# Termination, Deadlock, and Divergence

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Abstract. In this paper, a process algebra that incorporates explicit representations of successful termination, deadlock, and divergence is introduced and its semantic theory is analyzed. Both an operational and a denotational semantics for the language is given and it is shown that they agree. The operational theory is based upon a suitable adaptation of the notion of bisimulation preorder. The denotational semantics for the language is given in terms of the initial continuous algebra that satisfies a set of equations $E$, $CI_E$. It is shown that $CI_E$ is fully abstract with respect to our choice of behavioral preorder. Several results of independent interest are obtained; namely, the finite approximability of the behavioral preorder and a partial completeness result for the set of equations $E$ with respect to the preorder.

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## 1. *Introduction*

In this paper, we wish to develop a theory for a process algebra that incorporates some explicit representation of termination, deadlock, and divergence. We develop both an operational theory based on bisimulations, [30] and an equational theory similar to those for **CCS, ACP**, [8, 19, 20].

The theory of **ACP** [7, 8] deals with deadlock explicitly by introducing into the signature of the calculus a distinguished constant symbol $\delta$. Deadlock can also occur directly in processes. If $p$ can only perform actions from the set $H$, then the process $\partial_H(p)$ is considered to be the same as the deadlocked process $\delta$. But **ACP**, at least in its original formulation, does not have an explicit representation of successful termination.

On the other hand, **CCS** [26] has a single "terminated" process, *nil*, which stands for both successful termination and deadlock. This choice is justified by the fact that in **CCS** these two kinds of termination are experimentally indistinguishable, due to the restricted form of sequential composition, *action-prefixing*, present in the calculus. Since **ACP** allows sequential composition, this is no longer the case. Consider, for example, the process *nil*; $p$, where *nil*

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is now used to denote a successfully terminated process. Then, since *nil* is successfully terminated, *nil*; $p$ can perform any action which $p$ may perform. On the other hand, it is natural to assume that the process $\delta$; $p$ is deadlocked and will never perform any action. Thus, in the presence of sequential composition, there is an observable difference between the successfully terminated process *nil* and $\delta$.

One may express desirable properties of processes by means of equations. For example,

$$\delta; x = \delta$$

represents the fact that a deadlocked process can never proceed, and

$$nil; x = x$$

the fact that *nil* is a properly terminated process. Equational laws play a central role in the theory of **ACP**. **ACP** aims at isolating axioms expressing some a priori desirable properties that communicating systems should enjoy. A semantics for the resulting equational theory may then be obtained by constructing models for it, see, e.g., [8] and [9]; thus, establishing its logical consistency. In this paper, following previous work in the **CCS** literature [20, 22] we derive the equations for our language from an operational view of processes. In this approach, the emphasis is on operational semantics as a framework within which different intuitions about the behavior of processes may be discussed and compared. Sets of equations, for instance, complete equational characterizations of some notion of behavioral equivalence over processes, may then be derived from and justified using the operational semantics. In the remainder of the paper, we examine the equational theory induced by one possible choice of operational semantics for our language; a more detailed comparison between our approach and the philosophy underlying **ACP** may be found in the conclusions, where possible modifications to our operational semantics in order to obtain an equational theory closer to the **ACP** one are also discussed.

Many of the equations for our language are already well known either from **CCS** or **ACP**. However, the presence of the terminated process *nil* invalidates some of those from **ACP**. The equation

$$(x + y); z = x; z + y; z$$

is part of the theory of **ACP**, [8], but is not valid for our language, at least in its general form. In fact, if $x$ is $\delta$ and $y$ is *nil* then, assuming that $\delta + nil = \delta$, the left-hand side is equal to $\delta$; $z$, that is, $\delta$, whereas the right-hand side is equal to *nil*; $z + \delta$; $z$, that is, $z + \delta$. If $z$ is a nontrivial process, it is then reasonable to assume that $\delta$ and $z + \delta$ are different processes.

We also have within our language processes that may diverge internally. We let $\Omega$ be a process that can only diverge internally. Using the usual notation for recursive terms, this could also be represented by $rec\, x.\tau; x$, where $\tau$ is an internal unobservable move. The semantic identification of the totally undefined process $\Omega$ with the process that can only diverge internally $rec\, x.\tau; x$ is indeed open to debate. However, this choice may be supported both on behavioral and pragmatic grounds. In this paper, we follow Milner's experimental approach to the semantics of concurrent systems [26]. This approach is based upon the idea that two processes that cannot be distinguished by means of experimentation

based on observation should be deemed to be equivalent. With this in mind, it may be argued that the environment will never be able to elicit any information from a process that can only diverge internally by experimenting on it, that is, such a process contains no observable information. Following Scott's approach to semantics, a process that contains no information is considered less than any other process and thus identified with $\Omega$. A similar choice is present in the theory of denotational semantics for imperative sequential programming languages, where $\Omega$ is usually given the same denotation as the program **while** *true* **do** *skip* **od**. Such a program can only embark in infinite internal idling and as such represents a natural counterpart of the process $rec\,x.\tau;\ x$. Pragmatically, the choice of semantically identifying $\Omega$ and $rec\,x.\tau;\ x$ allows us to rely on the standard body of techniques of *continuous algebraic semantics* [13, 15], for instance, to give a denotational semantics for our language and provide powerful proof techniques for it. Our choice has some drawbacks in dealing with infinitary properties of processes such as *fairness*. However, a study of these properties is out of the scope of this paper and, in general, cannot be carried out within the framework of continuous semantics.

Obviously, we would expect *nil* and $\Omega$ to be different processes and we also demand that $\delta$ and $\Omega$ be different. The latter requirement is less defensible, but we are motivated by the *information-theoretic* view of computation as advocated by Scott [34]. Here the process that can only diverge ($\Omega$) contains no information and is therefore considered less than any other process. There is some information available about the process $\delta$; namely, that it is deadlocked; so $\Omega$ and $\delta$ should be considered different. In the presence of $\Omega$, and in particular taking Scott's approach to semantics, it is natural to express our theory in terms of *inequations*. One inequation is

$$\Omega \leq x,$$

and more generally the equations given above could be viewed as shorthand for two inequations, $t = u$ representing $t \leq u$ and $u \leq t$.

The main purpose of this paper is to show that an adequate semantic theory for a process algebra containing divergence, termination, and deadlock can be constructed using a suitable set of inequations, $E$. More specifically, we propose as a denotational semantics the initial continuous algebra generated by $E$, $CI_E$, [13, 15]. This is in contrast to [6], where metric spaces are used for this purpose in place of continuous partial orders. The advantage of the former is that all of the usual operators found in process algebra may be interpreted, whereas using metric spaces we can only readily interpret operators that are contractive. For instance, unguarded recursive definitions give rise to operators that are not contractive; in addition to this drawback, silent actions and abstraction operators have never been dealt with satisfactorily in this framework. Moreover, we can apply the existing and well-understood theory of algebraic continuous partial order's (cpo's); for example, to show the existence of $CI_E$ and to derive useful proof techniques such as Scott Induction [24].

In order to show that $CI_E$ is a reasonable model, we develop a behavioral or observational view of processes and prove that this coincides with the interpretation given by $CI_E$. This is given in terms of a variation on bisimulation equivalence [30]. To take divergence into account, we generalize bisimulation equivalence ($\approx$) to a preorder $\sqsupseteq$, which is often called pre-bisimulation preorder. Intuitively, $p \sqsupseteq q$ means that $p$ and $q$ are bisimilar except that at

times $p$ may diverge more frequently than $q$; in the absence of divergence $p \sqsubseteq q$ will imply $p \approx q$. This type of behavioral relation has been studied in [1], [2], [22], [27], and [36]. Here we modify it to take into consideration termination and deadlock and show that two processes are behaviorally related with respect to this new relation if and only if they are related in the equational model $CI_E$. In other words, $CI_E$ is *fully abstract* with respect to this new behavioral preorder. There may be other fully abstract models, but $CI_E$ is distinguished by being initial in the category of fully abstract models. In fact, it is initial in the category of models that are consistent with the behavioral preorder.

We now give a brief outline of the remainder of the paper. In Section 2, we define the language whose semantic properties will be investigated in the paper. The language is endowed with both an operational and a denotational semantics. The operational semantics is defined in Section 2.1 following standard lines by means of Plotkin's *Structural Operational Semantics* (**SOS**) [26, 32]. Section 2.1 also introduces several definitions and notational conventions which will be used throughout the paper. The denotational semantics for the language is given in Section 2.2. The definition is based on the well-known techniques of *Initial Algebra Semantics* [13, 15]; as already mentioned, we propose as a denotational model for the language the initial continuous $\Sigma$-algebra that satisfies a set of equations $E$, $CI_E$. The following sections are entirely devoted to showing that $CI_E$ is indeed a reasonable denotational model for our language. As argued by Milner [28], operational semantics should be the touchstone for assessing mathematical models for concurrent languages. The agreement between denotational models and operational ones is called *full abstraction* in [21], [25], and [31]. In this paper, we follow Milner and Plotkin's paradigm and justify the choice of our denotational model by showing that $CI_E$ is fully abstract with respect to a natural notion of an operational or behavioral preorder over our language. The behavioral preorder is introduced in Section 3, where several constraints that behavioral relations have to meet in order to be related to denotational ones are also discussed. In particular, it is argued in Section 3.1 that, in order to be related to $\leq_E$, a behavioral preorder should be *finitely approximable* [2, 16] and *closed with respect to all contexts*.

All the remaining sections of the paper are devoted to showing that our choice of a behavioral preorder, $\sqsubseteq_\omega^c$, possesses these two properties and coincides with $\leq_E$ over our language. Section 3.2 is devoted to an analysis of the preorder $\sqsubseteq_\omega$ and of its substitutive version $\sqsubseteq_\omega^c$. This analysis paves the way to the proof of our promised full abstraction result. The proof of full abstraction of $CI_E$ with respect to $\sqsubseteq_\omega^c$ over our language is outlined in Section 3.3 and relies on two main results:

—finite approximability of $\sqsubseteq_\omega^c$, and
—partial completeness of $\leq_E$ with respect to $\sqsubseteq_\omega^c$.

The proof of the partial completeness result is given in full detail in Section 3.3. It relies on the usual machinery used in the proofs of equational completeness for bisimulation-like relations [16, 19, 20, 36]. The proof of finite approximability of $\sqsubseteq_\omega^c$ occupies all of Section 4. It is given in two stages. The first, which is the topic of Section 4.1, consists of a modal characterization of the preorder $\sqsubseteq_\omega$ and is a simple adaptation of similar results present in the literature [2, 20, 27, 35]. The second employs this modal characterization to

prove that $\sqsubseteq_\omega$ is finitely approximable; this will allow us to conclude that $\sqsubseteq_\omega^c$ is finitely approximable as well.

We end with a conclusion in which we discuss the results of the paper and relationships with related work.

## 2. *The Language*

Let $Act$ be a countable set of atomic action symbols. It is assumed that $Act$ comes equipped with a bijection $\bar{\cdot}: Act \to Act$, which is its own inverse. The set $Act$ will be called the set of *observable actions* and will be ranged over by $a, b, \ldots$ . Let $\tau$ and $\delta$ be two distinguished symbols not occurring in $Act$. The symbol $\tau$ will stand for an internal, unobservable action; these actions will occur when processes communicate with each other. $Act_\tau =_{\text{def}} Act \cup \{\tau\}$ will be called the set of *actions* and will be ranged over by $\mu, \gamma \ldots$ . The symbol $\delta$ will stand for a *deadlocked process*, a process that cannot perform any move but is not successfully terminated. Successful termination will be denoted by the constant symbol *nil*.

The set of constant symbols in the process algebra we consider is completed by the symbol $\Omega$; as discussed in the introduction, $\Omega$ will stand for a process that can internally diverge. Alternatively, one may think of $\Omega$ as the totally undefined process, the process about which the environment has no information at all. $\Omega$ is not deadlocked and has not successfully terminated. The process combinators used to build new systems from existing ones will be the following:

— $+$ for *nondeterministic choice*,
— ; for *sequential composition*,
— $|$ for *parallel composition*,
— $\partial_H(\cdot)$ for the *encapsulation operator*. Intuitively, the process $\partial_H(p)$ behaves like $p$, but with actions in $H$ prohibited. A more detailed discussion of this operator may be found in, for example, [8].

Formally:

*Definition* 2.1. For each $n \in \omega$, let $\Sigma_n$, the set of operation symbols of arity $n$, be defined as follows:

$-\Sigma_0 = \{\text{nil}, \delta, \Omega\} \cup \text{Act}_\tau$
$-\Sigma_1 = \{\partial_H(\cdot) \mid H \subseteq \text{Act} \wedge H = \bar{H}\}$
$-\Sigma_2 = \{+, ;, |\}$
$-\Sigma_n = \emptyset$, for each $n > 2$.

The signature $\Sigma$ is defined as $\Sigma = \bigcup_{n \geq 0} \Sigma_n$.

Let $\mathtt{Var}$ be a countable set of variables, ranged over by $x, y, \ldots$ . The syntax of recursive terms over $\Sigma$ is then defined by

$$t ::= f(t_1, \ldots, t_k)\ (f \in \Sigma_k) \mid x \mid rec\, x.\ t.$$

We assume the usual notions of free and bound variables in terms, with $rec\, x.$ _ as the binding constructor. The set of recursive terms over $\Sigma$ will be denoted by $REC_\Sigma(\mathtt{Var})$ and will be ranged over by $t, u, \ldots$ . The set of closed recursive terms over $\Sigma$ will be denoted by $REC_\Sigma$ and will be ranged over by $p, q, p' \ldots$ . The set of syntactically finite processes (i.e., those not involving

occurrences of $rec\,x.t$) will be denoted by $FREC_\Sigma$ and will be ranged over by $d, e, d' \ldots$ .

Notationally, all the binary operators will be used in infix form, with the assumption that ; binds stronger than $|$, which in turn binds stronger than $+$. The constructor $rec\,x._$will have the lowest precedence among all the operators.

2.1. The Operational Semantics. The operational semantics for the language $REC_\Sigma$ consists of three different components. The first is an interpretation of $REC_\Sigma$ as a labelled transition system in Plotkin's **SOS** style, [26, 32]. This associates with each action symbol $\mu$ a binary infix relation. Intuitively, $p \xrightarrow{\mu} q$ means that $p$ may perform the action $\mu$ and thereby be transformed into $q$. The second is a *successful termination predicate* $\surd$, which will be written in a postfix manner. Intuitively, $p\surd$ if $p$ has terminated successfully, which will mean, among other things, that $p$ cannot perform any further actions. We would expect $nil\surd$ but not $\Omega\surd$, not $a; p\surd$ and not $\partial_{\{a\}}(a; p)\surd$. There is a choice of exactly how to define the termination predicate $\surd$. In what follows we will present one choice; another choice, which is more in keeping with the intuitions of [5], will be discussed in the conclusions. The final component is a *convergence predicate*, $\downarrow$. Intuitively, $p\downarrow$ means that the set of actions which $p$ can initially perform is fully specified. It will turn out that $nil\downarrow$ but not $\Omega\downarrow$.

*Definition* 2.2. Let $\surd$ be the least subset of $REC_\Sigma$ that satisfies:

(i) $\mathrm{nil} \in \surd$,
(ii) $p \in \surd$ implies $\partial_H(p) \in \surd$,
(iii) $p \in \surd$ and $q \in \surd$ imply $p + q, p; q, p \mid q \in \surd$,
(iv) $t[rec\,x.\ t/x] \in \surd$ implies $rec\,x.\ t \in \surd$.

In what follows, we write $p\surd$ iff $p \in \surd$. Note that the process $nil + \delta$ is not considered successfully terminated. Intuitively, the process is "stagnating" on a branch of its computation and the environment has no way of discarding this branch. In the semantic theory that we shall present in what follows, the process $nil + \delta$ will be equated to the deadlocked process $\delta$.

*Definition* 2.3. Let $\downarrow$ be the least subset of $REC_\Sigma$ that satisfies

(i) $\mathrm{nil}\downarrow, \delta\downarrow, \mu\downarrow$
(ii) $p\downarrow$ implies $\partial_H(p)\downarrow$
(iii) $p\downarrow, q\downarrow$ imply $(p + q)\downarrow, (p \mid q)\downarrow$
(iv) $t[rec\,x.\ t/x]\downarrow$ implies $rec\,x.\ t\downarrow$
(v) $p\surd, q\downarrow$ imply $(p; q)\downarrow$
(vi) $\neg(p\surd), p\downarrow$ imply $(p; q)\downarrow$.

Intuitively, $p\downarrow$ iff $p$ is a completely specified process, that is, if we can expand the recursive definition of $p$ a finite number of times to obtain at the top level all the possible moves of $p$. Clause (vi) of the definition of the predicate $\downarrow$ deserves some comment. It expresses the intuition that, if $p$ is not successfully terminated, $p; q$ is a completely specified process if $p$ is; in this case, in fact, the set of initial moves of $p; q$ is determined by that of $p$. $\uparrow$, the *divergence* predicate, will denote the complement of $\downarrow$, that is, $p\uparrow$ iff $\neg(p\downarrow)$.

*Example* 2.1. The following processes are divergent:

— $rec\,x.a + x$
— $rec\,x.a;\ x + \Omega$
— $rec\,x.a;\ x + rec\,x.a + a \mid x$.

The predicate $\downarrow$ is used to detect a form of "syntactic divergence". Roughly, $p\uparrow$ if $p$ contains unguarded recursive definitions [26], or unguarded occurrences of the divergent process $\Omega$. One can show that $p\surd$ implies $p\downarrow$ using induction on the proof of $p\surd$. Of course, the converse is not true; for instance, $a;\ q\downarrow$ but $a;\ q \notin \surd$.

*Definition* 2.4. For each $\mu \in Act_\tau$, let $\overset{\mu}{\rightarrow}$ be the least binary relation on $REC_\Sigma$ that satisfies the following axiom and rules:

(1) $\mu \overset{\mu}{\rightarrow}$ nil
(2) $p \overset{\mu}{\rightarrow} p'$ implies $p + q \overset{\mu}{\rightarrow} p',\ q + p \overset{\mu}{\rightarrow} p'$
(3) $p \overset{\mu}{\rightarrow} p'$ implies $p;\ q \overset{\mu}{\rightarrow} p';\ q$
(4) $p\surd,\ q \overset{\mu}{\rightarrow} q'$ imply $p;\ q \overset{\mu}{\rightarrow} q'$
(5) $p \overset{\mu}{\rightarrow} p'$ implies $p \mid q \overset{\mu}{\rightarrow} p',\ q \mid p \overset{\mu}{\rightarrow} q \mid p'$
(6) $p \overset{a}{\rightarrow} p',\ q \overset{\bar{a}}{\rightarrow} q'$ imply $p \mid q \overset{\tau}{\rightarrow} p' \mid q'$
(7) $p \overset{\mu}{\rightarrow} p',\ \mu \notin H$ imply $\partial_H(p) \overset{\mu}{\rightarrow} \partial_H(p')$
(8) $t[rec\,x.\ t/x] \overset{\mu}{\rightarrow} p'$ implies $rec\,x.\ t \overset{\mu}{\rightarrow} p'$.

For any $p$, let $Sort(p) = \{\mu \in Act_\tau \mid \exists \sigma \in Act_\tau^*,\ q \in REC_\Sigma\colon\ p \overset{\sigma\mu}{\rightarrow} q\}$, where, for $\sigma \in Act_\tau^*$, $\overset{\sigma}{\rightarrow}$ is defined in the natural way. One can check that, for each $p$, $Sort(p)$ is finite. That is, according to the terminology of [1] and [2], the transition system $\langle REC_\Sigma, Act_\tau, \rightarrow \rangle$ is *sort finite*. Some of our results will depend on this fact.

The three concepts defined above take no account of the special nature of $\tau$. Following Milner [26], $\tau$ is meant to be an internal invisible action. We now define three weaker versions of $\overset{\mu}{\rightarrow}$, $\surd$, and $\downarrow$ which use this assumption.

Let $\overset{\mu}{\Rightarrow}$ denote $(\overset{\tau}{\rightarrow})^* \circ \overset{\mu}{\rightarrow} \circ (\overset{\tau}{\rightarrow})^*$. So $p \overset{\mu}{\Rightarrow} q$ means that $p$ may evolve to $q$ performing the action $\mu$ and possibly silent moves. We also use the relation $\overset{\epsilon}{\Rightarrow}$, defined as $(\overset{\tau}{\rightarrow})^*$. In what follows, we write $p \overset{\tau^\omega}{\rightarrow}$ iff there exists a sequence $\langle p_i \mid i \geq 0\rangle$ such that $p_0 = p$ and $p_i \overset{\tau}{\rightarrow} p_{i+1}$, for each $i \geq 0$.

Let $Stable(p) = \{q \mid p \overset{\epsilon}{\Rightarrow} q \text{ and } q \overset{\tau}{\not\rightarrow}\}$. Then the weak counterpart to $\surd$ is defined by

$$p\surd\!\!\!\surd \text{ iff, for each } q \in Stable(p),\ q\surd.$$

For example, $nil\surd\!\!\!\surd$, $\tau + \delta\surd$, but not $\delta\surd$. This relation is characterized by

$$p\surd\!\!\!\surd \Leftrightarrow \begin{cases} \text{(i)} & p \overset{\tau}{\not\rightarrow} \text{ and } p\surd, \text{ or} \\ \text{(ii)} & p \overset{\tau}{\rightarrow} \text{ and, for each } q,\ p \overset{\tau}{\rightarrow} q \text{ implies } q\surd\!\!\!\surd. \end{cases}$$

Note that $rec\,x.\tau;\ x + a\sqrt{\!\!\sqrt{}}$, which is somewhat anomalous. However, we only apply the "weak tick" predicate $\sqrt{\!\!\sqrt{}}$ to processes that cannot perform an infinite sequence of $\tau$-actions and, for such processes, no such counterintuitive cases arise. Processes that can perform an infinite sequence of $\tau$-actions are semantically divergent, which brings us to our final weak predicate. Let $\Downarrow$ be the least predicate over $REC_\Sigma$ that satisfies

$$p\downarrow \text{ and } \left(\text{for each } q,\ p \xrightarrow{\tau} q \text{ implies } q \Downarrow\right) \text{ imply } p \Downarrow .$$

Intuitively, $p \Downarrow$ means that $p$ cannot perform $\tau$-actions indefinitely and a syntactically divergent process cannot be reached by performing these actions. Formally, one can prove

$$p \Downarrow \ \Leftrightarrow p \xrightarrow{\tau^\omega} \qquad \text{and} \qquad p \stackrel{\epsilon}{\Rightarrow} q \text{ implies } q\downarrow .$$

Note also that $p\sqrt{}$ implies $p \Downarrow$. This follows because we already know that $p\sqrt{}$ implies $p\downarrow$ and one can also show that it implies $p \xrightarrow{\mu}$ for no $\mu$, including $\tau$.

In the semantic preorder to be defined in Section 3 we use versions of $\Downarrow$ that are parameterized by actions:

—$p \Downarrow \tau$ if $p \Downarrow$,

$p \Downarrow a$ if $p \Downarrow$ and, for each $q$, $p \stackrel{a}{\Rightarrow} q$ implies $q \Downarrow$.

This concludes our operational description of a semantics for the language $REC_\Sigma$. It defines a Labelled Transition System with divergence and termination predicates $\langle REC_\Sigma,\ ACT_\tau \cup \{\epsilon\},\ \Rightarrow,\ \sqrt{\!\!\sqrt{}},\ \Downarrow\rangle$. In Section 3, this LTS will be used to define an operational preorder on processes.

2.2. Denotational Semantics. As pointed out in the introduction, the main purpose of this paper is to show that an adequate semantic theory for the process algebra described in the previous section can be constructed using a suitable set of inequations, $E$. Following Caurcelle and Nivat [11], Goguen et al. [13], Guessarian [15], and Hennessy [18], we propose as a denotational semantics for $REC_\Sigma$ the initial continuous algebra generated by a set of equations $E$, $CI_E$.

In order to show that $CI_E$ is a reasonable model for $REC_\Sigma$, in subsequent sections we develop a behavioral theory of processes and prove that this corresponds to the interpretation given by $CI_E$. In other words, $CI_E$ is fully abstract with respect to the behavioral preorder that we introduce in the next section. We assume the reader is familiar with the basic notions of continuous algebras (see, e.g., the above-quoted references); however, in what follows, we give a quick overview of the way a denotational semantics can be given to $REC_\Sigma(\mathtt{Var})$ following the standard lines of algebraic semantics [15]. The interested reader is invited to consult [18] for an explanation to the theory.

Let $\Sigma$ be the signature introduced in Definition 2.1 and $A$ be any $\Sigma$-cpo. A denotational semantics for the language $REC_\Sigma(\mathtt{Var})$ is given by the mapping

$$A[\![\,\cdot\,]\!]\colon REC_\Sigma(\mathtt{Var}) \to [\,ENV_A \to A\,],$$

where $ENV_A = [\mathtt{Var} \to A]$ is the set of $A$-environments, ranged over by the metavariables $\rho, \rho' \ldots$. As usual, $\rho[x \to a]$ will denote the environment,

A1 $x + y = y + x$
A2 $x + (y + z) = (x + y) + z$
A3 $x + x = x$
A4 $x + nil = x$
A5 $x + \delta = x$ if $x \notin \surd$
B1 $x; nil = x = nil; x$
B2 $\delta; x = \delta$
B3 $x;(y; z) = (x; y); z$
B4 $(x + y); z = x; z + y; z$ if $x, y \notin \surd$
C1 $\delta \mid \delta = \delta$
C2 Let $x \equiv \Sigma_{i \in I} \mu_i; x_i\{+\Omega\}$,

$$\delta \mid x = x \mid \delta = \begin{cases} \delta\{+\Omega\} & \text{if } I = \varnothing \\ \Sigma_{i \in I} \mu_i; (\delta \mid x_i)\{+\Omega\} & \text{otherwise} \end{cases}$$

EXP Let $x \equiv \Sigma_{i \in I} \mu_i; x_i\{+\Omega\}$
and $y \equiv \Sigma_{j \in J} \gamma_j; y_j\{+\Omega\}$,

$$x \mid y = \sum_{i \in I} \mu_i.(x_i \mid y) + \sum_{j \in J} \gamma_j;(x \mid y_j) + \sum_{(i,j)\ \mu_i = \bar{\gamma}_j} \tau;(x_i \mid y_j)\{+\Omega\}$$

E1 $\partial_H(nil) = nil$
E2 $\partial_H(\delta) = \delta$
E3 $\partial_H(\mu) = \begin{cases} \delta & \text{if } \mu \in H \\ \mu & \text{otherwise} \end{cases}$
E3 $\partial_H(x; y) = \partial_H(x); \partial_H(y)$
E5 $\partial_H(x + y) = \partial_H(x) + \partial_H(y)$
$\Omega_1$ $\Omega \leq x$
$\Omega_2$ $\tau;(x + \Omega) \leq x + \Omega$
$\Omega_3$ $\partial_H(\Omega) \leq \Omega$
$\Omega_4$ $\Omega; x \leq \Omega$
T1 $\mu; \tau = \mu$
T2 $\tau; x + x = \tau. x$
T3 $\mu;(x + \tau; y) = \mu;(x + \tau; y) + \mu; y$

NOTE: The summation notation in axiom EXP is justified by axioms A1–A5. In axiom EXP, an empty sum is understood as *nil*. $\{+\Omega\}$ indicates that $\Omega$ is an optional summand of a term and $\Omega$ is a summand of the right-hand side iff it is either a summand of $x$ or of $y$.

FIG. 1. The set of inequations $E$.

which is defined as follows:

$$\rho[x \to a](y) = \begin{cases} a & \text{if } x = y \\ \rho(y) & \text{otherwise.} \end{cases}$$

For completeness sake, we define $A[\![\cdot]\!]$ by structural induction on recursive terms, as follows:

(i) $A[\![x]\!]\rho = \rho(x)$,
(ii) $A[\![f(t_1, \ldots, t_k)]\!]\rho = f_A(A[\![t_1]\!]\rho, \ldots, A[\![t_k]\!]\rho)$ $(f \in \Sigma_k)$,
(iii) $A[\![rec\, x.\ t]\!]\rho = Y\lambda a.\, A[\![t]\!]\rho[x \to a]$,

where $Y$ denotes the least fixed-point operator.

Note that for each $p \in REC_\Sigma$, $A[\![p]\!]\rho$ does not depend on the environment $\rho$. The denotation of a closed term $p$ will be denoted by $A[\![p]\!]$ and we write $p \leq_A q$ iff $A[\![p]\!] \leq_A A[\![q]\!]$ and $p =_A q$ iff $p \leq_A q$ and $q \leq_A p$.

As already pointed out, a natural choice of $A$ would be the initial $\Sigma$-cpo $CI_E$ in the class of $\Sigma$-cpo's that satisfy some set of equations, or inequations, $E$ defined over the signature $\Sigma$. The equations that we consider will express desirable properties of processes; many of them are already well known from **CCS** or **ACP**. Some of the equations that are part of the theory of **ACP** have had to be modified due to our different treatment of successful termination. For example, note that by considering eq. (A4) for $x = \delta$, we obtain that $\delta + nil = \delta$. This identity captures the main difference between our *nil* and the empty process $\epsilon$ recently investigated in the literature on **ACP** [4, 5]: the intuition underlying it has been discussed after Definition 2.2. A more detailed comparison between our equations and the ones used in the theory of **ACP** with the empty process may be found in the conclusions. Let $\mathcal{C}(E)$ denote the category of $\Sigma$-cpo's that satisfy the equations in Figure 1 and continuous $\Sigma$-homomorphisms. The following result is then standard [11, 13, 15, 18].

PROPOSITION 2.1. *$\mathcal{C}(E)$ has an initial object $CI_E$.*

## 3. *The Behavioral Semantics*

3.1. THE BEHAVIORAL PREORDER. This section is devoted to an operational preorder that will be the behavioral counterpart of the denotational relation $\leq_{CI_E}$ (the ordering relation in the initial model $CI_E$) over $REC_\Sigma$. The existence of such a behavioral preorder, defined using a well-established mathematical tool, will reinforce $CI_E$ as a reasonable model for the language $REC_\Sigma$.

The behavioral preorder will be defined using a variation of bisimulation equivalence [26, 30], suitable for our language $REC_\Sigma$. Let *Rel* denote the set of binary relations over $REC_\Sigma$. We define a functional $\mathcal{F}: Rel \to Rel$, as follows:

given $\mathcal{R} \in Rel$, $p \mathcal{F}(\mathcal{R}) q$ iff, for each $\mu \in Act_\tau$,

(i) if $p \stackrel{\mu}{\Rightarrow} p'$, then, for some $q'$, $q \stackrel{\hat{\mu}}{\Rightarrow} q'$ and $p' \mathcal{R} q'$,
(ii) if $p \Downarrow \mu$, then
  (a) $q \Downarrow \mu$,
  (b) if $q \stackrel{\mu}{\Rightarrow} q'$ then, for some $p'$, $p \stackrel{\hat{\mu}}{\Rightarrow} p'$ and $p' \mathcal{R} q'$,
(iii) if $p \Downarrow$, then $p\surd \Leftrightarrow q\surd$.

The notation $\hat{\cdot}$ is used to simplify the definition: $\hat{\tau}$ stands for $\epsilon$ and $\hat{a}$ stands for $a$.

The functional $\mathcal{F}$ is one of the methods for adapting the usual definition functional of bisimulation equivalence. A number of variations are discussed in [1], [2], and [36]. There are also a number of ways of defining a behavioral preorder using $\mathcal{F}$. An established method is to take $\tilde{\sqsubseteq}$ to be the largest relation $\mathcal{R} \in Rel$ such that $\mathcal{R} \subseteq \mathcal{F}(\mathcal{R})$, [28, 29]. This relation is easily seen to be a preorder, that is, a reflexive and transitive relation, and is, in fact, the maximum fixed-point of the equation $\mathcal{R} = \mathcal{F}(\mathcal{R})$. The preorder $\tilde{\sqsubseteq}$ also satisfies many of the properties that we have already discussed in the introduction. For example, for every $p \in REC_\Sigma$, $\Omega \tilde{\sqsubseteq} p$; also $\delta$; $p \simeq \delta$ and *nil*; $p \simeq p$, where $\simeq$ is the kernel of $\tilde{\sqsubseteq}$, that is, $\simeq = \tilde{\sqsubseteq} \cap \tilde{\sqsubseteq}^{-1}$. The processes *nil* and $\delta$ are incomparable with respect to $\tilde{\sqsubseteq}$. In fact, it is easy to see that *nil* $\Downarrow$ and $\delta \Downarrow$, but nil$\surd$ whereas $\delta \not\in \surd$. Note also that $\delta + \Omega \simeq \Omega$. This follows from the definition but it is also perfectly reasonable. Intuitively, we would expect that, for every action $a$, $\delta + \Omega \tilde{\sqsubseteq} \delta + a$. But $\delta + a \simeq a$, so that $\delta + \Omega$ should be less than $a$ for each $a$; the only such process is $\Omega$.

Clause (iii) in the definition of $\mathcal{F}(\mathcal{R})$ takes care of deadlock cosiderations and there are a number of equivalent ways of stating it. Suppose we say that

$$p \textit{ must terminate if } p \Downarrow \text{ and } p \stackrel{\epsilon}{\Rightarrow} p' \stackrel{\tau}{\not\to} \text{ implies } p'\surd.$$

Then (iii) could be replaced by

(iiia)′ *p must terminate* implies *q must terminate*,
(iiib)′ $p \Downarrow$, *q must terminate* imply *p must terminate*.

Alternatively, suppose we say that $p$ is *deadlocked* if $p \Downarrow$, $p \xrightarrow{\mu}$ for no $\mu$, but $p \notin \surd$ and $p$ *may deadlock* if $p \stackrel{\epsilon}{\Rightarrow} p'$ for some $p'$ such that $p'$ is deadlocked. Then clause (iii) could also be replaced by

(iiia)′ $p$ *may deadlock* implies $q$ *may deadlock*,
(iiib)′ $p \Downarrow$, $q$ *may deadlock* imply $p$ *may deadlock*.

However, replacing clause (iii) with clauses such as

"if $p \Downarrow$, then $p\surd$ iff $q\surd$"

or

"if $p \Downarrow$, then $p$ is deadlocked iff $q$ is deadlocked"

would lead to a different semantic preorder. The terms $\tau; \delta$ and $\delta$ would be distinguished as would $a; \tau; \delta$ and $a; \delta$. Since $a; \tau$ and $a$ are identified, this would mean that the revised semantic preorder would not be preserved by ;.

An alternative method for using $\mathcal{F}$ to obtain a behavioral preorder is to apply it inductively, as follows:

— $\sqsubseteq_0 = REC_\Sigma \times REC_\Sigma$ (the top element in the lattice $(Rel, \subseteq)$),
— $\sqsubseteq_{n+1} = \mathcal{F}(\sqsubseteq_n)$,

and finally $\sqsubseteq_\omega = \bigcap_{n \geq 0} \sqsubseteq_n$.

The two relations $\sqsubseteq$ and $\sqsubseteq_\omega$ are in general different. For example, Abramsky [2], take the synchronization trees $p$ and $q$ defined as follows:

$$p \equiv a^\omega + \Omega, \qquad q \equiv \sum_{k \in \omega} a^k + \Omega.$$

Then, it is easy to see that $p \sqsubseteq_\omega q$, but $p \not\sqsubseteq q$. Two equivalent terms in our language are $rec\, x.\ a; x + \Omega$ and $rec\, x.\ x; a + a$, respectively. All the properties of $\sqsubseteq$ discussed above are also true of $\sqsubseteq_\omega$. In deciding which preorder to use, we take into account the type of semantic model we discussed in the previous section. We wish to define a behavioral preorder $\lesssim$ that satisfies

$$p \lesssim q \Leftrightarrow p \leq_{CI_E} q, \tag{1}$$

where $p \leq_{CI_E} q$ means $CI_E[\![p]\!] \leq CI_E[\![q]\!]$, for the set of inequations $E$ in Figure 1. This requirement induces certain constraints on $\lesssim$, the most important of which is called *finite approximability*. For any binary relation $\mathcal{R}$ over $REC_\Sigma$, let $\mathcal{R}^F$ be defined by

$p \mathcal{R}^F q$ if, for every finite term $d$, $d \mathcal{R} p$ implies $d \mathcal{R} q$.

We say that $\mathcal{R}$ is *finitely approximable* (*fa*) if $\mathcal{R} = \mathcal{R}^F$. Note that, for every transitive relation $\mathcal{R}$, $\mathcal{R} \subseteq \mathcal{R}^F$; thus, in order to show that such relations are *fa*, it is sufficient to prove that $\mathcal{R}^F \subseteq \mathcal{R}$. Intuitively, the finite approximability of a relation $\mathcal{R}$ means that $\mathcal{R}$ is essentially determined by how it behaves on finite terms. By the general construction of $CI_E$, [18], it follows that $\leq_{CI_E}$ is *fa* and therefore, to meet (1), we must also choose a behavioral preorder, which is also *fa*. The above example shows that $\sqsubseteq$ is not *fa*, as $p \not\sqsubseteq^F q$ but $p \sqsubseteq q$.

There is one further complication caused by requirement (1). The relation $\leq_{CI_E}$ is, by definition, *closed with respect to all contexts*. To explain this we need some notation. For any binary relation $\mathcal{R}$ over $REC_\Sigma$, let $\mathcal{R}$ be extended to $REC_\Sigma(\mathtt{Var})$ by

$$t\ \mathcal{R}\ u \text{ if, for every closed substitution } \rho,\ t\rho\ \mathcal{R}\ u\rho.$$

For any $\mathcal{R}$ over $REC_\Sigma(\mathtt{Var})$ define the new relation $\mathcal{R}^c$ by:

$$t\ \mathcal{R}^c u \text{ if, for every context } \mathcal{C}[\cdot] \text{ such that } \mathcal{C}[t] \text{ and } \mathcal{C}[u] \text{ are closed, } \mathcal{C}[t]\ \mathcal{R}\ \mathcal{C}[u].$$

Then $\mathcal{R}$ is said to be closed with respect to contexts if $\mathcal{R} = \mathcal{R}^c$. By construction, it follows that $\leq_{CI_E}$ is closed with respect to contexts. However, this is not true of $\sqsubseteq$ or $\sqsubseteq_\omega$. The usual counterexample associated with the **CCS** + operator [26] works:

$$a \sqsubseteq_\omega \tau; a \qquad \text{but} \qquad a + b \not\sqsubseteq_\omega \tau; a + b.$$

We may sum up this discussion by saying that in order to reflect the semantic ordering $\leq_{CI_E}$ behaviorally, it is necessary to choose a behavioral preorder that is both finitely approximable and preserved by contexts. We shall show that $\sqsubseteq_\omega$ is *fa* and therefore it is appropriate to take as our behavioral preorder $\sqsubseteq_\omega^c$, its closure with respect to all contexts. The proof that $\sqsubseteq_\omega$ is *fa* depends on the fact that our operational semantics is sort finite. In a transition system which is not sort finite $\sqsubseteq_\omega$ may not be *fa*. For instance, consider the following synchronization trees from [2]:

$$p \equiv a\Big(\sum_{n\in\omega} b_n nil + \Omega\Big) + \Omega,$$

$$q \equiv \sum_{n\in\omega} a\Big(\sum_{m\in\omega-\{n\}} b_m nil + \Omega\Big) + \Omega,$$

where, for each $n \neq m$, $b_n \neq b_m$. Then $p \sqsubseteq^F q$, but $p \not\sqsubseteq_2 q$.

The remainder of the paper is devoted to proving that our behavioral and denotational view of processes do agree on $REC_\Sigma$, that is, that, for $p, q \in REC_\Sigma$,

$$p \leq_{CI_E} q \Leftrightarrow p \sqsubseteq_\omega^c q.$$

In the next section, we analyze the preorder $\sqsubseteq_\omega^c$, giving an equivalent but more manageable definition. Using this equivalent formulation, we show that $\sqsubseteq_\omega^c$ satisfies all of the equations in $E$.

3.2. Analysis of the Preorder. In this section, we give a reformulation of $\sqsubseteq$ and use the more manageable definition to prove some of its properties. We are mainly interested in $\sqsubseteq_\omega$ but, as it turns out, most of the technical development concerns $\sqsubseteq$ rather than $\sqsubseteq_\omega$.

Let $\mathcal{G}$: $Rel \to Rel$ be the functional defined as follows:

For each $\mathcal{R} \in Rel$, $p\ \mathcal{G}(\mathcal{R})q$ iff, for each $\mu \in Act_\tau$,

(i) if $p \xrightarrow{\mu} p'$, then, for some $q'$, $q \stackrel{\hat{\mu}}{\Rightarrow} q'$ and $p'\ \mathcal{R}\ q'$
(ii) if $p \Downarrow \mu$, then
  (a) $q \Downarrow \mu$
  (b) if $q \xrightarrow{\mu} q'$, then, for some $p'$, $p \stackrel{\hat{\mu}}{\Rightarrow} p'$ and $p'\ \mathcal{R}\ q'$
(iii) if $p \Downarrow$, then $p\sqrt{\!\!\sqrt{}}$ iff $q\sqrt{\!\!\sqrt{}}$.

Let $\lesssim$ denote the maximum fixed-point of the functional $\mathcal{G}$, whose existence can be easily shown following standard lines [29].

PROPOSITION 3.1. *For* $p, q \in REC_\Sigma$, $p \sqsubseteq q \Leftrightarrow p \lesssim q$.

PROOF. Standard and thus omitted. □

This proposition allows us to investigate the properties of $\sqsubseteq$ using the technically simpler relation $\lesssim$. As a first application, we show that $\lesssim$, and consequently $\sqsubseteq$, is preserved by many of the operators of the calculus.

LEMMA 3.1. *If* $p \lesssim q$, *then*

(*a*) $p; r \lesssim q; r$
(*b*) $p \mid r \lesssim q \mid r$
(*c*) $\partial_H(p) \lesssim \partial_H(q)$.

PROOF. We examine only two of the operators leaving the remaining case to the reader.

(a) To show that $p \lesssim q$ implies $p; r \lesssim q; r$ it is sufficient to prove that the relation $\mathcal{R}$ defined as follows:

$$\mathcal{R} =_{\text{def}} \{(p; r, q; r) \mid p \lesssim q \text{ and } r \in REC_\Sigma\} \cup Id_{REC_\Sigma}$$

is a prebisimulation with respect to the functional $\mathcal{G}$, that is, $\mathcal{R} \subseteq \mathcal{G}(\mathcal{R})$. We only check that the clauses of the definition of the functional $\mathcal{G}$ are met for $(p; r, q; r)$ such that $p \lesssim q$.

(i) Assume $p; r \xrightarrow{\mu} x$. There are two cases to examine:

—$p \xrightarrow{\mu} p'$ and $x \equiv p'; r$. Then, as $p \lesssim q$, there exists $q'$ such that $q \stackrel{\hat{\mu}}{\Rightarrow} q'$ and $p' \lesssim q'$. Thus $q; r \stackrel{\hat{\mu}}{\Rightarrow} q'; r$ and $(p'; r, q'; r) \in \mathcal{R}$, by the definition of $\mathcal{R}$.

—$p\sqrt{}$ and $r \xrightarrow{\mu} x$. As $p\sqrt{}$, we may assume that $p \Downarrow$, $q\sqrt{\!\!\sqrt{}}$ and $q \Downarrow$. By the definition of $\sqrt{\!\!\sqrt{}}$, $\forall q' \in Stable(q)\ q'\sqrt{}$. Now, since $q \Downarrow$, $Stable(q) \neq \emptyset$. Let $q' \in Stable(q)$; then $q;\ r \stackrel{\epsilon}{\Rightarrow} q';\ r \xrightarrow{\mu} x$ and $(x, x) \in \mathcal{R}$.

(ii) Assume $(p; r) \Downarrow \mu$. First of all, we show that this implies $(q; r) \Downarrow \mu$. Note that $(p; r) \Downarrow \mu$ implies $p \Downarrow \mu$. As $p \lesssim q$, we have that $q \Downarrow \mu$. Suppose $(q; r) \Uparrow \mu$. We distinguish two cases:
($\mu = \tau$). This is equivalent to

$$q; r \xrightarrow{\tau^\omega} \quad \text{or} \quad \exists y\colon q; r \stackrel{\epsilon}{\Rightarrow} y \text{ and } y\uparrow.$$

Assume that $q; r \xrightarrow{\tau^\omega}$. As $q \Downarrow$, it must be the case that there exists $q'$ such that $q \stackrel{\epsilon}{\Rightarrow} q'$, $q'\surd$ and $r \xrightarrow{\tau^\omega}$. By induction on the derivation $q \stackrel{\epsilon}{\Rightarrow} q'$, we can show that there exists $p'$ such that $p \stackrel{\epsilon}{\Rightarrow} p'$ and $p' \lesssim q'$. As $q'\surd$ and $p \Downarrow$, $p'\not\surd$. This would mean $(p; r)\uparrow$, which contradicts the hypothesis.

Checking that the other possibility leads to a contradiction as well is omitted.

($\mu = a$) By definition, $(q; r) \Uparrow a$ iff $(q; r) \Uparrow$ or $\exists y$: $q; r \stackrel{a}{\Rightarrow} y$ and $y \Uparrow$. The case $(q; r) \Uparrow$ is dealt with as above. We assume that $\exists y$: $q; r \stackrel{a}{\Rightarrow} y$ and $y \Uparrow$. By the definition of $\Uparrow$, $y \Uparrow$ iff $y \xrightarrow{\tau^\omega}$ or $\exists \bar{y}$: $y \stackrel{\epsilon}{\Rightarrow} \bar{y}$ and $\bar{y}\uparrow$.

Thus, either $q; r \stackrel{a}{\Rightarrow} y \xrightarrow{\tau_\omega}$ or $q; r \stackrel{a}{\Rightarrow} y\uparrow$, for some $y$.

If $q; r \stackrel{a}{\Rightarrow} y \xrightarrow{\tau_\omega}$, then, as $q \Downarrow a$, $\forall q'$: $q \stackrel{a}{\Rightarrow} q'$, $q' \xrightarrow{\tau_\omega}$. Thus, there must exist $q'$ such that

$$\left(q \stackrel{a}{\Rightarrow} q'\surd \text{ and } r \xrightarrow{\tau^\omega}\right) \qquad \text{or} \qquad \left(q \stackrel{\epsilon}{\Rightarrow} q'\surd \text{ and } r \stackrel{a}{\Rightarrow} y \xrightarrow{\tau^\omega}\right).$$

In both cases, from $p \Downarrow a$ and $p \lesssim q$, we may deduce that $(p; r) \Uparrow a$. This goes against the hypothesis.

If $q; r \stackrel{a}{\Rightarrow} y\uparrow$, we proceed by analyzing the move $q; r \stackrel{a}{\Rightarrow} y$. There are three possibilities:

(a) $q \stackrel{a}{\Rightarrow} q'$ and $y \equiv (q'; r)\uparrow$. As $q \Downarrow a$ it must be the case that $q'\surd$ and $r\uparrow$. By induction on the derivation $q \stackrel{a}{\Rightarrow} q'$, we get $p'$ such that $p \stackrel{a}{\Rightarrow} p'$ and $p' \lesssim q'$. As $p \Downarrow a$ it must be the case that $p'\not\surd$ and $p' \Downarrow$. Hence, there exists $p''$ such that $p' \stackrel{\epsilon}{\Rightarrow} p''\surd$. Thus $p; r \stackrel{a}{\Rightarrow} (p''; r)\uparrow$. This contradicts the hypothesis that $(p; r) \Downarrow a$.

(b) $q \stackrel{a}{\Rightarrow} q'\surd$ and $r \stackrel{\epsilon}{\Rightarrow} y$.

(c) $q \stackrel{\epsilon}{\Rightarrow} q'\surd$ and $r \stackrel{a}{\Rightarrow} y$.

Both (b) and (c) follow the pattern of case (a). The $\mu$-moves of $q; r$ can be matched by those of $p; r$ as in case (i) above.

(iii) Assume $(p; r) \Downarrow$. We have to show that $(p; r)\not\surd$ iff $(q; r)\not\surd$. By clause (ii) above, $(q; r) \Downarrow$ and this, together with $p \lesssim q$, implies $p \Downarrow$ and $q \Downarrow$. It is easy to see that $(p; r)\not\surd$ implies $p\not\surd$ and $p\surd$ implies $r\not\surd$.

Assume now $(p; r)\not\surd$. By the above observation, we get that $p\not\surd$ and $p\surd$ implies $r\not\surd$. As $p \Downarrow$, $p\not\surd$ and $p \lesssim q$ imply $q\not\surd$. Hence, if $q; r \notin \not\surd$, it must be the case that, for some $x \in \mathit{Stable}(r)$, $q; r \stackrel{\epsilon}{\Rightarrow} q'; r \stackrel{\epsilon}{\Rightarrow} x$, where $q'\surd$ and $x \notin \surd$.
It is now easy to see that this would imply $p; r \notin \not\surd$, against the hypothesis. The proof of the converse implication is similar.

(b) It is sufficient to prove that $\mathscr{R} \subseteq \mathscr{G}(\mathscr{R})$, where

$$\mathscr{R} =_{\text{def}} \{(p \mid r, q \mid r) \mid p \lesssim q \text{ and } r \in REC_\Sigma\}.$$

Most of the proof is identical to that of the corresponding case of lemma 1 in [36, pp. 208–209]. The only new property to check is that $(p \mid r) \Downarrow$ implies $p \mid r \sqrt{\!\!\sqrt}$ iff $q \mid r \sqrt{\!\!\sqrt}$. Assume $p \mid r \Downarrow$ and $q \mid r \notin \sqrt{\!\!\sqrt}$. Then there exists $q' \mid r'$ such that $q \mid r \stackrel{\epsilon}{\Rightarrow} q' \mid r'$, $q' \mid r'$ is stable and $q' \mid r' \notin \sqrt{}$. This implies that either $q' \notin \sqrt{}$ or $r' \notin \sqrt{}$. As $q \mid r \stackrel{\epsilon}{\Rightarrow} q' \mid r'$, there exist sequences $\langle q_i \mid 0 \le i \le n \rangle$, $\langle r_i \mid 0 \le i \le n \rangle$ and $\langle a_i \mid 0 \le i \le n \rangle$ such that

$$q_0 \equiv q,\ r_0 \equiv r,\ \forall i < n\ q_i \stackrel{a_i}{\Rightarrow} q_{i+1} \text{ and } r_i \stackrel{\bar{a}_i}{\Rightarrow} r_{i+1},\ q_n \stackrel{\epsilon}{\Rightarrow} q' \text{ and } r_n \stackrel{\epsilon}{\Rightarrow} r'.$$

Since $p \mid r \Downarrow$ implies $q \mid r \Downarrow$, we have that $q_i \Downarrow$, for each $i$. Hence, we may inductively construct a sequence $\langle p_i \mid 0 \le i \le n \rangle$ such that $p_i \stackrel{a_i}{\Rightarrow} p_{i+1}$ and $p_i \lesssim q_i$. Thus, there exists $p'$ such that $p_n \stackrel{\epsilon}{\Rightarrow} p'$ and $p' \lesssim q'$. As $p \mid r \Downarrow$, $p' \Downarrow$. If $q' \notin \sqrt{}$, then $p' \notin \sqrt{\!\!\sqrt}$ and $p \mid r \notin \sqrt{\!\!\sqrt}$—a contradiction. If $r' \notin \sqrt{\!\!\sqrt}$, then, since $q' \mid r'$ is stable, $p' \Downarrow$ and $p' \lesssim q'$, there exists a stable state of the form $p'' \mid r'$ such that $p' \stackrel{\epsilon}{\Rightarrow} p''$. Now, $p'' \mid r' \notin \sqrt{}$ implies $p \mid r \notin \sqrt{\!\!\sqrt}$, again a contradiction. The converse implication is similar.

Checking that $\lesssim$ is preserved by $\partial_H(\cdot)$ is left to the reader. □

Since it is well known from the theory of bisimulation equivalence for **CCS** [29] and **ACP**, [8], $\lesssim$ is not preserved by $+$. For example, $nil \lesssim \tau$, but it is not the case that $a + nil \lesssim a + \tau$. In fact, $(a + nil) \Downarrow$ but $a + \tau \stackrel{\tau}{\rightarrow} nil$ and $a + nil \not\lesssim nil$. However, following Milner [26], we have a standard way of associating a precogruence with $\lesssim$. It is sufficient to close $\lesssim$ with respect to all the operators in $\Sigma$. The resulting precogruence, which we denote by $\lesssim^{fc}$, is known to be the largest $\Sigma$-precongruence contained in $\lesssim$. Note that $\lesssim^{c}$ and $\lesssim^{fc}$ are a priori different. In the latter, we only close with respect to contexts built from the operators in $\Sigma$, but in the former we also close with respect to contexts involving $rec\,x.\_$. We eventually prove that they coincide, but, for the moment, we concentrate on $\lesssim^{fc}$.

Let us now define the following preorder over $REC_\Sigma$:

$$p \lesssim^{+} q \Leftrightarrow \forall r \in REC_\Sigma\ p + r \lesssim q + r.$$

By analogy with one of the characterizations of the congruence associated with bisimulation equivalence [29], we might expect that $\lesssim^{+}$ and $\lesssim^{fc}$ coincide over $REC_\Sigma$. In order to prove that this is indeed the case, it will be useful to introduce an alternative characterization of $\lesssim^{+}$. The following definition is adapted from [36]:

*Definition* 3.1. For each $p, q \in REC_\Sigma$, $p \lesssim^{*} q$ iff

(i) $\forall a \in Act$, if $p \stackrel{a}{\rightarrow} p'$, then, for some $q'$, $q \stackrel{a}{\Rightarrow} q'$ and $p' \lesssim q'$,

(ii) if $p \stackrel{\tau}{\Rightarrow} p'$, then

  (a) $p' \Downarrow$ implies, for some $q'$, $q \stackrel{\tau}{\Rightarrow} q'$ and $p' \lesssim q'$,

  (b) $p' \Uparrow$ implies, for some $q'$, $q \stackrel{\epsilon}{\Rightarrow} q'$ and $p' \lesssim q'$,

(iii) $p \Downarrow \mu$ implies

  (a) $q \Downarrow \mu$,

  (b) if $q \stackrel{\mu}{\rightarrow} q'$, then, for some $q'$, $p \stackrel{\mu}{\Rightarrow} p'$ and $p' \lesssim q'$,

(iv) if $p \Downarrow$, then $p \sqrt{\!\!\sqrt}$ iff $q \sqrt{\!\!\sqrt}$.

The relation $\lesssim^*$ is easily seen to be a preorder. The following theorem states that $\lesssim^*$ and $\lesssim^+$ coincide over $REC_\Sigma$. The proof of the theorem uses the following technical lemma.

LEMMA 3.2. *If* $p \Uparrow$, $p \lesssim q$ *and* $\neg\exists p'$: $p \stackrel{\tau}{\Rightarrow} p' \wedge p' \Downarrow$, *then* $p \lesssim q + r$, *for each* $r \in REC_\Sigma$.

PROOF. It is sufficient to check that the relation

$$\mathcal{R} =_{\text{def}} \lesssim \cup \left\{ (p', q + r) \mid p \stackrel{\epsilon}{\Rightarrow} p' \text{ and } p' \lesssim q \right\}$$

is such that $\mathcal{R} \subseteq \mathcal{F}(\mathcal{R})$. Checking the clauses of the definition of $\lesssim$ is routine. □

THEOREM 3.1. *For each* $p, q \in REC_\Sigma$, $p \lesssim^* q$ *iff* $p \lesssim^+ q$.

PROOF. The proof is identical to the one of the corresponding result in [36, Lemma 3]. Thus, we just check the cases not covered by that lemma.

($\Leftarrow$) Assume that $p \not\lesssim^* q$ because

$$p \Downarrow \text{ and } \left[ (p\surd \wedge q \not\in \surd) \vee (p \not\in \surd \wedge q\surd) \right].$$

We have to find a process $r \in REC_\Sigma$ such that $p + r \not\lesssim q + r$. Assume, without loss of generality, that $p \Downarrow$, $p\surd$, and $q \not\in \surd$. Take $r \equiv nil$. Then $(p + nil) \Downarrow$ and $(p + nil)\surd$ whilst, as it may be easily checked, $(q + nil) \not\in \surd$. Hence, $p + nil \not\lesssim q + nil$ and $p \not\lesssim^+ q$.

($\Rightarrow$) Suppose that $p \lesssim^* q$. We show that, for each $r \in REC_\Sigma$, $(p + r) \Downarrow$ implies $(p + r)\surd$ iff $(q + r)\surd$.

Now, $(p + r) \Downarrow$ implies $p \Downarrow$. Since $p \Downarrow$ and $p \lesssim^* q$, we have that $p\surd$ iff $q\surd$. The claim then follows from the fact that, for each $p, r \in REC_\Sigma$,

$$Stable(p + r) = \begin{cases} \{p + r\} & \text{if } p \stackrel{\tau}{\not\rightarrow} \text{ and } r \stackrel{\tau}{\not\rightarrow} \\ \{x \in Stable(p) \cup Stable(r) & \\ \mid p \stackrel{\tau}{\Rightarrow} x \text{ or } r \stackrel{\tau}{\Rightarrow} x\} & \text{otherwise.} \end{cases}$$ □

As an easy corollary of the above theorem, we get that $\lesssim^* \subseteq \lesssim$. In fact,

$$p \lesssim^* q \Leftrightarrow p \lesssim^+ q \Rightarrow p + nil \lesssim q + nil \Leftrightarrow p \lesssim q.$$

The next lemma establishes the fact that $\lesssim^*$ is a $\Sigma$-precongruence.

LEMMA 3.3. $\lesssim^*$ *is a* $\Sigma$*-precongruence.*

PROOF. We examine each operator separately.

(;) Assume $p \lesssim q$ and $r \in REC_\Sigma$. We check that the clauses of the definition of $\lesssim^*$ are met by $p; r$ and $q; r$.

(i) Suppose that $p; r \stackrel{a}{\rightarrow} x$. By the operational semantics, there are two cases to examine:

—$p \stackrel{a}{\rightarrow} p'$ and $x \equiv p'; r$. Then, as $p \lesssim^* q$, $q \stackrel{a}{\Rightarrow} q'$ and $p' \lesssim q'$, for some $q'$. By the operational semantics, $q; r \stackrel{a}{\Rightarrow} q'; r$ and, as by Lemma 3.1 $\lesssim$ is preserved by;, $p'; r \lesssim q'; r$.

—$p\surd$ and $r \stackrel{a}{\rightarrow} x$. Now $p\surd$ implies $p \Downarrow$, and since $p \lesssim^* q$, we may deduce that

$$q \Downarrow, q\not\surd \quad \text{and} \quad \forall \mu \in Act_\tau, q \stackrel{\mu}{\not\rightarrow}.$$

Since $q \stackrel{\tau}{\not\rightarrow}$, we get that $Stable(q) = \{q\}$. Thus, $q\surd$. Hence, $q; r \stackrel{a}{\rightarrow} x$ and $x \lesssim x$.

(ii) Suppose that $p; r \stackrel{\tau}{\Rightarrow} x$. Following the definition of $\lesssim^*$, we distinguish two cases:

—$x \Downarrow$. We proceed by analyzing the move $p; r \stackrel{\tau}{\Rightarrow} x$.

(1) $p \stackrel{\tau}{\Rightarrow} p'$ and $x \equiv p'; r$. Then, $x \Downarrow$ implies $p' \Downarrow$. Since $p \lesssim^* q$, there exists $q'$ such that $q \stackrel{\tau}{\Rightarrow} q'$ and $p' \lesssim q'$. By the operational semantics, $q; r \stackrel{\tau}{\Rightarrow} q'; r$ and, by Lemma 3.1, $p'; r \lesssim q'; r$.

(2) $p \stackrel{\epsilon}{\Rightarrow} p'\surd$ and $r \stackrel{\tau}{\Rightarrow} x$. Left to the reader.

—$x \Uparrow$. This case can be checked using the pattern used in the above case.

(iii) Assume $(p; r) \Downarrow \mu$. Reasoning as in the corresponding case of Lemma 3.1, we get that $(q; r) \Downarrow \mu$. Suppose that $q; r \stackrel{\mu}{\rightarrow} x$. By the operational semantics, there are two cases to examine:

—$q \stackrel{\mu}{\rightarrow} q'$ and $x \equiv q'; r$. Since $(p; r) \Downarrow \mu$ and $(q; r) \Downarrow \mu$, we get that $p \Downarrow \mu$ and $q \Downarrow \mu$. Since $p \lesssim^* q$, there exists $p'$ such that $p \stackrel{\mu}{\Rightarrow} p'$ and $p' \lesssim q'$. By the operational semantics, $p; r \stackrel{\mu}{\Rightarrow} p'; r$ and, by Lemma 3.1, $p'; r \lesssim q'; r$.

—$q\surd$ and $r \stackrel{\mu}{\rightarrow} x$. Since $(p; r) \Downarrow \mu$, we get that $p \Downarrow$. Thus, $p \lesssim^* q$ and $q\surd$ imply $p\surd$. By the operational semantics, $p; r \stackrel{\mu}{\rightarrow} x$ and $x \lesssim x$.

(iv) As in the corresponding case of Lemma 3.1.

(+) To prove that $\lesssim^*$ is preserved by $+$, it is sufficient to notice that $+$ is associative with respect to $\lesssim$ and use the alternative characterization of $\lesssim^*$ given by the previous theorem.

(|) The proof is identical to that of the corresponding case of Lemma 4 in [36]. Checking clause (iv) of the definition of $\lesssim^*$ can be done as in Lemma 3.1.

Checking that $\lesssim^*$ is preserved by $\partial_H(\cdot)$ is left to the reader. □

As a corollary of the above result, we get that $\lesssim^*$, and consequently $\lesssim^+$, coincides with $\lesssim^{fc}$.

COROLLARY 3.1. *The relations* $\lesssim^{fc}$, $\lesssim^+$, *and* $\lesssim^*$ *all coincide over* $REC_\Sigma$.

PROOF. By Theorem 3.1, we know that $\lesssim^* = \lesssim^+$. By the definition of $\lesssim^{fc}$ we derive that $\lesssim^{fc} \subseteq \lesssim^+$. Moreover, by the above lemma, $\lesssim^*$ is a $\Sigma$-precongruence and this implies $\lesssim^* \subseteq \lesssim^{fc}$, as $\lesssim^{fc}$ is the largest $\Sigma$-precongruence contained in $\lesssim$ and $\lesssim^* \subseteq \lesssim$. □

The results that we have presented so far seem to imply that $\lesssim^*$ is a suitable notion of semantic preorder for the language $REC_\Sigma$. However, difficulties arise when we try to relate $\lesssim^*$ with the denotational model $CI_E$ outlined in Section 2. This is discussed more fully in the conclusions.

As we have already seen, $\lesssim$ (and consequently $\sqsubseteq$) is not finitely approximable [2]. Fortunately, however, we are able to relate $\sqsubseteq$, $\sqsubseteq_\omega$, and $\sqsubseteq_\omega^F$ over an important subset of $REC_\Sigma^2$; in fact, our next aim is to show that the three preorders coincide over $FREC_\Sigma \times REC_\Sigma$. The following technical result is standard.

FACT 3.1. *For each* $n \in \omega$, $\sqsubseteq_{n+1} \subseteq \sqsubseteq_n$.

THEOREM 3.2. *For each* $d \in FREC_\Sigma$, $p \in REC_\Sigma$, $d \sqsubseteq p$ *iff* $d \sqsubseteq_\omega p$.

PROOF. The "only if" implication is easily seen to hold by induction on $n$. To prove that $d \sqsubseteq_\omega p$ implies $d \sqsubseteq p$, we define the depth of a finite process as follows:

$$\mathtt{dt}(d) =_{\text{def}} \begin{cases} 2 & \text{if } \nexists \mu \in Act_\tau \colon d \overset{\mu}{\Rightarrow} \\ 1 + max\{\mathtt{dt}(d') \mid \exists \mu \colon d \overset{\mu}{\Rightarrow} d'\} & \text{otherwise.} \end{cases}$$

This is well defined because all the $d'$'s are finite (hence $\{d' \mid d \overset{\mu}{\Rightarrow} d'\}$ is finite for each $\mu \in Act_\tau$) and the transition system that we are considering is sort-finite. Note that, for each $d \in FREC_\Sigma$, $\mu \in Act_\tau$, $d \overset{\mu}{\Rightarrow} d'$ implies $\mathtt{dt}(d') \leq \mathtt{dt}(d) - 1$. By induction on $\mathtt{dt}(t)$, we prove that $d \sqsubseteq_{\mathtt{dt}(d)} p$ implies $d \sqsubseteq p$.

*Base case.* $\mathtt{dt}(d) = 2$. Assume $\mathtt{dt}(d) = 2$ and $d \sqsubseteq_2 p$. We show that the relation

$$\mathcal{R} =_{\text{def}} \{(d, p') \mid p \overset{\epsilon}{\Rightarrow} p'\}$$

is a prebisimulation. Consider $(d, p') \in \mathcal{R}$; we proceed to check the defining clauses of the functional $\mathcal{F}$. Clause (i) is trivially met as $\mathtt{dt}(d) = 2$ implies $d \overset{\mu}{\not\Rightarrow}$ for each $\mu \in Act_\tau$. Assume $d \Downarrow \mu$. Since $d \sqsubseteq_2 p$, we have that $p \Downarrow \mu$. It is easy to see that this implies $p' \Downarrow \mu$.

Assume $d \Downarrow \mu$, $p' \Downarrow \mu$ and $p' \overset{\mu}{\Rightarrow} p''$. As $d \sqsubseteq_2 p$ we have that $p \Downarrow \mu$. Hence, it must be the case that $\mu = \tau$ (otherwise, $p \overset{\epsilon}{\Rightarrow} p' \overset{\mu}{\Rightarrow} p''$ whilst, as $\mathtt{dt}(d) = 2$, $d \overset{\mu}{\not\Rightarrow}$. This would contradict the hypothesis that $d \sqsubseteq_2 p$). Thus, $(d, p'') \in \mathcal{R}$ by the definition of $\mathcal{R}$. To check clause (iii), assume that $d \Downarrow$. Then, since $d \sqsubseteq_2 p$, $p \Downarrow$ and $d\surd$, iff $p\surd$. Suppose

$$(d\surd \text{ and } p' \not\in \surd) \quad \text{or} \quad (d \not\in \surd \quad \text{and} \quad p'\surd).$$

If $d\surd$ and $p' \not\in \surd$, then $p \not\in \surd$, contradicting the hypothesis. If $d \not\in \surd$ and $p'\surd$, then $p \overset{\tau}{\Rightarrow} p'$ and $d \not\sqsubseteq_1 p'$, again contradicting the hypothesis that $d \sqsubseteq_2 p$. Hence, $d\surd$ iff $p'\surd$. Thus, $\mathcal{R} \subseteq \mathcal{F}(\mathcal{R})$ and, as $(d, p) \in \mathcal{R}$, $d \sqsubseteq p$.

*Inductive step.* Assume $\mathtt{dt}(d) > 2$ and $d \sqsubseteq_{\mathtt{dt}(d)} p$. Then:

(i) if $d \stackrel{\mu}{\Rightarrow} d'$ then there exists $p'$ such that $p \stackrel{\hat{\mu}}{\Rightarrow} p'$ and $d' \sqsubseteq_{\mathtt{dt}(d)-1} p$. Since $\mathtt{dt}(d') \leq \mathtt{dt}(d) - 1$, we have that $d' \sqsubseteq_{\mathtt{dt}(d)-1} p'$ implies $d' \sqsubseteq_{\mathtt{dt}(d')} p'$. By induction this implies that $d' \sqsubseteq p'$.

(ii) Assume $d \Downarrow \mu$. Then, $p \Downarrow \mu$ as $d \sqsubseteq_{\mathtt{dt}(d)} p$. Assume now that $p \stackrel{\mu}{\Rightarrow} p'$. Then reasoning as in case (i), we can find $d'$ such that $d \stackrel{\mu}{\Rightarrow} d'$ and $d' \sqsubseteq p'$.

(iii) Follows from the assumption that $d \sqsubseteq_{\mathtt{dt}(d)} p$.

Thus, $d \sqsubseteq_\omega p \Rightarrow d \sqsubseteq_{\mathtt{dt}(d)} p \Rightarrow d \sqsubseteq p$. □

COROLLARY 3.2. *For each* $d \in FREC_\Sigma$, $p \in REC_\Sigma$, $d \sqsubseteq p \Leftrightarrow d \sqsubseteq_\omega p \Leftrightarrow d \sqsubseteq^F_\omega p$.

PROOF. An easy consequence of the above theorem and of the definition of $\sqsubseteq^F_\omega$. □

As a consequence of the above result we have that $\sqsubseteq^{fc}_\omega$ and $\lesssim^*$ coincide over $FREC_\Sigma \times REC_\Sigma$. This explains our interest in $\lesssim^*$; it is technically more manageable than $\sqsubseteq^F_\omega$ and it will be used in the next section to show that for $d \in FREC_\Sigma$, $p \in REC_\Sigma$

$$d \sqsubseteq^{fc}_\omega p \Leftrightarrow CI_E[\![d]\!] \leq CI_E[\![p]\!].$$

This result can be lifted to the entire language in a fairly standard way, once we know that $\sqsubseteq_\omega$ is finitely approximable.

As a final result in this section, we examine the equations in $E$ and show that they are satisfied by $\sqsubseteq^{fc}_\omega$. Let $\leq_E$ denote the least $\Sigma$-precongruence over $REC_\Sigma$ that satisfies the equations $E$.

PROPOSITION 3.2. *For* $p, q \in REC_\Sigma$, $p \leq_E q$ *implies* $p \sqsubseteq^{fc}_\omega q$.

PROOF. Since $\sqsubseteq^{fc}_\omega$ contains $\lesssim^*$ and $\lesssim^*$ is a $\Sigma$-precongruence, it is sufficient to establish that $\lesssim^*$ satisfies all of the equations $E$. This we leave to the reader. □

3.3. FULL-ABSTRACTION. In this section we outline the proof of full-abstraction, namely

$$p \sqsubseteq^{c}_\omega q \Leftrightarrow CI_E[\![p]\!] \leq CI_E[\![q]\!]. \tag{2}$$

For convenience, we abbreviate $CI_E[\![p]\!]$ to $[\![p]\!]_E$. The first point to note is that it is sufficient to prove (2) for $\sqsubseteq^{fc}_\omega$ rather than $\sqsubseteq^{c}_\omega$. For, in this case, we can show that $\sqsubseteq^{fc}_\omega$ and $\sqsubseteq^{c}_\omega$ coincide: we already know that $\sqsubseteq^{c}_\omega \subseteq \sqsubseteq^{fc}_\omega$ and the fact that $\sqsubseteq^{fc}_\omega$ coincides with the preorder generated by the model means that it is preserved by contexts, that is, $\sqsubseteq^{fc}_\omega = \sqsubseteq^{fc,c}_\omega$. We know $\sqsubseteq^{fc}_\omega \subseteq \sqsubseteq_\omega$, from which it now follows that $\sqsubseteq^{fc}_\omega \subseteq \sqsubseteq^{c}_\omega$.

One crucial property of $\sqsubseteq^{fc}_\omega$ that we require is *finite approximability*.

THEOREM 3.3. $\sqsubseteq^{fc}_\omega$ *is finitely approximable.*

The proof of this theorem is quite involved and uses a characterization of $\sqsubseteq_\omega$ in terms of a modal property language similar to that in [2] and [20]. The next section is entirely devoted to the exposition of the proof. Since it is independent of the rest of the paper, we assume the theorem in the remainder of this section.

Another property that we establish is a partial completeness result, namely:

THEOREM 3.4. *For $d \in FREC_\Sigma$, $p \in REC_\Sigma$, $d \lesssim^* p$ implies $[\![d]\!]_E \leq [\![p]\!]_E$.*

Using these two theorems we now show how to establish full-abstraction (2). The proof actually requires some general results about the semantic mappings defined in Section 2.2, which may be found in [11], [15], and [18]. The first states that for any $p \in REC_\Sigma$ there exists an infinite sequence of *finite approximations* $p^n \in FREC_\Sigma$ such that, for any interpretation $A$,

$$A[\![p]\!] = \bigsqcup_{n \geq 0} A[\![p^n]\!].$$

The second states that for interpretations of the form $CI_E$, for each $d$, $e \in FREC_\Sigma$,

$$[\![d]\!]_E \leq [\![e]\!]_E \quad \text{if, and only if,} \quad d \leq_E e.$$

Finally, every finite approximation $p^n$ may be generated syntactically from $p$. Let $<$ be the least $\Sigma$-precongruence over $REC_\Sigma$ which satisfies

$$\begin{array}{lll} (\Omega) & \Omega & \leq x \\ (\mathbf{Rec}) & rec\,x.\,t & = t[rec\,x.\,t/x]. \end{array}$$

Then, for every $n \geq 0$, $p^n < p$. These general results may now be applied to prove:

THEOREM 3.5 (FULL-ABSTRACTION). *For $p, q \in REC_\Sigma$, $[\![p]\!]_E \leq [\![q]\!]_E$ iff $p \sqsubseteq^{fc}_\omega q$.*

PROOF. Suppose $[\![p]\!]_E \leq [\![q]\!]_E$. Since we may assume that $\sqsubseteq^{fc}_\omega$ is *fa*, it is sufficient to show that, for finite $d$, if $d \sqsubseteq^{fc}_\omega p$, then $d \sqsubseteq^{fc}_\omega q$. From the partial completeness result and the coincidence of $\lesssim^*$ and $\sqsubseteq^{fc}_\omega$ over $FREC_\Sigma \times REC_\Sigma$, we may assume that, for such a $d$, $[\![d]\!]_E \leq [\![p]\!]_E$ and therefore $[\![d]\!]_E \leq [\![q]\!]_E$. Now, since $d \in FREC_\Sigma$, $[\![d]\!]_E$ is a finite element in the algebraic cpo $CI_E$. This means that, for some $n \geq 0$, $[\![d]\!]_E \leq [\![q^n]\!]_E$, that is, $d \leq_E q^n$. From Proposition 3.2 it follows that $d \sqsubseteq^{fc}_\omega q^n$. Now, it is trivial to check that both of the laws $(\Omega)$ and $(Rec)$ are satisfied by $\lesssim^*$, that is, $< \subseteq \lesssim^*$, from which it follows that $q^n \sqsubseteq^{fc}_\omega q$. We may therefore conclude that $d \sqsubseteq^{fc}_\omega q$.

The converse is even more straightforward as it is an immediate consequence of the partial completeness theorem. Let us recall that $\leq_{CI_E}$ denotes the relation over $REC_\Sigma \times REC_\Sigma$ defined by $p \leq_{CI_E} q$ iff $[\![p]\!]_E \leq [\![q]\!]_E$. By the construction of $CI_E$, it is a finitely approximable relation. So, it is sufficient to show that if $d \sqsubseteq^{fc}_\omega p$, for $d \in FREC_\Sigma$, then $d \leq_{CI_E} p$. But, as $\sqsubseteq^{fc}_\omega$ and $\lesssim^*$ coincide over $FREC_\Sigma \times REC_\Sigma$, this is precisely the statement of Theorem 3.4. □

There may be other models which are fully abstract with respect to $\sqsubseteq^{fc}_\omega$ over $REC_\Sigma$. However, $CI_E$ is characterized by being initial in the category of fully abstract models and continuous $\Sigma$-homomorphisms. A $\Sigma$-cpo $A$ is called *consistent* if, for each $p, q \in REC_\Sigma$,

$$p \sqsubseteq^{fc}_\omega q \quad \text{implies} \quad A[\![p]\!] \leq A[\![q]\!].$$

The next theorem states that $CI_E$ is initial in the category of consistent models and continuous $\Sigma$-homomorphisms. Since every fully abstract model is obviously consistent, this implies that $CI_E$ is, up to isomorphism, the initial fully abstract model.

THEOREM 3.6. *Let $A$ be a consistent $\Sigma$-cpo. Then there exists a unique continuous $\Sigma$-homomorphism $h_A$: $CI_E \to A$.*

PROOF. Assume $A$ is a consistent $\Sigma$-cpo. By the full abstraction theorem, $[\![p]\!]_E \leq [\![q]\!]_E$ iff $p \sqsubseteq_{\omega}^{fc} q$, for each $p$, $q \in REC_\Sigma$. As $A$ is consistent, for each $p$, $q \in REC_\Sigma$, $[\![p]\!]_E \leq [\![q]\!]_E$ implies $A[\![p]\!] \leq A[\![q]\!]$. Thus, $A \in \mathscr{C}(\leq_{CI_E})$, the category of $\Sigma$-cpo's that satisfy the relation $\leq_{CI_E}$ and continuous $\Sigma$-homomorphisms. As $CI_E$ is initial in $\mathscr{C}(\leq_{CI_E})$ and $A \in \mathscr{C}(\leq_{CI_E})$, we thus have that there exists a unique continuous $\Sigma$-homomorphism $h_A$: $CI_E \to A$. □

So we have reduced full-abstraction to two theorems, Theorem 3.3 and Theorem 3.4. As already stated, the former is the subject of the next section, so in the remainder of this subsection we prove the latter. The proof of the partial completeness theorem follows the lines of the completeness theorems for finite terms in [16], [20], and [36], except that some care must be taken in the form of induction used—one of our terms may be infinite.

We first show that all finite terms may be reduced to a suitable normal form. The following facts will be useful in the syntactic manipulations to follow:

LEMMA 3.4. *The following are derived laws of the set of inequations $E$:*

$(D1)$ $\tau;\Omega = \Omega$,
$(D2)$ $\tau;(x + \Omega) = x + \Omega$.

PROOF. Left to the reader. The proof of (D2) uses axioms ($\Omega$1), ($\Omega$2), and (T2). □

*Definition* 3.2. The set of *normal forms* (*nf*s) is the least subset of $REC_\Sigma$ that satisfies:

(i) $\delta$ is an nf,
(ii) $\sum \mu_i; p_i\{+\Omega\}$ is an nf if
  (a) each $p_i$ is a nf, and
  (b) if, for some $i$, $\mu_i$ is $\tau$, then $p_i \Downarrow$.

The notation $\{+\Omega\}$ is used, as usual, to indicate that $\Omega$ is an optional summand.

Note that, according to the definition, if $n$ is an nf, then $n \Uparrow$ iff $\Omega$ is a summand of $n$.

PROPOSITION 3.3 (NORMALIZATION). *For each $d \in FREC_\Sigma$, there exists an nf, $\mathtt{nf}(d)$, such that $\mathtt{nf}(d) =_E d$ and $depth(d) = depth(\mathtt{nf}(d))$.*

PROOF. By induction on the *depth* of $d$. We proceed by case analysis on the structure of $d$. We examine only two cases leaving the others to the reader.

$d \equiv e; f$ By the inductive hypothesis, $e =_E \mathtt{nf}(e)$ and $f =_E \mathtt{nf}(f)$. If $\mathtt{nf}(e) \equiv \delta$, then $d \equiv e; f =_E \delta; f =_E \delta$, which is an nf. If $\mathtt{nf}(e) \equiv nil$, then $d \equiv e;$

$f =_E \mathrm{nf}(f)$. Otherwise,

$$\begin{aligned} e; f &=_E \Big(\sum \mu_i; e_i\{+\Omega\}\Big); f, \\ &=_E \sum (\mu_i; e_i); f\{+\Omega; f\}, && \text{by repeated application of B4,} \\ &=_E \sum \mu_i; (e_i; f)\{+\Omega\}, && \text{by } \Omega 1, \Omega 4, \text{ and repeated application of B3.} \end{aligned}$$

As the depth of $e_i; f$ is less than that of $e; f$, for each $i$, we may apply the inductive hypothesis to obtain an nf $\pi_i$ such that $\pi_i =_E e_i; f$, for each $i$. Thus, $e; f =_E \sum \mu_i; \pi_i\{+\Omega\}$. Assume now that $\mu_j = \tau$ and $\pi_j \Uparrow$ for some $j$. Then $\Omega$ is a summand of $\pi_j$. Let $\pi_j$ be $\bar{\pi} + \Omega$. Then,

$$\begin{aligned} e; f &=_E \sum_{i \neq j} \mu_i; \pi_i\{+\Omega\} + \tau; (\bar{\pi} + \Omega) \\ &=_E \sum_{i \neq j} \mu_i; \pi_i\{+\Omega\} + \bar{\pi} + \Omega && \text{by induction and D2.} \end{aligned}$$

This procedure can be iterated to obtain an nf.

$d \equiv e \mid f$ By the inductive hypothesis, $e =_E \mathrm{nf}(e)$ and $f =_E \mathrm{nf}(f)$. If $\mathrm{nf}(e \equiv \delta$, then there are two cases to examine:

—$\mathrm{nf}(f) \equiv \delta$. Then, $e \mid f =_E \delta \mid \delta =_E \delta$, by axiom C1.
—$\mathrm{nf}(f) \equiv \sum_{i \in I} \mu_i; f_i\{+\Omega\}$. We distinguish two cases:
If $I = \varnothing$, then, by axiom C2, $e \mid f =_E \delta\{+\Omega\}$. A normal form may now be obtained by possibly applying axioms A1 and A5.
If $I \neq \varnothing$, then, by axiom C2,

$$e \mid f =_E \delta \mid \Big(\sum_{i \in I} \mu_i; f_i\{+\Omega\}\Big) =_E \sum_{i \in I} \mu_i; (\delta \mid f_i)\{+\Omega\}.$$

By the inductive hypothesis, for each $i \in I$, there exists a normal form $\pi_i$ such that $\pi_i =_E \delta \mid f_i$. Thus $e \mid f =_E \sum_{i \in I} \mu_i; \pi_i\{+\Omega\}$. Assume now that there exists $j \in I$ such that $\mu_j = \tau$ and $\pi_j \Uparrow$. Then, as $\pi_j$ is an nf, $\pi_j =_E \pi + \Omega$, where $\pi \Downarrow$. Thus,

$$\tau; \pi_j =_E \tau; (\pi + \Omega) =_E \pi + \Omega \qquad \text{by D2.}$$

Iterating this procedure we may generate a normal form.

So by symmetry, we may assume that $\mathrm{nf}(e)$ and $\mathrm{nf}(f)$ are both different from $\delta$ and have the form $\mathrm{nf}(e) \equiv \sum_{i \in I} \mu_i; e_i\{+\Omega\}$ and $\mathrm{nf}(f) \equiv \sum_{j \in J} \gamma_j; f_j\{+\Omega\}$. Then

$$\begin{aligned} e \mid f &=_E \Big(\sum_{i \in I} \mu_i; e_i\{+\Omega\}\Big) \mid \Big(\sum_{j \in J} \gamma_j; f_j\{+\Omega\}\Big) \\ &=_E \sum_{i \in I} \mu_i; \big(e_i \mid \mathrm{nf}(f)\big) + \sum_{j \in J} \gamma_j; \big(\mathrm{nf}(e) \mid f_j\big) \\ &\quad + \sum_{(i,j).\ \mu_i = \bar{\gamma}_j} \tau; \big(e_i \mid f_j\big)\{+\Omega\}, \end{aligned}$$

by applying axiom Exp. By the inductive hypothesis, for each $i \in I$, $j \in J$, and $(i, j)$ such that $\mu_i = \bar{\gamma}_j$ there exist normal forms $\pi_i$, $\pi_j$, and $\pi_{ij}$ such that

$$\pi_i =_E e_i \mid \mathrm{nf}(f), \qquad \pi_j =_E \mathrm{nf}(e) \mid f_j, \qquad \text{and} \qquad \pi_{ij} =_E e_i \mid f_j.$$

Thus, $e \mid f =_E \sum_{i \in I} \mu_i; \pi_i + \sum_{j \in J} \gamma_j; \pi_j + \sum_{(i,j):\ \mu_i = \bar{\gamma}_j \tau} \tau; \pi_{ij}\{+\Omega\}$. Assume now that there exists $i_1 \in I$ such that $\mu_{i_1} = \tau$ and $\pi_{i_1} \Uparrow$. Then, as $\pi_{i_1}$ is an nf, $\pi_{1_1} =_E \bar{\pi} + \Omega$, where $\bar{\pi} \Downarrow$. Thus,

$$\mu_{i_1}; \pi_{i_1} =_E \tau; (\bar{\pi} + \Omega) =_E \bar{\pi} + \Omega, \quad \text{by D2}.$$

The same applies to each $j \in J$ such that $\gamma_j = \tau$ and $\pi_j \Uparrow$ and to each $\pi_{ij}$ such that $\pi_{ij} \Uparrow$. Iterating this procedure we may generate an nf. □

The above proposition tells us that we can deal with the set of normal forms instead of $FREC_\Sigma$. This is first applied in proving the following technical result:

FACT 3.2. *For each $d \in FREC_\Sigma$, $d \Downarrow$ and $d \Uparrow a$ imply $d =_E d + a; \Omega$.*

PROOF. By the above proposition we may assume, without loss of generality, that $d$ is an nf. The proof is by induction on the depth of the nf $d \equiv \sum_{i \in I} \mu_i; d_i$. By the definitions of the divergence predicates, $d \Uparrow a$ and $d \Downarrow$ imply $d \stackrel{a}{\Rightarrow} e$ for some $e \Uparrow$. There are two cases to examine:

(1) $d \stackrel{a}{\rightarrow} e$ and $e \Uparrow$. Then, there exists $i \in I$ such that $\mu_i = a$ and $d_i \equiv e$. Since $e$ is an nf, $e \Uparrow$ implies $e =_E e + \Omega$. Hence

$$\begin{aligned} d &=_E d + a; (e + \Omega) \\ &=_E d + a; (e + \tau; \Omega) && \text{by (D1) and substitutivity} \\ &=_E d + a; (e + \tau; \Omega) + a; \Omega && \text{by (T3)} \\ &=_E d + a; \Omega. \end{aligned}$$

(2) $d \stackrel{\tau}{\rightarrow} d' \stackrel{a}{\Rightarrow} e$ and $e \Uparrow$. Then, $d' \Uparrow a$. Moreover, as $d$ is an nf and $d \Downarrow$, $d'$ is an nf and $d' \Downarrow$. Thus, by the inductive hypothesis, $d' =_E d' + a; \Omega$. We can then calculate

$$\begin{aligned} d &=_E d + \tau; d' && \text{as } \mu_i = \tau \text{ and } d_i \equiv d', \text{ for some } i \in I, \\ &=_E d + \tau; (d' + a; \Omega) \\ &=_E d + \tau; (d' + a; \Omega) + d' + a; \Omega && \text{by } (T2) \\ &=_E d + a; \Omega && \text{by } (T2) \text{ and } (A1)\text{–}(A3) \end{aligned}$$

This completes the proof. □

We would not expect arbitrary terms from $REC_\Sigma$ to have normal forms since the process of normalization may not terminate. However, for weakly

convergent terms, we have a weaker form of normal forms, called *head normal forms*; these look like normal forms at the topmost level.

*Definition* 3.3. The set of *head normal forms* (*hnf*s) is the least subset of $REC_\Sigma$ that satisfies:

$-\delta$ is an hnf,
$-\sum \mu_i; p_i$ is an hnf if, for each $i$, $\mu_i = \tau$ implies $p_i$ is an hnf.

It is easy to check that if $h$ is an hnf, then $h \Downarrow$. It would not be reasonable to expect all terms to be reducible to hnfs using the equations $E$. For example, $p \equiv rec\,x.a; x$ is not an hnf, and applying equations to it will not help. However, if we are allowed to expand recursive definitions, that is, use the axiom (Rec), then $p$ can be rewritten to $a; rec\,x.\ a; x$, which is an hnf. Let us use $\preceq_{Er}$ to denote the precongruence obtained by allowing the use of axiom (Rec). In this extended rewrite system, all weakly convergent terms may be reduced to hnfs.

PROPOSITION 3.4. *For each $p \in REC_\Sigma$, such that $p \Downarrow$, there exists an hnf $\mathrm{h}(p)$ such that $p =_{Er} \mathrm{h}(p)$.*

PROOF. We assume that, for each $q$ such that $p \stackrel{\tau}{\Rightarrow} q$, $q$ has an hnf. The proof then proceeds by induction on why $p\downarrow$. We examine only two cases of the inductive definition of $\downarrow$.

$p \equiv q; r$ Then $(q; r)\downarrow$ iff (i) $q\surd$ and $r\downarrow$ or (ii) $q \notin \surd$ and $q \downarrow$.

If (i) holds, then, by induction, $r =_{Er} \mathrm{h}(r)$. It is easy to show that $q\surd$ implies $q =_{Er} nil$. Hence $q; r =_{Er} nil; \mathrm{h}(r) =_{Er} \mathrm{h}(r)$.

If (ii) holds, then, by induction, $q =_E \delta$ or $q =_{Er} \mathrm{h}(q) \equiv \sum \mu_i; q_i$. If $q =_E \delta$, then apply D2 to obtain an hnf. Otherwise, $q; r =_{Er} \mathrm{h}(q); r$. Recall that all hnfs are weakly convergent. Then

$$\begin{aligned} \mathrm{h}(q); r \equiv \left(\sum \mu_i; q_i\right); r &=_{Er} \sum (\mu_i; q_i); r && \text{by repeated use of B4} \\ &=_{Er} \sum \mu_i; (q_i; r) && \text{by repeated use of B3.} \end{aligned}$$

Assume that there exists $j$ such that $\mu_j = \tau$. Then, $\mathrm{h}(q); r \stackrel{\tau}{\rightarrow} (nil; q_j); r$. By hypothesis, there exists an hnf $\bar{h}$ such that $\bar{h} =_{Er} (nil; q_j); r =_{Er} q_j; r$, by B3 and B1. Thus, each $q_j; r$ such that $\mu_j = \tau$ has an hnf $\bar{h}_j$. Let $J = \{j \mid \mu_j = \tau\}$. Then

$$q; r =_E \sum_{j \in J} \tau; \bar{h}_j + \sum_{j \notin J} \mu_j; (q_j; r),$$

which is an hnf.

$p \equiv rec\,x.\ t$. Then $(rec\,x.\ t)\downarrow$ iff $t[rec\,x.\ t/x]\downarrow$. By the inductive hypothesis, there exists an hnf $h$ such that

$$t[rec\,x.\ t/x] =_{Er} h.$$

Thus, by axiom (Rec),

$$rec\,x.\ t =_{Er} t[rec\,x.\ t/x] =_{Er} h.$$

The other cases can be checked as in Proposition 3.3. □

A standard ingredient in the proofs of the equational characterization of bisimulation type relations is the so-called *derivation lemma* [19, 20, 36]. We also need such a lemma and, although it is not strictly necessary, it will be convenient to extend the set of equations with

(X1) $(x + \mu; y) \mid z = (x + \mu; y) \mid z + \mu; (y \mid z)$,
(X2) $(x + a; y) \mid (z + \bar{a}; w) = (x + a; y) \mid (z + \bar{a}; w) + \tau; (y \mid w)$,
(X3) $z \mid (x + \mu; y) = z \mid (x + \mu; y) + \mu; (z \mid y)$.

These equations are satisfied by the interpretation $CI_E$ and so including them is harmless. Let $F$ denote the extended set of equations obtained by augmenting $E$ with axioms (X1), (X2), (X3), and (Rec).

PROPOSITION 3.5. *For $p \in REC_\Sigma$, $p \overset{\mu}{\Rightarrow} q$ implies $p =_F p + \mu; q$.*

PROOF. By induction on the derivation, $p \overset{\mu}{\Rightarrow} q$.

*Basis.* $p \overset{\mu}{\rightarrow} q$. We proceed by a subinduction on why $p \overset{\mu}{\rightarrow} q$. Most cases are straightforward, so we just give the proof for a few selected cases in which the auxiliary axioms are used.

—$p \equiv p_1 \mid p_2$, $p_1 \overset{\mu}{\rightarrow} q_1$ and $q \equiv q_1 \mid p_2$. By the subinductive hypothesis, $p_1 =_F p_1 + \mu; q_1$. Hence

$$\begin{aligned} p_1 \mid p_2 &=_F (p_1 + \mu; q_1) \mid p_2 \\ &=_F (p_1 + \mu; q_1) \mid p_2 + \mu; (q_1 \mid p_2) \qquad \text{by X1} \\ &=_F p_1 \mid p_2 + \mu; (q_1 \mid p_2). \end{aligned}$$

—$p \equiv p_1 \mid p_2$, $p_1 \overset{a}{\rightarrow} q_1$ and $p_2 \overset{\bar{a}}{\rightarrow} q_2$. Then, by the subinductive hypothesis, $p_1 =_F p_1 + a; q_1$ and $p_2 =_F p_2 + \bar{a}; q_2$. Thus

$$\begin{aligned} p_1 \mid p_2 &=_F (p_1 + a; q_1) \mid (p_2 + \bar{a}; q_2) \\ &=_F (p_1 + a; q_1) \mid (p_2 + \bar{a}; q_2) + \tau; (q_1 \mid q_2) \qquad \text{by X2} \\ &=_F p_1 \mid p_2 + \tau; (q_1 \mid q_2). \end{aligned}$$

—$p \equiv rec\, x.t$ and $t[rec\, x.t/x] \overset{\mu}{\rightarrow} q$. By the subinductive hypothesis, $t[rec\, x.t/x] =_F t[rec\, x.t/x] + \mu; q$. The claim now follows by axiom (Rec).

*Inductive step.* We distinguish two cases:

(i) $p \overset{\mu}{\rightarrow} p' \overset{\tau}{\Rightarrow} q$. By induction, $p =_F p + \mu; p'$ and $p' =_F p' + \tau; q$. Thus

$$\begin{aligned} p &=_F p + \mu; (p' + \tau; q) \\ &=_F p + \mu; (p' + \tau; q) + \mu; q \qquad \text{by T3} \\ &=_F p + \mu; q. \end{aligned}$$

(ii) $p \xrightarrow{\tau} p' \stackrel{\mu}{\Rightarrow} q$. By induction, $p =_F p + \tau; p'$ and $p' =_F p' + \mu; q$. Thus

$$\begin{aligned} p &=_F p + \tau; (p' + \mu; q) \\ &=_F p + \tau; (p' + \mu; q) + p' + \mu; q \qquad \text{by T2} \\ &=_F p + \tau; p' + p' + \mu; q \\ &=_F p + \tau; p' + \mu; q \qquad \text{by T2} \\ &=_F p + \mu; q. \end{aligned}$$

This completes the inductive argument. □

To prove the partial completeness result for $\leq_E$ over $FREC_\Sigma \times REC_\Sigma$, we need one further technical result. This is an adaptation of a result originally shown for observational equivalence [29], and further adapted in [36] to prebisimulations.

LEMMA 3.5. *For each* $p, q \in REC_\Sigma$, $p \lesssim q$ *implies* $p \lesssim^* q$ *or* $\tau; p \lesssim^* q$ *or* $p \lesssim^* \tau; q$.

PROOF. Assume $p, q \in REC_\Sigma$ and $p \lesssim q$. There are three possibilities:

(i) $\exists p'$: $p \xrightarrow{\tau} p'$ and $p' \lesssim q$,
(ii) $\exists q'$: $q \xrightarrow{\tau} q'$ and $p \lesssim q'$,
(iii) neither (i) or (ii) holds.

If (i) holds, then we show that $p \lesssim^* \tau; q$. This follows easily by noting that now the move $p \xrightarrow{\tau} p'$ matches $\tau; q \xrightarrow{\tau} nil; q \simeq^* q$ (where $\simeq^*$ denotes the kernel of $\lesssim^*$). Moreover, $q\!\!\downarrow$ iff $\tau; q\!\!\downarrow$.

If (ii) holds, then one may show that $\tau; p \lesssim^* q$. This follows easily from an argument that is based on the following observations:

$-\tau; p \Downarrow \tau$ implies $p \Downarrow \tau$. Then, as $p \lesssim q$, $q \Downarrow \tau$. Moreover, the move $q \xrightarrow{\tau} q'$ matches $\tau; p \xrightarrow{\tau} nil; p \simeq^* p$.
$-\tau; p \Downarrow$ iff $p \Downarrow$. Moreover, $\tau; p\!\!\downarrow$ iff $p\!\!\downarrow$.

If (iii) holds, then it is easy to show that $p \lesssim^* q$. □

The above lemma will be used in the proof of relative completeness to relate processes with respect to $\lesssim$ and $\lesssim^*$, that is, in each case in which we know that $p \lesssim q$ we distinguish three cases:

$$p \lesssim^* q, \qquad \tau; p \lesssim^* q, \qquad \text{and} \qquad p \lesssim^* \tau; q.$$

The proof of completeness will use induction over a binary relation $\ll$ over $FREC_\Sigma \times REC_\Sigma$. The relation $\ll$ is defined as follows:

*Definition* 3.4. For each $(d, p)$, $(d', p') \in FREC_\Sigma \times REC_\Sigma$, $(d, p) \ll (d', p')$ iff

(i) $depth(d) < depth(d')$, or
(ii) $depth(d) = depth(d')$, $p' \Downarrow$ and $p' \stackrel{\tau}{\Rightarrow} p$.

To apply induction over $\ll$, we need to know that $\ll$ is a *well-founded* relation over $FREC_\Sigma \times REC_\Sigma$, that is, for each $(d, p) \in FREC_\Sigma \times REC_\Sigma$, there does not exist an infinite descending chain $(d, p) \gg (d_0, p_0) \gg (d_1, p_1) \cdots$. However, the well-foundedness of $\ll$ is an easy consequence of the finiteness of $depth(d)$, $d \in FREC_\Sigma$, the depth of the derivation tree of $d$, and of the fact that $p \Downarrow$ implies $p \overset{\tau^\omega}{\nrightarrow}$. We have now got all the technical machinery that we need to prove the partial completeness result.

THEOREM 3.7 (PARTIAL COMPLETENESS). *For each* $d \in FREC_\Sigma$, $p \in REC_\Sigma$, $d \lesssim^* p$ *implies* $[\![d]\!]_E \leq [\![p]\!]_E$.

PROOF. First, note that the interpretation $CI_E$ satisfies all of the equations in $F$. As a result $p \leq_F q$ implies $[\![p]\!]_E \leq [\![q]\!]_E$ and so it is sufficient to show that $d \lesssim^* p$ implies $d \leq_F p$. The proof is by induction over $\ll$. By Proposition 3.3, we may assume that $d$ is a normal form. By definition of normal form, $d$ is either $\delta$ or $\Sigma \mu_i; d_i\{+\Omega\}$.

If $d \equiv \delta$, then $\delta \lesssim^* p$ implies $p \overset{\mu}{\nrightarrow}$, for each $\mu \in Act_\tau$, $p \notin \surd$ and $p \Downarrow$. Then, by Proposition 3.4, there exists a hnf $h$ such that $h =_{Er} p$. By the above observations we get that, as $=_{Er}$ is sound with respect to $\simeq^*$, it must be the case that $h \equiv \delta$. Hence, $p =_{Er} \delta$. This implies $p =_F \delta$.

Thus, we may assume that $d \equiv \Sigma \mu_i; d_i\{+\Omega\}$. For technical reasons that will be clear in the remainder of the proof, it will be convenient to isolate the case in which $d$ is of the form $\tau; e$ for some nf $e$.

— $d \equiv \tau; e$. Since $d$ is a normal form we can assume $d \Downarrow$. The proof of the statement $d \leq_F p$ is divided into two steps:

(i) We prove that $\tau; e + p \leq_F p$. As $\tau; e \lesssim^* p$ and $\tau; e \overset{\tau}{\rightarrow} nil; e \simeq^* e$ then, as $e \Downarrow$, there exists $p'$ such that $p \overset{\tau}{\Rightarrow} p'$ and $e \lesssim p'$. By Lemma 3.5, $e \lesssim p'$ implies

$$e \lesssim^* p \quad \text{or} \quad \tau; e \lesssim^* p' \quad \text{or} \quad e \lesssim^* \tau; p'.$$

It is easy to see that:

$$\begin{aligned} (e, p') &\ll (\tau; e, p), \quad \text{as} \quad depth(e) < depth(\tau; e), \\ (\tau; e, p') &\ll (\tau; e, p), \quad \text{as} \quad p \Downarrow \text{ and } p \overset{\tau}{\Rightarrow} p', \text{ and} \\ (e, \tau; p') &\ll (\tau; e, p) \quad \text{as} \quad depth(e) < depth(\tau; e); \end{aligned}$$

thus we may apply induction to obtain

$$e \leq_F p' \quad \text{or} \quad \tau; e \leq_F p' \quad \text{or} \quad e \leq_F \tau; p'.$$

In each case, we obtain, by possibly applying axiom T1, that $\tau; e \leq_F \tau; p'$. Since $p \overset{\tau}{\Rightarrow} p'$, we may apply Proposition 3.5 to deduce that $p + \tau; p' =_F p$. Hence, $\tau; e + p \leq_F p + \tau; p' =_F p$.

(ii) We prove $\tau; e \leq_F \tau; e + p$. As $p \Downarrow$, by Proposition 3.4, there exists an hnf $h$ such that $p =_{Er} h$. This obviously implies $p =_F h$. Moreover, as $\tau; e \lesssim^* p$, $h$ must have the form $\Sigma \mu_i; p_i$. We show that, for each $i$, $\tau; e \leq_F \tau; e + \mu_i; p_i$. We distinguish two cases:

(1) $\mu_i = \tau$. Then $p \xrightarrow{\tau} t\,nil$; $p_i \simeq {}^*p_i$. Since $\tau; e \Downarrow \tau$, there exists $e'$ such that $\tau; e \overset{\tau}{\Rightarrow} e'$ and $e' \lesssim p_i$. Once again, by Lemma 3.5, $e' \lesssim p_i$ implies

$$e' \lesssim {}^*p_i \quad \text{or} \quad \tau; e' \lesssim {}^*p_i \quad \text{or} \quad e' \lesssim {}^*\tau; p_i.$$

Moreover, reasoning as in case (i), we may apply induction to obtain

$$e' \leq_F p_i \quad \text{or} \quad \tau; e' \leq_F p_i \quad \text{or} \quad e' \leq_F \tau; p_i.$$

In each case, we get, by possibly applying T1, that $\tau; e' \leq_F \tau; p_i$. By Proposition 3.5, $\tau; e =_F \tau; e + \tau; e' \leq_F \tau; e + \tau; p_i$.

(2) $\mu_i = a$. We distinguish two subcases:

(A) If $\tau; e \Downarrow a$, then there exists $e'$ such that $\tau; e \overset{a}{\Rightarrow} e'$ and $e' \lesssim p_i$. By Lemma 3.5, $e' \lesssim p_i$ implies

$$e' \lesssim {}^*p_i \quad \text{or} \quad \tau; e' \lesssim {}^*p_i \quad \text{or} \quad e' \lesssim {}^*\tau; p_i.$$

In each case $depth(e') < depth(\tau; e)$ and $depth(\tau; e') < depth(\tau; e)$; thus, we may apply induction to obtain

$$e' \leq_F p_i \quad \text{or} \quad \tau; e' \leq_F p_i \quad \text{or} \quad e' \leq_F \tau; p_i.$$

In each case, by possibly applying T1, $a; e' \leq_F a; p_i$. By Proposition 3.5, $\tau; e =_F \tau; e + a; e' \leq_F \tau; e + a; p_i$;

(B) if $\tau; e \Uparrow a$, then, by Fact 3.2 and the definition of $=_F$, $\tau; e \Downarrow$ and $\tau; e \Uparrow a$ imply $\tau; e =_F \tau; e + a; \Omega$. This implies that

$$\tau; e =_F \tau; e + a; \Omega \leq_F \tau; e + a; p_i.$$

This completes the proof of the fact that $\tau; e \leq_F \tau; e + \mu_i; p_i$, for each $i$.

Combining (i) and (ii), we get that $\tau; e \leq_F \tau; e + \sum \mu_i; p_i =_F \tau; e + p \leq_F p$, and this finishes the proof for the case $d \equiv \tau; e$.

—We now consider the general case, $d \equiv \sum \mu_i; d_i\{+\Omega\}$. To show that $d \leq_F p$, we follow the pattern used above.

(i) We prove, first of all, that $\mu_i; d_i + p \leq_F p$, for each $i$. As $d$ is an nf and $d \lesssim {}^*p$, then $d \xrightarrow{\mu_i} nil; d_i \simeq {}^*d_i$ implies that there exists $p'$ such that $p \overset{\mu_i}{\Rightarrow} p'$ and $d_i \lesssim p'$ (note that if $\mu_i = \tau$, then, as $d$ is an nf, $d_i \Downarrow$). By Lemma 3.5, $d_i \lesssim p'$ implies

$$d_i \lesssim {}^*p' \quad \text{or} \quad \tau; d_i \lesssim {}^*p' \quad \text{or} \quad d_i \lesssim {}^*\tau; p'.$$

If $d_i \lesssim {}^*p'$ or $d_i \lesssim {}^*\tau; p'$, we have that $depth(d_i) < depth(d)$; thus, we may apply induction to obtain $d_i \leq_F p'$ or $d_i \leq_F \tau; p'$. If $\tau; d_i \lesssim {}^*p'$, then we might not be able to apply induction (e.g., consider the case in which $depth(d) = depth(\tau; d_i)$ and $p \Uparrow$). However, by the case $d \equiv \tau; e$ examined above, we know that, as $\tau; d_i$ is itself an nf,

$\tau; d_i \leq_F p'$. So in each of the three cases, by possibly applying T1, we obtain $\mu_i; d_i \leq_F \mu_i; p'$.

Hence, $\mu_i; d_i + p \leq_F \mu_i; p' + p$ and, by Proposition 3.5, $p =_F p + \mu_i; p'$. Thus, for each $i$, $\mu_i; d_i + p \leq_F p$.

(ii) We now show that $d \leq_F d + p$.

(A) If $\Omega$ is a summand of $d$, then $d =_F d + \Omega \leq_F d + p$ and we are done.

(B) If $\Omega$ is not a summand of $d$, then $d \Downarrow$. Since $d \lesssim^* p$ and $d \Downarrow$, we have that $p \Downarrow$. Thus, by Proposition 3.4, there exists an hnf $\sum \mu_i; p_i$ such that $p =_F \sum \mu_i; p_i$. As in case (ii) of the part of the proof concerning the case $d \equiv \tau; e$, we can show that, for each $i$, $d \leq_F d + \mu_i; p_i$. Thus, $d \leq_F d + p$.

Combining (i) and (ii), we get that $d \leq_F d + p \leq_F p$.

This completes the proof. □

## 4. *Finite Approximability*

This section is entirely devoted to proving that $\sqsubseteq_{\omega}^{fc}$ is finitely approximable. Using the fact that $\sqsubseteq_{\omega}^{fc}$ coincides with $\sqsubseteq_{\omega}^{+}$, it is quite easy to see that it is sufficient to establish that $\sqsubseteq_{\omega}$ is finitely approximable. This is carried out in two stages. The first consists in a modal characterization of $\sqsubseteq_{\omega}$; we define a set $\mathcal{L}$ of modal formulas, $\phi$ and a satisfaction relation $\models$ with the property that

$$p \sqsubseteq_{\omega} q \quad \text{iff} \quad \{\phi \mid p \mid = \phi\} \subseteq \{\phi \mid q \mid = \phi\}.$$

This is the topic of Section 4.1 and is a simple modification of similar results in [2], [20], [27], and [35]. In Section 4.2, we show that satisfying a particular modal formula $\phi$ depends on a finite amount of information. More precisely, if $p \models \phi$, then there is a finite term $d(p, \phi)$ from $FREC_{\Sigma}$ such that $d(p, \phi) \models \phi$ and $d(p, \phi) \sqsubseteq_{\omega} p$. Intuitively, $d(p, \phi)$ represents the finite part of $p$ which ensures that $p$ satisfies $\phi$. These two results are combined in the final theorem of the paper, Theorem 4.3, which establishes that $\sqsubseteq_{\omega}$ and $\sqsubseteq_{\omega}^{F}$ coincide.

4.1. Modal Characterization of $\sqsubseteq_{\omega}$. We introduce a modal language which is a slight reformulation of the program logics introduced in [2], [20], [27], and [35]. The added atomic formulas will reflect the extra *deadlock structure*, which is present in the definition of our transition system semantics for $REC_{\Sigma}$.

*Definition* 4.1. Let $\mathcal{L}$ be the least class of formulas generated by the following clauses:

$-\nabla, \Delta, T, \perp \in \mathcal{L}$,
$-\phi, \psi \in \mathcal{L}$ imply $\phi \wedge \psi, \phi \vee \psi \in \mathcal{L}$,
$-\alpha \in Act \cup \{\epsilon\}$, $\phi \in \mathcal{L}$, imply $\langle \alpha \rangle \phi, [\alpha]\phi \in \mathcal{L}$.

The metavariables $\phi, \psi, \varphi \cdots$ will range over $\mathcal{L}$.

The satisfaction relation $\models \subseteq REC_\Sigma \times \mathscr{L}$ is defined as follows:

$$
\begin{aligned}
p &\models T && \text{always,}\\
p &\models \bot && \text{never,}\\
p &\models \nabla && \Leftrightarrow \quad p \Downarrow \text{ and } p\surd,\\
p &\models \Delta && \Leftrightarrow \quad p \Downarrow \text{ and } p \notin \surd,\\
p &\models \phi \wedge \psi && \Leftrightarrow \quad p \models \phi \text{ and } p \models \psi,\\
p &\models \phi \vee \psi && \Leftrightarrow \quad p \models \phi \text{ or } p \models \psi,\\
p &\models \langle \alpha \rangle \phi && \Leftrightarrow \quad \exists q\colon p \overset{\alpha}{\Rightarrow} Iq \text{ and } q \models \phi,\\
p &\models [\alpha]\phi && \Leftrightarrow \quad p \Downarrow \alpha \text{ and } \forall q\colon p \overset{\alpha}{\Rightarrow} q \; q \models \phi.
\end{aligned}
$$

Here $p \overset{\epsilon}{\Rightarrow} q$ is used to mean $p \overset{\tau}{\Rightarrow} q$ or $p \equiv q$; moreover, $p \Downarrow \epsilon$ iff $p \Downarrow$. The set of modal formulas in $\mathscr{L}$ satisfied by a process $p \in REC_\Sigma$, $\mathscr{L}(p)$, is given by:

$$\mathscr{L}(p) =_{\text{def}} \{\phi \in \mathscr{L} \mid p \models \phi\}.$$

The *modal depth* of a formula $\phi$, $\mathrm{md}(\phi)$, is defined by structural recursion as follows:

$$
\begin{aligned}
\mathrm{md}(T) &= \mathrm{md}(\bot) = 0,\\
\mathrm{md}(\nabla) &= \mathrm{md}(\Delta) = 1,\\
\mathrm{md}(\phi \wedge \psi) &= \mathrm{md}(\phi \vee \psi) = max\{\mathrm{md}(\phi), \mathrm{md}(\psi)\},\\
\mathrm{md}(\langle \alpha \rangle \phi) &= \mathrm{md}([\alpha]\phi) = 1 + \mathrm{md}(\phi).
\end{aligned}
$$

For each $\phi$, $\mathrm{md}(\phi)$ measures the maximum depth of the nesting of modalities and atomic formulas $\Delta$ and $\nabla$ in $\phi$. For example, $\mathrm{md}(\langle a \rangle([b]T \vee [c][d]\nabla)) = 4$ and $\mathrm{md}(\langle a \rangle \Delta \wedge [b]T) = 2$.

THEOREM 4.1 (MODAL CHARACTERIZATION THEOREM). *For $p, q \in REC_\Sigma$, $p \sqsubseteq_\omega q$ iff $\mathscr{L}(p) \subseteq \mathscr{L}(q)$.*

PROOF. Let $\mathscr{L}^{(n)}$ denote the subset of modal formulas $\phi \in \mathscr{L}$ such that $\mathrm{md}(\phi) \leq n$. To prove the "only if" implication, it is sufficient to show that, for each $n \in \omega$, $\phi \in \mathscr{L}^{(n)}$,

$$p \sqsubseteq_n q \qquad \text{and} \qquad p \models \phi \text{ imply } q \models \phi.$$

We proceed by induction on $n$. The claim is trivial for $n = 0$. Assume $p \sqsubseteq_{n+1} q$; we show that, for each $\phi \in \mathscr{L}^{(n+1)}$, $p \models \phi$ implies $q \models \phi$. The proof proceeds by structural induction over $\phi$. We consider only two cases and leave the others to the reader.

$\phi \equiv \nabla$. Obviously $\nabla \in \mathscr{L}^{(n+1)}$. Assume $p \models \nabla$. By the definition of $\models$, $p \models \nabla$ iff $p \Downarrow$ and $p\surd$. Since $p \sqsubseteq_{n+1} q$, we have that $q \Downarrow$ and $q\surd$. Thus, $q \models \nabla$.

$\phi \equiv [\alpha]\psi$. Assume $p \models [\alpha]\psi$. Then, by the definition of $\models$, $p \Downarrow \alpha$ and, for each $p'$ such that $p \overset{\alpha}{\Rightarrow} p'$, $p \models \psi$. Since $p \sqsubseteq_{n+1} q$, $p \Downarrow \alpha$ implies $q \Downarrow \alpha$. Moreover, it is easy to see that $q \overset{\alpha}{\Rightarrow} q'$ implies

$$\exists p'\colon p \overset{\alpha}{\Rightarrow} p' \text{ and } p' \sqsubseteq_n q'.$$

By the outer inductive hypothesis, $q' \models \psi$, for each $q'$ such that $q \overset{\alpha}{\Rightarrow} q'$. Thus $q \models [\alpha]\psi$.

We now show the converse implication. In fact, we show a stronger result that depends on the sort finiteness of the operational semantics for $REC_\Sigma$. Assume $p, q \in REC_\Sigma$. Let $A$ be a finite subset of $Act \cup \{\epsilon\}$ such that $\epsilon \in A$ and $(Sort(p) \cup Sort(q)) - \{\tau\} \subseteq A$. Such an $A$ exists by the sort finiteness of our transition system semantics for $REC_\Sigma$. For each $n \in \omega$, let $\mathscr{L}^{(A,n)}$ denote the subset of modal formulas $\phi$ over $A$ of modal depth at most $n$. By induction on $n$, we prove that

$$p \not\simeq_n q \text{ implies } \exists \phi \in \mathscr{L}^{(A,n)}\colon p \models \phi \text{ and } q \not\models \phi.$$

The base case is vacuous. Assume $p \not\simeq_{n+1} q$. Then, one of the following cases occurs:

(1) $p \overset{\mu}{\Rightarrow} p'$ and, for all $q'$, $q \overset{\hat{\mu}}{\Rightarrow} q'$ implies $p' \not\simeq_n q'$. By the inductive hypothesis, for each $q'$ such that $q \overset{\hat{\mu}}{\Rightarrow} q'$ there exists $\phi_{q'} \in \mathscr{L}^{(A,n)}$ such that $p \models \phi_{q'}$ and $q \not\models \phi_{q'}$. The set

$$\Gamma = \left\{ \phi_{q'} \mid q \overset{\hat{\mu}}{\Rightarrow} q' \right\}$$

might be infinite, for example, if $q \Uparrow \mu$; however, since $A$ is a finite set, $\mathscr{L}^{(A,n)}$ is finite up to logical equivalence $\equiv_\lambda$. Hence $\Gamma / \equiv_\lambda$ is finite as well, and there exists a formula $\psi \in \mathscr{L}^{(A,n)}$ that is logically equivalent to $\wedge\Gamma$. Take $\phi = \langle \hat{\mu} \rangle \psi$. Then, $\mathrm{md}(\phi) \leq n + 1$ and $p \models \phi$. On the other hand, $q \not\models \phi$.

(2) $p \Downarrow \alpha$ and $q \Uparrow \alpha$. We distinguish two cases: if $q \overset{\alpha}{\Rightarrow}$, then $[\alpha]T \in \mathscr{L}^{(A,n+1)}$ and $p \models [\alpha]^T$ while $q \not\models [\alpha]T$. Otherwise, it must be the case that $q \Uparrow$. Hence, $p \models [\epsilon]T$ while $q \not\models [\epsilon]T$.

(3) $p \Downarrow \mu$, $q \Downarrow \mu$, $q \overset{\mu}{\Rightarrow} q'$ and, for all $p'$, $p \overset{\hat{\mu}}{\Rightarrow} p'$ implies $p' \not\simeq_n q'$. By the inductive hypothesis, for each $p'$ such that $p \overset{\hat{\mu}}{\Rightarrow} p'$ there exists $\phi_{p'} \in \mathscr{L}^{(A,n)}$ such that $p' \models \phi_{p'}$ and $q' \not\models \phi_{p'}$. The set of formulas $\Gamma = \{\phi_{p'} \mid p \overset{\hat{\mu}}{\Rightarrow} p'\}$ might be infinite, however, reasoning as above, we may deduce the existence of a formula $\psi \in \mathscr{L}^{(A,n)}$, which is logically equivalent to $\vee\Gamma$. Take

$$\phi = [\hat{\mu}]\psi.$$

It is easy to see that $\mathrm{md}(\phi) \leq n + 1$ and $p \models \phi$. On the other hand, $q \not\models \phi$.

(4) $p \Downarrow$ and $\neg(p \surd$ iff $q \surd)$. We may assume, without loss of generality, that $q \Uparrow$; otherwise, case (2) applies. There are two subcases to examine:

(a) $p \Downarrow$, $q \Downarrow$, $p \surd$ and $q \not\surd$. Then, $p \models \nabla$ while $q \not\models \nabla$.
(b) $p \Downarrow$, $q \Downarrow$, $p \not\surd$ and $q \surd$. Then, $p \models \Delta$ while $q \not\models \Delta$.

This completes the proof. □

4.2. Finitary Properties. Let us recall that the finitary part of the preorder $\sqsubseteq_\omega$, $\sqsubseteq_\omega^F$, is defined as follows:

$$p \sqsubseteq_\omega^F q \text{ iff, } \forall d \in FREC_\Sigma,\ d \sqsubseteq_\omega p \overset{d}{\Rightarrow} \sqsubseteq_\omega q.$$

As an easy consequence of Theorem 3.2, we have that $\sqsubseteq_{\omega}^{F} = \sqsubseteq^{F}$. Moreover, it is easy to establish that $p \sqsubseteq_{\omega} q$ implies $p \sqsubseteq_{\omega}^{F} q$. We now show that the converse implication also holds. As stated at the beginning of Section 4, the key point of the proof of the inclusion $\sqsubseteq_{\omega}^{F} \subseteq \sqsubseteq_{\omega}$ will be a construction that, given $p \in REC_{\Sigma}$ and $\phi \in \mathcal{L}(p)$, will generate a finite process $d(p, \phi)$ such that $d(p, \phi) \models \phi$ and $d(p, \phi) \sqsubseteq p$. In order to give the construction of $d(p, \phi)$, we need a technical definition that uses the precongruence $\leq_{Er}$ defined immediately before Proposition 3.4:

*Definition* 4.2. Let $p \in REC_{\Sigma}$, $d \in FREC_{\Sigma}$ and suppose that $d \leq_{Er} p$. It follows that $CI_E[\![d]\!] \leq CI_E[\![p]\!]$. Since $CI_E$ is an algebraic cpo, there exists some $n \geq 0$ such that $CI_E[\![d]\!] \leq CI_E[\![p^n]\!]$. Since both are finite terms, this implies $d \leq_E p^n$ and therefore $d \leq_{Er} p^n$. So, for $d$, $e \in FREC_{\Sigma}$ such that $d \leq_{Er} p$ and $e \leq_{Er} p$, let $d \vee e$ denote the least principal approximation $p^k$ such that $d \leq_{Er} p^k$ and $e \leq_{Er} p^k$.

This operator will be used in the construction of $d(p, \phi)$. In what follows, we consider $\mathcal{L}$ modulo logical equivalence. Note that, because of this assumption and the law

$$[\alpha] \bigwedge_{i \in I} \phi_i = \bigwedge_{i \in I} [\alpha] \phi_i,$$

when considering formulas of the form $[\alpha]\phi$ we can restrict ourselves to the cases in which $\phi$ is either $T$ or has the form $\bigvee_{i \in I} \phi_i$, where each $\phi_j$ is either of the form $\langle \beta \rangle \psi$ or $[\beta]\psi$, for some $\beta \in Act \cup \{\epsilon\}$ and $\psi \in \mathcal{L}$.

The construction $d(-, -)$: $\{(p, \phi) \mid p \models \phi\} \to FREC_{\Sigma}$ will be given by induction over the relation $\lessdot \subseteq (\mathcal{L} \times REC_{\Sigma})^2$ defined as follows:

$$(\phi, q) \lessdot (\psi, p) \Leftrightarrow \begin{cases} \text{(i) } \mathrm{ht}(\phi) < \mathrm{ht}(\psi), \text{ or} \\ \text{(ii) } \mathrm{ht}(\phi) = \mathrm{ht}(\psi),\ p \Downarrow \text{ and } p \stackrel{\tau}{\Rightarrow} q, \end{cases}$$

where the *height* of a formula $\phi$, $\mathrm{ht}(\phi)$, is easily defined by structural recursion on $\phi$ [2]. Of course, for the following inductive construction to make sense we have to ensure that $\lessdot$ is a well founded relation. However, this follows from the fact that $\mathrm{ht}(\phi)$ is finite for each $\phi \in \mathcal{L}$ and the fact that $p \Downarrow$ implies that $\{q \mid p \stackrel{\tau}{\Rightarrow} q\}$ is finite. We can now state and prove the main theorem of this section.

THEOREM 4.2. *For each $p \in REC_{\Sigma}$, $\phi \in \mathcal{L}(p)$, there exists a finite process $d(p, \phi)$ such that*:

*(i)* $d(p, \phi) \models \phi$, *and*
*(ii)* $d(p, \phi) \leq_{Er} p$.

PROOF. The proof of the theorem is constructive. By induction on the relation $\lessdot$ we construct, for each $p \in REC_{\Sigma}$ and $\phi \in \mathcal{L}(p)$, a finite process that meets the statement of the theorem. We proceed by structural recursion on the formula $\phi$.

$p \models \phi \equiv T$. Then, $d(p, T) \equiv \Omega$.

$\phi \equiv \bot$ . Vacuous.

$p \vDash \phi \equiv \nabla$. Recall that, by the definition of $\vDash$, $p \vDash \nabla$ iff $p \Downarrow$ and $p\sqrt{}$. As $p \Downarrow$, $p =_{Er} h(p)$ for some hnf $h(p)$. By the soundness of $=_{Er}$, $p \simeq {}^{*}h(p)$. Moreover, as $p\sqrt{}$, $h(p) \equiv \sum_{i \in I} \tau; p_i + \sum_{j \in J} a_j; p_j$. Since $\precsim^{*} \subseteq \underset{\sim}{\sqsubseteq}_{\omega}$, by the modal characterization theorem

$$\sum_{i \in I} \tau; p_i + \sum_{j \in J} a_j; p_j \vDash \nabla.$$

Take $d(p, \nabla) \equiv \sum_{i \in I} \tau; d(p_i, \nabla) + \sum_{j \in J} a_j; \Omega$.

—We show that $d(p, \nabla) \vDash \nabla$. By the inductive hypothesis, $d(p_i, \nabla) \vDash \nabla$, for each $i \in I$. This implies that $d(p_i, \nabla) \Downarrow$ and $d(p_i, \nabla)\sqrt{}$, for each $i \in I$. Thus, $d(p, \nabla) \Downarrow$ and $d(p, \nabla)\sqrt{}$ (note that, as $p \vDash \nabla$, $I = \varnothing$ implies $J = \varnothing$).

—By induction $d(p_i, \nabla) \precsim_{Er} p_i$, for each $i \in I$, from which it follows that $d(p, \nabla) \precsim_{Er} h(p) =_{Er} p$.

$p \vDash \phi \equiv \Delta$. By the definition of $\vDash$, $p \vDash \Delta$ iff $p \Downarrow$ and $p \notin \sqrt{}$. As $p \Downarrow$, we may assume, without loss of generality, that $p$ has either the form $\delta$ or $\sum_{i \in I} \tau; p_i + \sum_{j \in J} a_j; p_j$. If $p$ is $\delta$, then $\delta(p, \Delta) \equiv \delta$.

If $p$ is $\sum_{i \in I} \tau; p_i + \sum_{j \in J} a_j; p_j$ then partition $I$ into $I_1 =_{\text{def}} \{i \in I \mid p_i\sqrt{}\}$ and $I_2 =_{\text{def}} \{i \in I \mid p_i \notin \sqrt{}\}$. Note that, as $p \notin \sqrt{}$, $I \neq \varnothing$ implies $I_2 \neq \varnothing$. Then

$$d(p, \Delta) \equiv \sum_{i \in I_1} \tau; d(p_i, \nabla) + \sum_{i \in I_2} \tau; d(p_i, \Delta) + \sum_{j \in J} a_j; \Omega.$$

—We show that $d(p, \Delta) \vDash \Delta$. The claim is trivial when $I = \varnothing$. Assume now $I \neq \varnothing$. Then, $I_2 \neq \varnothing$; thus, $d(p, \Delta) \overset{\tau}{\rightarrow} nil; d(p_k, \Delta)$, for some $k \in I_2$. By the inductive hypothesis, $d(p_k, \Delta) \vDash \Delta$. Thus, $d(p_k, \Delta) \notin \sqrt{}$. It is easy to see that $d(p, \Delta) \Downarrow$; in fact, by the inductive hypothesis, for each $i \in I_1 \cup I_2$, $d(p_i, *) \Downarrow$, $* \in \{\Delta, \nabla\}$. Thus $d(p, \Delta) \vDash \Delta$.

—To prove that $d(p, \Delta) \precsim_{Er} p$ it is sufficient to note that, by induction, $d(p_i, \nabla) \precsim_{Er} p_i$, for each $i \in I_1$, and $d(p_i, \Delta) \precsim_{Er} p_i$, for each $i \in I_2$.

$p \vDash \phi \equiv \phi_1 \vee \phi_2$. By the definition of $\vDash$, $p \vDash \phi_1 \vee \phi_2$ iff $p \vDash \phi_1$ or $p \vDash \phi_2$. Assume, without loss of generality, that $p \vDash \phi_1$. Then, $d(p, \phi) \equiv d(p, \phi_1)$. Both the statements of the theorem then follow by induction.

$p \vDash \phi_1 \wedge \phi_2$. Then, $p \vDash \phi_1$ and $p \vDash \phi_2$. Take $d(p, \phi) \equiv d(p, \phi_1) \vee d(p, \phi_2)$. We show that $d(p, \phi) \vDash \phi$. By the inductive hypothesis, $d(p, \phi_i) \vDash \phi_i$, $i = 1, 2$. By the definition of $\vee$, $d(p, \phi_i) \precsim_{Er} d(p, \phi_1) \vee d(p, \phi_2)$, $i = 1, 2$. Since $\underset{\sim}{\sqsubseteq}$ and $\underset{\sim}{\sqsubseteq}_{\omega}$ coincide over $FREC_{\Sigma} \times REC_{\Sigma}$ by Theorem 3.2, $d(p, \phi_i) \underset{\sim}{\sqsubseteq}_{\omega} d(p, \phi_1) \vee d(p, \phi_2)$, $i = 1, 2$. By the modal characterization theorem, $d(p, \phi_1) \vee d(p, \phi_2) \vDash \phi_i$, $i = 1, 2$. Hence, $d(p, \phi) \vDash \phi_1 \wedge \phi_2$.

$p \vDash \langle \alpha \rangle \phi$. By the definition of $\vDash$, $p \vDash \langle \alpha \rangle \phi$ iff there exists $q$ such that $p \overset{\alpha}{\Rightarrow} q$ and $q \vDash \phi$. Then:

—if $\alpha = \epsilon$, then $d(p, \langle \epsilon \rangle \phi) \equiv \tau; d(q, \phi) + \Omega$;
—if $\alpha = a$, then $d(p, \langle a \rangle \phi) \equiv a; d(q, \phi) + \Omega$.

In both cases, it is easy to see that both the statements of the theorem are met by $d(p, \langle \alpha \rangle \phi)$. The details are omitted.

$p \vDash [\epsilon]\phi$. By the definition of $\vDash$, $p \vDash [\epsilon]\phi$ iff $p \Downarrow$ and, for each $q$ such that $p \stackrel{\epsilon}{\Rightarrow} q$, $q \vDash \phi$. As $p \Downarrow$, we may assume, without loss of generality, that $p$ is an hnf. If $p$ is $\delta$, then $d(p, [\epsilon]\phi) \equiv \delta$ and both the statements are trivially seen to hold.

Assume now that $p \equiv \sum_{i \in I} \tau; p_i + \sum_{j \in J} a_j; p_j$. As previously remarked, we may assume that up to logical equivalence $\phi$ is either $T$ or is of the form $\bigvee_{h \in H} \phi_h$, where each $\phi_h$ is of the form $\langle \alpha \rangle \phi$ or $[\alpha]\psi$, for some $\alpha \in Act \cup \{\epsilon\}$ and $\psi \in \mathscr{L}$.

—If $\phi$ is $T$, then $d(p, [\epsilon]T) \equiv \sum_{i \in I} \tau; d(p_i, [\epsilon]T) + \sum_{j \in J} a_j; \Omega$. To see that $d(p, [\epsilon]T) \vDash [\epsilon]T$, it is sufficient to prove that $d(p, [\epsilon]T) \Downarrow$. This follows from the fact that, by the inductive hypothesis, $d(p_i, [\epsilon]T) \vDash [\epsilon]T$ and thus $d(p_i, [\epsilon]T) \Downarrow$, for each $i \in I$. The proof of the fact that $d(p, [\epsilon]T) \leq_{Er} p$ is routine and is omitted.

—If $\phi \equiv \bigvee_{h \in H} \phi_h$, then $p \vDash \bigvee_{h \in H} \phi_h$. This is because there exists $k \in H$ such that $p \vDash \phi_k$. We proceed by analyzing the form of $\phi_k$:

$\phi_k \equiv [a]\psi$. Then, $p \Downarrow a$ and, for each $q$ such that $p \stackrel{a}{\Rightarrow} q$, $q \vDash \psi$. Take $d(p, [\epsilon]\phi) \equiv d(p, [a]\psi)$. First of all, note that $d(p, [a]\psi) \leq_{Er} p$ follows immediately by the inductive hypothesis. We now show that $d(p, [a]\psi) \vDash [\epsilon]\bigvee_{h \in H} \phi_h$. Of course, $d(p, [a]\psi) \Downarrow$ as, by the inductive hypothesis, $d(p, [a]\psi) \vDash [a]\psi$. Assume now that $d(p, [a]\psi) \stackrel{\epsilon}{\Rightarrow} x$. If $x \equiv d(p, [a]\psi)$, then, by the inductive hypothesis, $d(p, [a]\psi) \vDash [a]\psi = \phi_k$. Hence, $d(p, [a]\psi) \vDash \bigvee_{h \in H} \phi_h$. If $d(p, [a]\psi) \stackrel{\tau}{\Rightarrow} x$, then it is easy to see that $x \vDash [a]\psi$ as well. Thus, $x \vDash \bigvee_{h \in H} \phi_h$. This establishes that $d(p, [a]\psi) \vDash [\epsilon]\bigvee_{h \in H} \phi_h$.

$\phi_k \equiv \langle a \rangle \psi$. Then, $p \vDash \langle a \rangle \psi$ iff there exists $j_1 \in J$ such that $a_{j_1} = a$ and $p_{j_1} \stackrel{\epsilon}{\Rightarrow} q \vDash \psi$, for some $q$. Take

$$d(p, \phi) \equiv \sum_{i \in I} \tau; d(p_i, \phi) + \sum_{j:\, a_j \neq a} a_j; \Omega + a; d(p_{j_1}, \langle \epsilon \rangle \psi) + a; \Omega.$$

Note that, by construction and the inductive hypothesis, $d(p, \phi) \leq_{Er} p$. We show that $d(p, \phi) \vDash [\epsilon]\bigvee_{h \in H} \phi_h$. Obviously, $d(p, \phi) \Downarrow$ as, by the inductive hypothesis, $d(p_i, \phi) \Downarrow$, for each $i \in I$. Assume that $d(p, \phi) \stackrel{\epsilon}{\Rightarrow} x$. Then there are two cases to examine:

(A) $x \equiv d(p, \phi)$. Then, $d(p, \phi) \stackrel{a}{\Rightarrow} nil; d(p_{j_1}, \langle \epsilon \rangle \psi)$. Since by the inductive hypothesis $d(p_{j_1}, \langle \epsilon \rangle \psi) \vDash \langle \epsilon \rangle \psi$, we have that $d(p, \phi) \vDash \langle a \rangle \psi$ and this implies $d(p, \phi) \vDash \bigvee_{h \in H} \phi_h$.

(B) $d(p_i, \phi) \stackrel{\epsilon}{\Rightarrow} x$ for some $i \in I$. By the inductive hypothesis, $d(p_i, \phi) \vDash \phi$ and this implies $x \vDash \bigvee_{h \in H} \phi_h$.

Hence, $d(p, \phi) \vDash \phi$.

$\phi_k \equiv [\epsilon]\psi$. Take $d(p, \phi) \equiv d(p, [\epsilon]\psi)$. Proving that $d(p, [\epsilon]\psi)$ meets the statement of the theorem is done exactly as in the case $\phi_k \equiv [a]\psi$.

$\phi_k \equiv \langle\epsilon\rangle\psi$. Take

$$d(p, \phi) \equiv d(p, \langle\epsilon\rangle\psi) \vee \left( \sum_{i\in I} \tau; d(p_i, \phi) + \sum_{j\in J} a_j; \Omega \right).$$

Note that, by the inductive hypothesis, $d(p, \langle\epsilon\rangle\psi) \leq_{Er} p$ and, as $d(p_i, \phi) \leq_{Er} p_i$ $(i \in I)$, we also have that $\sum_{i\in I}\tau; d(p_i, \phi) + \sum_{j\in J} a_j; \Omega \leq_{Er} p$. Thus, by the definition of $\vee$, $d(p, \phi) \leq_{Er} p$ and we have checked the second part of the statement of the theorem.

We are left to show that $d(p, \phi) \vDash \phi$. By the inductive hypothesis, $d(p_i, \phi) \vDash \phi$, for each $i \in I$. Thus, for each $i \in I$, $d(p_i, \phi) \Downarrow$. By the definition of $\Downarrow$, this implies that $(\sum_{i\in I}\tau; d(p_i, \phi) + \sum_{j\in J} a_j; \Omega \Downarrow$ and therefore that $d(p, \phi) \Downarrow$. Moreover, $d(p, \langle\epsilon\rangle\psi) \equiv d(p, \phi_k) \vDash \phi_k$, by the inductive hypothesis. Since $d(p, \phi_k) \tilde{\sqsubseteq}_\omega d(p, \phi)$, by the modal characterization theorem we get that $d(p, \phi) \vDash \phi_k$. Hence, by the definition of $\vDash$, $d(p, \phi) \vDash \bigvee_{h\in H} \phi_h$.

Assume now that $d(p, \phi) \overset{\tau}{\Rightarrow} y$. Then, since $\sum_{i\in I}\tau; d(p_i, \phi) + \sum_{j\in J} a_j; \Omega \tilde{\sqsubseteq} d(p, \phi)$, there exists some $i \in I$ and $x$ such that $d(p_i, \phi) \overset{\epsilon}{\Rightarrow} x$ and $x \tilde{\sqsubseteq} y$. Since $d(p_i, \phi) \vDash [\epsilon]\bigvee_{h\in H}\phi_h$ we have that $x \vDash \bigvee_{h\in H}\phi_h$. By the modal characterization theorem, the finiteness of $x$ and $y$ and Theorem 3.2, this implies $y \vDash \bigvee_{h\in H}\phi_h$. Thus, we have shown that $d(p, \phi) \vDash \phi$.

The proof of the case $\phi \equiv [\epsilon]\bigvee_{h\in H}\phi_h$ is thus complete.

$p \vDash [a]\phi$. Since $p \vDash [a]\phi$ we have that $p \Downarrow a$. This implies that $p \Downarrow$ and thus we may assume, without loss of generality, that $p$ is an hnf. If $p \equiv \delta$, then $d(p, [a]\phi) \equiv \delta$.

If $p \equiv \sum_{i\in I}\tau; p_i + \sum_{j\in J} a_j; p_j$ then

$$d(p, [a]\phi) \equiv \sum_{i\in I} \tau; d(p_i, [a]\phi) + \sum_{j:\, a_j = a} a; d(p_j, [\epsilon]\phi) + \sum_{j:\, a_j \neq a} a_j; \Omega.$$

Both the statements of the theorem are easily seen to hold when $d(p, [a]\phi) \equiv \delta$. Assume now that the other case occurs. First of all, note that $d(p, [a]\phi)$ is well defined as

—$p \vDash [a]\phi$ implies $p_i \vDash [a]\phi$, for each $i \in I$, and
—$p \vDash [a]\phi$ implies $p_j \vDash [\epsilon]\phi$, for each $j \in J$ such that $a_j = a$.

By the inductive hypothesis, $d(p_i, [a]\phi) \vDash [a]\phi$, for each $i \in I$, and $d(p_j, [\epsilon]\phi) \vDash [\epsilon]\phi$, for each $j \in J$ such that $a_j = a$. Thus, by the definition of $\vDash$, $d(p_i, [a]\phi) \Downarrow a$ and $d(p_j, [\epsilon]\phi) \Downarrow$. It is easy to see that this

implies $d(p, [a]\phi) \Downarrow a$. Assume now that $d(p, [a]\phi) \stackrel{a}{\Rightarrow} x$. Then either there exists $i \in I$ such that $d(p_i, [a]\phi) \stackrel{a}{\Rightarrow} x$ or, for some $j$ such that $a_j = a$, $d(p_j, [\epsilon]\phi) \stackrel{\epsilon}{\Rightarrow} x$. In both cases, by the inductive hypothesis and the definition of $\models$, we get that $x \models \phi$. We have thus shown that $d(p, [a]\phi) \models [a]\phi$.

Finally note that, by the inductive hypothesis, $d(p_i, [a]\phi) \leq_{Er} p_i$ and $d(p_j, [\epsilon]\phi) \leq_{Er} p_j$, for each $i \in I$ and $j$ such that $a_j = a$. Therefore, by construction, $d(p, [a]\phi) \leq_{Er} p$. This completes the proof of the theorem. □

*Example* 4.1. As an example of application of the construction of $d(p, \phi)$ given in the above proof, we shall give $d(p, \phi)$ for $p \equiv rec\, x.\ (x; a + a)$ and $\phi \equiv \phi_1 \wedge \phi_2$, with $\phi_1 \equiv \langle a \rangle [a] \perp$ and $\phi_2 \equiv \langle a \rangle \langle a \rangle T$. It is easy to see that $p \models \phi$ as

(a) $p \models \phi_1$ because $p \stackrel{a}{\rightarrow} nil \models [a] \perp$, and

(b) $p \models \phi_2$ because $p \stackrel{a}{\rightarrow} nil; a \models \langle a \rangle T$.

By the construction of the above theorem, $d(p, \phi_1) \equiv a; nil + \Omega$ and $d(p, \phi_2) \equiv a; (a; \Omega + \Omega) + \Omega$. Hence,

$$d(p, \phi) \equiv d(p, \phi_1) \vee d(p, \phi_2) = p^2 =_{Er} \Omega + a + a; a.$$

After this rather delicate and lengthy proof, we have all the technical machinery that is needed to prove that the preorder $\sqsubseteq_\omega$ is finitely approximable over $REC_\Sigma$.

THEOREM 4.3. *For each* $p, q \in REC_\Sigma$, $p \sqsubseteq_\omega q$ *iff* $p \sqsubseteq_\omega^F q$.

PROOF. We have already recalled that $p \sqsubseteq_\omega q$ implies $p \sqsubseteq_\omega^F q$. We now show that the converse implication also holds. Assume that $p \sqsubseteq_\omega^F q$ and $p \models \phi$, $\phi \in \mathcal{L}$. Then, by Theorem 4.2, $d(p, \phi) \models \phi$ and $d(p, \phi) \leq_{Er} p$. By the soundness of $\leq_{Er}$ with respect to $\sqsubseteq_\omega$, $d(p, \phi) \sqsubseteq_\omega p$.

Since $p \sqsubseteq_\omega^F q$, $d(p, \phi) \sqsubseteq_\omega q$. Since, by Theorem 4.2, $d(p, \phi) \models \phi$, by the modal characterization theorem $q \models \phi$. Thus, $\mathcal{L}(p) \subseteq \mathcal{L}(q)$ and, by the modal characterization theorem, this implies that $p \sqsubseteq_\omega q$. □

Because of the coincidence of $\sqsubseteq$ and $\sqsubseteq_\omega$ over $FREC_\Sigma \times REC_\Sigma$, we have that $\sqsubseteq_\omega$ is indeed the finitary part of the prebisimulation preorder $\sqsubseteq$, $\sqsubseteq^F$.

## 5. *Conclusion*

In this paper, we have developed a semantic theory for a process algebra that incorporates some explicit representation of successful termination, deadlock, and divergence. The process algebra that we have considered has been endowed with both an operational and a denotational semantics and the two semantic views of processes have been shown to agree. Namely, we have shown that the denotational model that we have proposed in this paper, the initial continuous algebra that satisfies a set of equations $CI_E$, is fully abstract with respect to a natural operational preorder over the language. The proof of the full abstraction theorem relies on several results of independent interest; namely, the finite approximability of the behavioral preorder and a partial completeness result for the set of inequations $E$ with respect to the preorder. The proof of the finite approximability of the behavioral preorder is one of the main novelties of the

paper; it relies on a characterization of the behavioral preorder in terms of a modal logic and makes a fundamental use of a novel construction that produces, for each term $p$ and modal formula $\phi$ satisfied by $p$, a finite approximant of $p$, $d(p, \phi)$, which satisfies $\phi$. We believe that the pattern followed in the proof of Theorem 4.3 provides a general technique to establish the finite approximability of bisimulation-like preorders that afford logical characterizations in terms of modal logics allowing finite conjunctions and disjunctions only.

As pointed out in [16] and [18], our choice of a denotational semantics for the language studied in this paper gives us a complete axiomatic proof system (albeit a nonrecursively enumerable one) for closed terms of the language. Moreover, as our denotational model is based upon the well-known theory of algebraic cpos, rather than metric spaces as in [6], we may obtain effective proof systems for the language by using induction rules such as Scott Induction and Fixed-Point Induction [24]. Other advantages of using the theory of cpos rather than metric spaces are that all of the usual operators found in process algebras may be readily interpreted, as no restriction need be placed on the recursive definitions allowed in the language $REC_\Sigma$, and that features like silent actions and encapsulation/abstraction operators may be smoothly dealt with within it. On the other hand, using metric spaces, we can only readily interpret operators that are *contractive* and this requirement imposes restrictions on their applicability. For instance, it is well known that unguarded recursive definitions give rise to operators which are not contractive. Moreover, essential features of process algebras like silent actions and abstraction/encapsulation operators have never been dealt with satisfactorily in this framework.

The language we have considered in this paper incorporates features from **CCS** and **ACP**. It extends **ACP** by allowing an explicit representation of successful termination and divergence; moreover, our language allows for general recursive definitions. The auxiliary operators which **ACP** uses to axiomatize $|$ (namely, $\lfloor$, for *left-merge*, and $|_c$, for the *communication merge*) could be added to our language without affecting the results of the paper. The language extends **CCS** as it allows general sequential composition and an explicit representation of deadlock (as opposed to successful termination). However, the signature of **CCS** contains a family of *relabelling operators* $\_[R]$, where $R$: $Act_\tau \to Act_\tau$ is a function such that $R(\bar{a}) = \overline{R(a)}$ and $R(\tau) = \tau$. The introduction of such an operator in the signature of our language would cause some problems. To see that, we recall that our results about the finite approximability of the behavioral preorder $\underset{\sim}{\sqsubseteq}_\omega^c$ depend on the sort-finiteness of our transition system semantics for the language (see Section 4). However, if $Act$ is infinite, this is no longer the case. To show this, consider an enumeration $\{a_0, a_1, \ldots, a_i, \ldots\}$ of the set of observable actions $Act$. Using the enumeration of $Act$, we may define a relabelling $S$ such that $S(a_i) = a_{i+1}$, for each $i \in \omega$. Take the process $p$ defined as follows, [2]:

$$p \equiv rec\, x.\ a_0 + x[S].$$

Then it is easy to see that the unguarded recursive definition and the generality of $S$ give rise to a process that is not sort-finite. In fact, $p \xrightarrow{a_i}$ for each $i \in \omega$. As a consequence of these observations, our behavioral preorder would not be finitely approximable and $CI_E$ would not be fully abstract with respect to it.

However, it may be argued that one rarely, if ever, needs relabelling operators of such a generality. In practice, relabelling functions are usually assumed to be constant on all but finitely many actions in $Act_\tau$. If we allow only this kind of relabellings in our language, then the resulting transition system semantics will again be sort-finite, [1, 2], and thus all of the results of the paper will carry through to this extended language.

An interesting point to note is that most of the technical analysis of our operational preorder $\sqsubseteq_\omega^c$ has been carried out by using $\lesssim$ and $\lesssim^*$. Since $\lesssim$ is technically simpler than $\sqsubseteq$, it might be that some of the results of the paper could have been obtained in a simpler way by using $\lesssim_\omega^c$ as behavioral preorder. However, this is not the case. It turns out that no denotational model of the form $CI_{E'}$, for any set of equations $E'$, can be fully abstract with respect to $\lesssim_\omega^c$. This is because in such a model all the syntactically finite terms, i.e. terms from $FREC_\Sigma$, are interpreted as semantically finite elements. That is, if $d \in FREC_\Sigma$ and $d \leq_{CI_{E'}} p$ then, for some finite approximation of $p$, $p^n$, $d \leq_{CI_{E'}} p^n$. This property does not hold of $\lesssim_\omega^c$ and so $\lesssim_\omega^c$ can coincide with $\leq_{CI_{E'}}$ for no set of equations $E'$. As a counterexample, consider the two synchronization trees:

$$\begin{aligned} d &\equiv b; a + \Omega, \\ p &\equiv \sum_{k \geq 1} b; q(k) + \Omega, \end{aligned}$$

where

$$\begin{aligned} q(1) &\equiv a + \tau; a; c, \\ q(k+1) &\equiv a + \tau; q(k), \qquad k \geq 1. \end{aligned}$$

Note that, for each $k \geq 1$, $a \lesssim_k q(k)$ and therefore $d \lesssim_\omega p$. The finite approximations to $p$ are all of the form

$$p^m \equiv \sum_{1 \leq k \leq m} b; q(k) + \Omega.$$

However, for each $m \geq 1$, $d \not\lesssim_\omega p^m$. In fact, $d \not\lesssim_{m+3} p^m$ because, for each $k$, $a \not\lesssim_{k+2} q(k)$.

We end this conclusion with a brief comparison with related work. Several term model constructions [25], for **CCS**- and **SCCS**-like languages have been proposed in the literature (see e.g., [16] and [22]). In each of these papers, a denotational semantics is given to the languages considered by means of the initial continuous algebra that satisfies a set of equations $E$. The denotational model is then shown to be fully abstract with respect to a behavioral preorder. In [12], DeNicola and Hennessy show how the denotational models of the *testing equivalences* they introduce have a natural representation in terms of a particular class of trees, the *acceptance trees* of [17]. In [2], Abramsky takes a language-independent standpoint and analyzes the general relationships between strong prebisimulation, $\sqsubseteq$, over transition systems and its finitary part, $\sqsubseteq^F$. Abramsky also shows how his general results may be used to obtain a fully abstract model wth respect to (the finitary part of) strong prebisimulation over a version of **SCCS** [28], with only finite summations and relates his model to the one in [16]. In [36], a behavioral relation similar to $\lesssim$ is studied and applied to **CCS**; complete axiomatizations are given for finite and regular processes. In many ways, the present paper may be considered as an extension of this work, employing ideas from [22]. It provides the first comprehensive treatment of a

weak version of prebisimulation and, in addition, it establishes a mathematical setting within which the notions of termination, divergence, and deadlock may be compared and contrasted. Similar motivations are at the heart of [9]. There, Bergstra et al. present several axiomatic systems to reason about successful termination, deadlock, and divergence in the theories of both bisimulation and failures equivalence [10]. Models for the equational theories are exhibited; thus, proving their logical consistency. Apart from a systematic analysis of axiom systems and semantic models dealing with the notions of abstraction and divergence, the paper presents a new failure semantics that allow *fair abstraction of unstable divergence*. This semantics does not always consider divergence as catastrophic, as it is done in [10], for example, and a weak form of *Koomen's Fair Abstraction Rule* [3] holds for it. The theory presented in [9] however, only deals with a language without parallel features and no completeness result, relating the axiomatic systems and the equivalences presented in the paper, is shown. In [33], a modal logic similar to the one employed in Section 4 of this paper is used to construct an information system [34] that generates a complete partial order of synchronization trees [26], $\mathscr{P}_c(T_\Sigma)$. The elements of $\mathscr{P}_c(T_\Sigma)$, called *forests*, are sets of synchronization trees closed with respect to strong observational equivalence [26] and a suitable metric [14]. Some operations, among which a general notion of sequential composition dealing with deadlock and successful termination, are defined over $\mathscr{P}_c(T_\Sigma)$ and used to give a denotational semantics for a **CSP**-like language. However, the paper, being mostly concerned with a study of the mathematical properties of the space $\mathscr{P}_c(T_\Sigma)$, does not attempt an operational justification of the denotational semantics or an equational characterization of the congruence induced by it.

The dichotomy deadlock/successful termination has been dealt with in a diferent fashion in **CSP** [10, 23] and the latest papers on **ACP** [4, 5]. Both these process algebras introduce an explicit constant standing for successful termination, SKIP in **CSP** and $\epsilon$ in **ACP**. These constants obey the following operational rules:

$-\texttt{SKIP} \xrightarrow{\surd} \texttt{STOP}$, and

$-\epsilon \xrightarrow{\surd} \delta$,

where STOP and $\delta$ are the constants used to denote deadlock in **CSP** and **ACP**, respectively. The intuition captured by the above-given rules is that successful termination is an action in the behavior of a process, the action processes perform when they terminate. On the other hand, a deadlocked process like STOP or $\delta$ is one that cannot perform any move, not even a successful termination one. This is reflected in the equational laws satisfied by, for example, $\epsilon$ in **ACP**. For instance, in the equational theory of **ACP** with the empty process $\epsilon$, the equation

$$\delta + \epsilon = \epsilon$$

replaces our $\delta + nil = \delta$. Indeed, in that theory $\delta$ always gets canceled in a sum context, that is, the equation

$$x + \delta = x$$

holds without any conditions on $x$. However, the equation

$$x + \epsilon = x$$

no longer holds (contrary to what happens for our *nil*). In the theory of **ACP**, these equations express some a priori considerations about the properties that

concurrent, communicating systems are expected to have and are used to describe the intended semantics of processes. The consistency of such axiomatic descriptions of the semantics of processes is then shown by exhibiting models for the axioms. In this paper, following Milner [26], we have taken the view that operational semantics should be the touchstone for assessing mathematical models of concurrent processes. In this approach, operational semantics is used as a framework within which different intuitions about the behavior of processes may be expressed and compared. Equational theories, for example, complete axiomatizations of some notion of behavioral equivalence or preorder, are then derived from and justified by the operational description of processes. An operational description of the semantics of processes allows us to discuss different intuitions about successful termination and deadlock. The approach we have followed in this paper is based upon the intuition that both deadlocked processes and successfully terminated ones do not perform any move and that the only way of behaviorally distinguishing them is to observe their behavior in contexts built using sequential composition. However, we can revise our framework in at least two ways so as to give an operational understanding to the **ACP** theory of $\epsilon$. One involves changing the interpretation of the termination predicate $\surd$. Following the intuition underlying the **ACP** treatment of the successfully terminated process $\epsilon$, $p\surd$ may be read as *p has a termination possibility* or *p may terminate*, as it is done in [5]. A termination predicate which is more in line with the **ACP** theory may then be defined by changing rule (iii) of Definition 2.2 to

$$p\surd \text{ implies } (p + q)\surd \quad \text{and} \quad (q + p)\surd.$$

Another possibility involves the introduction of a special action, $\surd$, and defining $\epsilon$ to be $\surd; \delta$. In both cases we would obtain the **ACP** laws for $\epsilon$. Alternatively, we could revise the language by replacing *nil* with $\epsilon$. Our results carry through to the revised language after simple modifications to the operational semantics, the set of equations $E$, the behavioral preorder and the modal logic considered in Section 4. These changes are needed in order to take into account the different nature between $\epsilon$ and *nil*. This shows that the proof techniques employed in the paper to prove our full abstraction result are indeed quite general and easily adapted to capture different intuitive notions of successful termination in a language.

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