetric continuous n-linear mappings is isometrically isomorphic to the space $L(ⓢ^n X, Y)$, where $ⓢ^n X$ is the symmetric n-fold completed projective tensor product of X with itself. $L_s(^nX, Y)$ is also isomorphic to the Banach space $P(^nX, Y)$ of n-homogeneous continuous polynomials from X to Y, where each element $P \in P(^nX, Y)$ is defined as $P(x) = A(x, \ldots, x)$ for a unique element $A \in L_s(^nX, Y)$. For basic properties of tensor products, we refer to [3] (See also [11]). See [4] for any unexplained notation and definitions from infinite dimensional holomorphy.

Received November 26, 1984.

[1]Research supported in part by FAPESP and CNPG-(Brazil), when the author was visiting Kent State University, Kent, Ohio.

Our proof that $P(^nT^*)$ is Tsirelson-like will show somewhat more. Specifically, our main result is that for every $p \in (1, \infty)$, every linear continuous mapping R: $\hat{\otimes}_\varepsilon^n T \to l_p$ is compact where $\hat{\otimes}_\varepsilon^n T$ is the completed n-fold injective tensor product of T with itself. We will show how this implies that $P(^nT^*)$ is Tsirelson-like and also derive other consequences of this result for spaces of polynomials and analytic functions. A basic tool which we use is a lemma which states that if X is a Banach space such that every continuous linear operator from X to l_p is compact, then every continuous linear operator from X to l_q is compact for all $q < p$. We recall the classical result (for example, see [10]) that every continuous linear operator from l_p to l_q is compact, whenever $q < p$. Therefore it is natural to ask whether the following more general result holds. Given three Banach spaces X, Y, and Z, such that all continuous linear operators from X to Y and from Y to Z are compact, does it follow that every continuous linear operator from X to Z is compact. At the end of this note, we give a counterexample due to J. Bourgain.

We begin by recalling the following result which is essentially proved in [1].

PROPOSITION 1. *$L(^nT^*)$ is reflexive for every $n \in N$.*

As a consequence, the isomorphic space $L(T^*, L(^{n-1}T^*))$ of linear mappings of T^* to $L(^{n-1}T^*)$ is reflexive. Since all spaces involved here have the approximation property and T is reflexive, we conclude that every such linear mapping is compact and therefore $L(^nT^*) \cong T \hat{\otimes}_\varepsilon L(^{n-1}T^*)$. Continuing by induction, we see that $L(^nT^*) \cong \hat{\otimes}_\varepsilon^n T$. Note that by the defining property of the projective tensor product, $L(^nT^*)$ is also isomorphic to $(\otimes_\pi^n T^*)^*$. Also it is well known [6] that the completed injective tensor product of Banach spaces with basis has a basis.

LEMMA 2. *Every continuous linear operator S: $L(^nT^*)^* \to l_1$ is compact.*

Proof. Let (x_j) be an arbitrary bounded sequence in $(L^nT^*)^*$. Without loss, we may assume that (x_j) converges weakly to a point x_0 since $L(^nT^*)$ is reflexive. Therefore (Sx_j) converges weakly, and hence in norm, to Sx_0 in l_1, which completes the proof. Q.E.D.

LEMMA 3. *Let P: $L(^nT^*)^* \to l_1$ be a continuous k-homogeneous polynomial. Then P is compact; that is, P takes bounded subsets of $L(^nT^*)^*$ to relatively compact subsets of l_1.*

Proof. Let A be the symmetric k-linear mapping associated to P,

$$A: \mathop{\times}_{1}^{k} L(^nT^*)^* \to l_1,$$

where $\overset{k}{\underset{1}{\times}} E$ denotes the product of E with itself k times. Using the reflexivity of $L(^nT^*)$, we see that A is a k-linear mapping,

$$A\colon \overset{k}{\underset{1}{\times}} \left(\hat{\otimes}_\pi^n T^*\right) \to l_1.$$

As such, there is a unique continuous linear mapping associated to A,

$$\tilde{A}\colon \hat{\otimes}_\pi^k\left(\hat{\otimes}_\pi^n T^*\right) \to l_1.$$

However, the domain of $\tilde{A}$ is isomorphic to $L(^{nk}T^*)^*$, and so $\tilde{A}$ is compact by Lemma 2. Hence A and P are compact. Q.E.D.

LEMMA 4. *Let $q \in N$ and let S: $L(^nT^*)^* \to l_q$ be a continuous linear mapping. Then S is compact.*

Proof. Define P_q: $l_q \to l_1$ by $P_q(x) = (x_1^q, x_2^q, \dots)$. It is not difficult to show that a bounded set C in l_q is relatively compact if and only if $P_q(C)$ is relatively compact in l_1. Using this, let us assume that $S(B)$ is not relatively compact, where B is the unit ball of $L(^nT^*)^*$. But then $P_q \circ S$: $L(^nT^*)^* \to l_1$ is a q-homogeneous non-compact polynomial, contradicting Lemma 3. Q.E.D.

An immediate consequence of Lemma 4 is that $L(^nT^*)$ contains no isomorphic copy of l_p for any $p > 1$. Indeed, if $L(^nT^*)$ contained an isomorphic copy of some l_p, then the adjoint R of this isomorphism R: $L(^nT^*)^* \to l_{p'}$, would be surjective, where $1/p + 1/p' = 1$. But then if q is any integer larger than p', $i \circ R$: $L(^nT^*)^* \to l_q$ would have dense range, contradicting Lemma 4. However, in order to obtain the stronger result mentioned in the introduction, we shall need to extend Lemma 4 to the case of all real numbers $q > 1$, using a sliding hump argument.

LEMMA 5. *Suppose a Banach space X has the property that for some $p > 1$, $L(X, l_p) = K(X, l_p)$. Then $L(X, l_q) = K(X, l_q)$ for all $q \in [1, p]$. Here, $K(X, l_p)$ denotes the compact linear operators from X to l_p.*

Proof. If the conclusion is false then for some $q, 1 \leq q < p$, there is a non-compact linear operator $S \in L(X, l_q)$, and so there is a bounded sequence (c^j) in $S(X_1)$ with no convergent subsequence. (Here, $X_1 = \{x \in X$: $\|x\| \leq 1\}$. Also, for each point $y \in l_q$ and each integer k,

$$\Pi^k(y) = (y_1, \dots, y_k, 0, 0, \dots) \in l_q\big).$$

Without loss of generality, we may assume that for some $\delta > 0$, $\|c^j - c^k\|_q >$

2δ whenever $j \neq k$. By a diagonal process, we may assume further that for each n, $(c_n^j)_j$ converges to some number c_n. Therefore, taking $b^j = c^j - c^{j+1}$, we may assume that each b_j is in $S(X_1), 2\delta \leq \|b^j\|_q \leq 1$, and $b_n^j \to 0$ as $j \to \infty$, for each n. We claim that there are increasing sequences $(j_n), (k_n)$ such that for all n,

$$(*) \qquad \|(\Pi^{k_{n+1}} - \Pi^{k_n})(b^{j_n})\|_q > \delta.$$

Indeed, since $\Pi^n(b^1) \to b^1$ as $n \to \infty$, there is some $k_1 \in N$ such that $\|\Pi^{k_1}(b^1)\|_q > 3\delta/2$. Let $j_1 = 1$. Choose $j_2 \in N$ such that $\|\Pi^{k_1}(b^{j_2})\|_q < \delta/2$. Next, choose $k_2 \in N$, $k_2 > k_1$, such that $\|\Pi^{k_2}(b^{j_2})\|_q > 3\delta/2$. Hence

$$\|(\Pi^{k_2} - \Pi^{k_1})(b^{j_2})\|_q \geq \|\Pi^{k_2}(b^{j_2})\|_q - \|\Pi^{k_1}(b^{j_2})\|_q > 3\delta/2 - \delta/2 = \delta.$$

Continuing this process, we find the required sequences $(j_n), (k_n)$ satisfying $(*)$.

Define T: $l_q \to l_p$ by $T(x) = (T_n(x))_n$, where

$$T_n(x) = \sum_{i=k_n+1}^{k_{n+1}} \overline{b_i^{j_n}} |b_i^{j_n}|^{q-2} x_i.$$

Note that by Hölder's inequality,

$$\begin{aligned} \sum_{i=k_n+1}^{k_{n+1}} |b_i^{j_n}|^{q-1}|x_i| &\leq \left(\sum_{i=k_n+1}^{k_{n+1}} \left(|b_i^{j_n}|^{q-1}\right)^{q'} \right)^{1/q'} \left(\sum_{i=k_n+1}^{k_{n+1}} |x_i|^q \right)^{1/q} \\ &= \left(\sum_{i=k_n+1}^{k_{n+1}} |b_i^{j_n}|^q \right)^{1/q'} \left(\sum_{i=k_n+1}^{k_{n+1}} |x_i|^q \right)^{1/q} \\ &\leq \left(\sum_{i=k_n+1}^{k_{n+1}} |x_i|^q \right)^{1/q} \end{aligned}$$

since we always have $\|b^j\|_q \leq 1$. Therefore,

$$\begin{aligned} \|Tx\|_p^p &= \sum_{n=1}^{\infty} \left| \sum_{i=k_n+1}^{k_{n+1}} b_i^{j_n} |\bar{b}_i^{j_n}|^{q-2} x_i \right|^p \\ &\leq \sum_{n=1}^{\infty} \left[\sum_{i=k_n+1}^{k_{n+1}} |x_i|^q \right]^{p/q}. \end{aligned}$$

Since $p \geq q$, we see that $\|Tx\|_p^p \leq 1$ and so T is a continuous linear operator.

Also, for each fixed r, and $m > r$,

$$\|T(b^{j_m}) - T(b^{j_r})\|_p^p \geq |T_r(b^{j_m} - b^{j_r})|^p$$
$$= \left| \sum_{i=k_r+1}^{k_{r+1}} \left(\bar{b}_i^{j_r} |b_i^{j_r}|^{q-2} b_i^{j_m} - |b_i^{j_r}|^q \right) \right|^p .$$

Since $b_i^{j_m} \to 0$ as $j_m \to \infty$ for all i, there is $m_0 > r$ such that

$$\left| \bar{b}_i^{j_r} |b_i^{j_r}|^{q-2} b_i^{j_m} \right| \leq \frac{1}{2} |b_i^{j_r}|^q, \; k_r + 1 \leq i \leq k_{r+1} \quad \text{for all } m \geq m_0.$$

Therefore

$$\|T(b^{j_m}) - T(b^{j_r})\|_p^p \geq \left| \sum_{i=k_r+1}^{k_{r+1}} |b_i^{j_r}|^q - \sum_{i=k_r+1}^{k_{r+1}} \bar{b}_i^{j_r} |b_i^{j_r}|^{q-2} b_i^{j_m} \right|^p$$
$$\geq \left(\sum_{i=k_r+1}^{k_{r+1}} |b_i^{j_r}|^q - \left| \sum_{i=k_r+1}^{k_{r+1}} \bar{b}_i^{j_r} |b_i^{j_r}|^{q-2} b_i^{j_m} \right| \right)^p$$
$$\geq \left(\sum_{i=k_r+1}^{k_{r+1}} |b_i^{j_r}|^q - \sum_{i=k_r+1}^{k_{r+1}} |\bar{b}_i^{j_r}| \, |b_i^{j_r}|^{q-2} |b_i^{j_m}| \right)^p$$
$$\geq \left(\sum_{i=k_r+1}^{k_{r+1}} |b_i^{j_r}|^q - \frac{1}{2} \sum_{i=k_r+1}^{k_{r+1}} |b_i^{j_r}|^q \right)^p$$
$$= \frac{1}{2^p} \left(\sum_{i=k_r+1}^{k_{r+1}} |b_i^{j_r}|^q \right)^p$$
$$= \frac{1}{2^p} \|(\Pi^{k_{r+1}} - \Pi^{k_r+1})(b^{j_r})\|_q^{pq}$$
$$> \frac{\delta^{qp}}{2^p} \quad \text{for all } m \geq m_0.$$

Consequently we can find a set $N_1 \subset \mathbf{N}$ and a constant c such that

$$\|T(b^{j_m}) - T(b^{j_k})\|_p > c \quad \text{for all } m, k \in N_1, m \neq k.$$

Thus, $\{T(b^{j_m})\colon m \in N_1\}$ is not relatively compact in l_p, and so $T \circ S \notin K(X, l_p)$, a contradiction. Q.E.D.

Now, if $L({}^nT^*)$ contained an isomorphic copy of some l_p, then the adjoint of the inclusion mapping would be a continuous linear surjection of $L({}^nT^*)^*$

onto $l_{p'}$. However, Lemmas 4 and 5 show that every linear operator from $L(^nT^*)^*$ to $l_{p'}$ is compact. Thus we have proved the following.

THEOREM 6. *The space $L(^nT^*)$ is a reflexive Banach space with basis which does not contain an isomorphic copy of any l_p space.*

The next result has both good and bad aspects, since although it shows that $L(^nT^*)$ is not quite as "good" as Tsirelson's space T, it also proves that it cannot be isomorphic to it.

PROPOSITION 7. *$L(^nT^*)$ does not have an unconditional basis for any $n > 1$.*

Proof. By [12], T^* is finitely universal and thus is sufficiently Euclidean [7, p. 37]. By [7, 3.4], $(T^* \hat{\otimes}_\pi T^*)^* = L(^2T^*)$ does not have local unconditional structure, and in particular, $L(^2T^*)$ cannot have an unconditional basis. In general, since T^* is a complemented subspace of $E = \hat{\otimes}_\pi^n T^*$, E is sufficiently Euclidean. Applying [7, 3.4] again, we conclude that $(E \hat{\otimes}_\pi T^*)^* = L(^{n+1}T^*)$ does not have local unconditional structure, and so does not have an unconditional basis. Q.E.D.

COROLLARY 8. *For all $n \in N$ and $p \in (1, \infty)$, $L(^nT^*, l_p)$ is reflexive.*

Proof. This is a simple consequence of the proof of Theorem 6. Indeed

$$L(^nT^*, l_p) = L\left(\hat{\otimes}_\pi^n T^*, l_p\right) = L\left(L(^nT^*)^*, l_p\right)$$

by the defining property of the projective tensor product and the above remarks. Since both factors are reflexive and have the approximation property, an application of the above lemmas and [8] completes the proof. Q.E.D.

Corollary 8 implies an improvement of the main result of [1].

COROLLARY 9. *For all $p \in (1, \infty)$, $(H(T^*, l_p), \tau_w)$ is reflexive.*

Proof. The proof is an immediate application of [5]. Indeed, by Corollary 8, $P(^nT^*, l_p)$ is reflexive for every n, since this space is a complemented subspace of $L(^nT^*, l_p)$. Since $(H(T^*, l_p), \tau_w)$ is barreled and $(P(^nT^*, l_p))_{n=0}^\infty$ is a shrinking equi-Schauder decomposition of $(H(T^*, l_p), \tau_w)$, an application of [5, cf. 9] completes the proof. Q.E.D.

Finally, we remark that Lemma 5 shows that there are non-trivial examples of triples (X, Y, Z) of Banach spaces with the property that if every continuous linear operator from X to Y is compact and if every continuous linear operator from Y to Z is compact, then every continuous linear operator from X to Z is compact. We are grateful to J. Bourgain for showing us that such a transitive relation fails in general. Indeed, if one takes $X = Z = l_2$, and Y the space of Bourgain-Delbaen [cf. 2] with $\alpha = 2/3$, then every operator from X to Y and from Y to X is compact.

BIBLIOGRAPHY

1. R. ALENCAR, R. ARON and S. DINEEN, *A reflexive space of holomorphic functions in infinitely many variables*, Proc. Amer. Math. Soc., vol. 90 (1984), pp. 407–411.
2. J. BOURGAIN and F. DELBAEN, *A class of special $\mathscr{L}_\infty$-spaces*, Acta. Math. vol. 145 (1981), pp. 155–176.
3. J. DIESTEL and J.J. UHL, *Vector measures*, Math. Surveys, no. 15, Amer. Math. Soc., Providence, R.I., 1977.
4. S. DINEEN, *Complex analysis in locally convex spaces*, Math. Studies, no. 57, North Holland, 1981.
5. S. DINEEN, *Locally convex topologies on $H(U)$*. Ann. Inst. Fourier (Grenoble), vol. 23 (1973), pp. 19–54.
6. B.R. GELBAUM and J. GIL DE LAMADRID, *Bases of tensor products of Banach spaces*, Pacific J. Math., vol. 11, (1961), pp. 1281–1286.
7. Y. GORDON and D.R. LEWIS, *Absolutely summing operators and local unconditional structures*, Acta Math., vol. 133 (1974), pp. 27–48.
8. J.R. HOLUB, *Reflexivity of $L(E, F)$*, Proc. Amer. Math. Soc., vol. 39 (1973), 175–177,
9. N. KALTON, *Schauder decompositions in locally convex spaces*, Proc. Cambridge Philos. Soc., vol. 68 (1970), p. 377.
10. H.R. PITT, *A note on bilinear forms*, J. London Math. Soc., vol. 11 (1936), pp. 174–180.
11. R.A. RYAN, *Applications of topological tensor products to infinite dimensional holomorphy*, Thesis, Trinity College, Dublin, 1980.
12. B.S. TSIRELSON, *Not every Banach space contains an imbedding of l_p or C_0*, Functional Anal. Appl., vol. 8 (1974), pp. 138–141.

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