Terminological Logics with Modal Operators

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Abstract

Terminological knowledge representation formalisms can be used to represent objective, time-independent facts about an application domain. Notions like belief, intentions, and time which are essential for the representation of multi-agent environments can only be expressed in a very limited way. For such notions, modal logics with possible worlds semantics provides a formally well-founded and well-investigated basis. This paper presents a framework for integrating modal operators into terminological knowledge representation languages. These operators can be used both inside of concept expressions and in front of terminological and assertional axioms. We introduce syntax and semantics of the extended language, and show that satisfiability of finite sets of formulas is decidable, provided that all modal operators are interpreted in the basic logic K, and that the increasing domain assumption is used.

1 Introduction

Terminological knowledge representation languages in the style of KL-ONE [Brachman and Schmolze, 1985] have been developed as a structured formalism to describe the relevant concepts of a problem domain and the interactions between these concepts. Various terminological systems have been designed and implemented that are based on the ideas underlying KL-ONE (see [Woods and Schmolze, 1992] for an overview). Representing knowledge of an application domain with such a kind of system amounts to introducing the terminology of this domain via concept definitions, and then describing (an abstraction of) the relevant part of the "world" by listing the facts that hold in this part of the world. In a traditional terminological system, such a description is rigid in the sense that it does not allow for the representation of notions like time, or beliefs of different agents. In systems modeling aspects of intelligent agents, however, intentions, beliefs, and time-dependent facts play an important role.

Modal logics with possible worlds semantics is a formally well-founded and well-investigated framework for the representation of such notions. The present paper is concerned with integrating modal operators (for time, belief, etc.) into a terminological formalism. The first task is to find an appropriate semantics for the combined language. In addition, if such a language should be used in a system, one must design algorithms for the important inference problems (such as consistency of knowledge bases) for the language.

Several approaches have been proposed for the combination of terminological formalisms with notions like time or beliefs. A very simple possibility to represent beliefs of agents is realized in the partition hierarchy SB-PART [Kobsa, 1989], which is an extension of the SB-ONE system. In this approach, each agent may have its own set of terminological axioms *(TBox),* and these TBoxes can be ordered hierarchically. However, this extension lacks a formal semantics and it does not allow for representing properties of belief, such as introspection, or interactions between beliefs of different agents. A more formal approach is used in M-KRYPTON [Saffiotti and Sebastiani, 1988], where a sub-language of the KRYPTON representation language is extended by modal operators B_i, which can be used to represent the beliefs of agent *i.* Properties of beliefs are taken into consideration by using the well-known modal logic KD45. Due to the undecidable base language, however, [Saffiotti and Sebastiani, 1988] just introduces a formal semantics, without giving any inference algorithms for the extended language. In [Schild, 1991], it has been shown that terminological systems already have a strong connection to modal logic. In fact, the concept language *ACC* is nothing but a syntactic variant of the propositional multi-modal logic $K_{(m)}$. Building upon this observation, [Schild, 1993] augments *ACC* by tense operators. The two approaches that come next to the one we shall introduce below are described in [Laux, 1994a; 1994b] and in [Ohlbach and Baader, 1993]. Both extend *ACC* by modal operators, but with different emphasis. The differences between these approaches and ours are clarified in the next section.

2 Classification

When extending a terminological knowledge representation language by modalities for belief, time, etc. one has various degrees of freedom. Before describing the specific choices made in this article, we shall informally explain the different alternatives.

For simplicity, assume that we are interested in time and belief operators only. Thus, in addition to the objects we have time points and belief worlds. This means that the domain of an interpretation is the Cartesian product $D = D_{object} \times D_{time} \times D_{beltef}$ of the set of objects, the set of time points, and the set of belief worlds. Concepts are no longer just sets of objects; their interpretation also depends on the actual belief world and time point. Thus, they can be seen as subsets of *D,* and not just as subsets of D_{objtct}. Roles operate on objects, whereas modalities for time (like future or *tomorrow)* operate on time points, and modalities for belief (like *hel-John)* operate on belief worlds. As for concepts, however, the interpretation of roles and modalities depends on all dimensions. Thus, a role *loves* is interpreted as a function from *D* into $2^{D_{object}}$ relates any individual in *Dobject* (say John) with a set of individuals (the individuals John loves), but this set depends on the actual time point and belief world. Modalities like *future* are treated analogously. Modal operators can now be used both inside of concept expressions and in front of concept definitions and assertions. For example, we can describe the set of individuals that love a woman that—according to John's belief—is pretty by the concept expression 3 *loves.(*Woman H *[bel-John]Pretty)*, and we can express that—according to John's belief—a happy husband is one married to a woman whom he (John) believes to be pretty by the terminological axiom

[bel-Joh n] (*Happy-h usband -*
3 **m** *arried- to.(Worn an* ⊓ *[bel- Joh n] Pre tty)).*

The assertion *[bel-John](future) (Peter married-to Mary)* says that John believes that, at some point in the future, Peter will be married to Mary.

With the usual interpretation of the Boolean operators, of value and exists restrictions on roles, and of box and diamond operators for the modalities, this yields a rnulti-dimensional version of the multi-modal logic K_m. As described until now, this logic is a strict sub-language of the one introduced in [Ohlbach and Baader, 1993]. The restriction lies in the fact that, unlike in [Ohlbach and Baader, 1993], we do not consider roles and modalities that have a complex structure (such as *[wants]own,* where the modality wants is used to modify the role own). There are several reasons why this approach is not, yet satisfactory. First, the object and the other dimensions are treated analogously. This means, for example, that the interpretation of the modality *future* depends not only on the actual time point, but also on the current object and the belief world. Whereas the dependence from the belief world may seem to be quite reasonable, it is rather counterintuitive that the future time points reached from time t_0 are different, depending on whether we are interested in the individual Sue or Mary. Thus, it seems to be more appropriate to treat the object dimension in a special way: whereas the interpretation of roles should depend on the actual time point etc., the interpretation of modalities should not depend on the object under consideration.

The need for a special treatment of the object dimension can also be motivated by considering the semantics of concept definitions (and assertions). In [Ohlbach and Baader, 1993], concept definitions are required to hold for all objects, time points, and belief worlds. This is a straightforward generalization of the treatment of definitions in terminological languages, where a definition *C* = *D* must hold for *all* objects, i.e., in a model of *C — D* all objects *o* must satisfy that *o* belongs to the interpretation of *C* iff it belongs to the interpretation of *D.* For the other dimensions, however, this differs from the usual definition of models in modal logics, where a formula is only required to hold in one world.

Another problem is that not only the roles, but also all the other modalities are just interpreted in the basic logic K, i.e., they are not required to satisfy specific axioms for belief or time. In the present paper, we shall not take into account this last aspect, but we shall treat the object dimension in a special way, thus eliminating the problems mentioned above. In [Laux, 1994a; 1994b] both aspects are considered. However, modal operators are not allowed to occur inside of concept expressions, which considerably simplifies the algorithmic treatment of the formalism. The difference to [Ohlbach and Baader, 1993] is, on the one hand, the special treatment of the object dimension. In addition, [Ohlbach and Baader, 1993] does not consider assertions, and even though concept definitions are introduced, they are not handled by the satisfiability algorithm. On the other hand, [Ohlbach and Baader, 1993] allows for very complex roles and modalities, which are not considered here.

3 Syntax and Semantics of *ACCM*

First, we present the syntax of our multi-dimensional modal extension of the concept language *ACC.* As for *ACC,* we assume a set of concept names, a set of role names, and a set of object names to be given. Beside the object dimension (which will be treated differently from the other dimensions), we assume that there are *v* > 1 additional dimensions (such as time points, epistemic alternatives, or intensional states). In each dimension, there can be several modalities, which can be used in box and diamond operators. For example, in the dimension *time points* we could have *future* and *tomorrow,* **and** in the dimension *belief worlds* we could have *belief-John* and *belief-Mary.* If *o* is a modality of dimension t we write *dim(o) — i.* In this case, *[o]* and *(o)* are *modal operators* of dimension *i.*

Definition 3.1 (Syntax) Concepts *of ACCM ARE* ^tn^~ *ductively defined as follows. Each concept name is a concept, and* T *and* _L *are concepts. If C* **and** *D are concepts, R is a role name,* **and** *o is a modality then* *C H D* (concept conjunction), *CUD* (concept disjunction), -C (concept negation), ∀*R.C* (value restriction), ∃*R.C* (exists restriction), *[o]C* (box operator), *and (o)C* (diamond operator) *are concepts.*

Terminological axioms *of ACCM ARE of the form m (C = D) where C and D are concepts of* ***ACCM And*** *m is a (possibly empty) sequence of modal operators.* Assertional axioms *of ACCM ore of the form* **m** *(xRy) or* m *(x : C)* *where x and y are object names, R is a role name, C is a concept, and m is a (possibly empty) sequence of modal operators. An ACCM*-formula *is either a terminological or an assertional axiom.*

Traditional terminological systems impose severe restrictions on the admissible sets of terminological axioms: *(1)* The concepts on the left-hand sides of axioms must be concept names, *(2)* concept names occur at most once as left-hand side of an axiom (unique definitions), and *(3)* there are no cyclic definitions. The effect of these restrictions is that terminological axioms are just macro definitions (introducing names for large descriptions), which can simply be expanded before starting the reasoning process. Unrestricted terminological axioms are a lot harder to handle algorithmically (see, e.g., [Buchheit *et a*/., 1993]), but they are very useful for expressing constraints on concepts that are required to hold in the application domain. In the presence of modal operators, the requirement of having unique definitions is not appropriate anyway. For example, Peter may have a definition of *Happy-husband* that is quite different from John's definition. Thus, it is desirable to have different definitions $m_1(A = C)$ and $m_2(A = D)$ of the same concept name *A* for different modal sequences mi and m_2- Even though m_1 and m_2 are different, there can be interactions between these definitions. For example, m_1 could be of the form (o) and m_2 of the form *[o]* . Thus, it is not a priori clear how the requirement of "unique definitions" can be adapted to the case of terminological axioms with modal prefix. To avoid these problems, we consider the more general case where arbitrary axioms are allowed.

Let us now turn to the semantics of *ACCM* ■ The modal operators will be interpreted by a Kripke-style possible worlds semantics. Thus, for each dimension i we need a set of possible worlds D_i),. Modalities of dimension i correspond to accessibility relations on D_i, which may, however, depend on the other dimensions as well. Concepts and roles are interpreted in an object domain, but this interpretation also depends on the modal dimensions.

Definition 3.2 (Semantics) *A* Kripke structure $K = (\mathcal{W}, \Gamma, K_I)$ *consists of a set $\mathcal{W}$ of possible worlds, a set of accessibility relations Γ, and a K-interpretation K_I over $\mathcal{W}$, which are given as follows. First, the set of possible worlds $\mathcal{W}$ is the Cartesian product of non-empty domains $D_1, \ldots, D_\nu$, one for each dimension. Second, Γ contains for each modality o of dimension i an accessibility relation γ_o, which is a function $\gamma_o : \mathcal{W} \to 2^{D_i}$. We also write $((d_1, \ldots, d_i, \ldots, d_\nu), (d_1, \ldots, d'_i, \ldots, d_\nu)) \in \gamma_o$ for $d'_i \in \gamma_o(d_1, \ldots, d_i, \ldots, d_\nu)$. Finally, the K-interpretation K_I consists of a domain Δ^{K_I} and an interpretation function $\cdot^{K_I}$. The domain is the union of non-empty domains $\Delta^{K_I}(w)$ for all worlds $w \in \mathcal{W}$. The interpretation function associates (i) with each object name x an element $x^{K_I} \in \Delta^{K_I}$, (ii) with each concept name A and world $w \in \mathcal{W}$ a set $(A, w)^{K_I} \subseteq \Delta^{K_I}(w)$, (iii) with the top concept and the bottom concept the sets $(\top, w)^{K_I} = \Delta^{K_I}(w)$ and $(\bot, w)^{K_I} = \emptyset$ (for each world w), and (iv) with each role name R and world $w \in \mathcal{W}$ a binary relation $(R, w)^{K_I} \subseteq \Delta^{K_I}(w) \times \Delta^{K_I}(w)$.*

$$
\begin{aligned}
(C \sqcap D, w)^{K_I} &= (C, w)^{K_I} \cap (D, w)^{K_I} \\
(C \sqcup D, w)^{K_I} &= (C, w)^{K_I} \cup (D, w)^{K_I} \\
(\neg C, w)^{K_I} &= \Delta^{K_I}(w) \setminus (C, w)^{K_I} \\
(\forall R.C, w)^{K_I} &= \{\delta \in \Delta^{K_I}(w) \mid \delta' \in (C, w)^{K_I} \\
&\qquad \text{for each } \delta' \text{ with } (\delta, \delta') \in (R, w)^{K_I}\} \\
(\exists R.C, w)^{K_I} &= \{\delta \in \Delta^{K_I}(w) \mid \delta' \in (C, w)^{K_I} \\
&\qquad \text{for some } \delta' \text{ with } (\delta, \delta') \in (R, w)^{K_I}\} \\
([o]\, C, w)^{K_I} &= \{\delta \in \Delta^{K_I}(w) \mid \delta \in (C, w')^{K_I} \\
&\qquad \text{for each } w' \text{ with } (w, w') \in \gamma_o\} \\
(\langle o \rangle\, C, w)^{K_I} &= \{\delta \in \Delta^{K_I}(w) \mid \delta \in (C, w')^{K_I} \\
&\qquad \text{for some } w' \text{ with } (w, w') \in \gamma_o\}
\end{aligned}
$$

Observe that, for each concept C and world w, we have $(C, w)^{K_I} \subseteq \Delta^{K_I}(w)$. Two $\mathcal{ALC}_\mathcal{M}$ concepts C and D are called *equivalent* iff for all Kripke structures $K = (\mathcal{W}, \Gamma, K_I)$ and all worlds $w \in \mathcal{W}$ we have $(C, w)^{K_I} = (D, w)^{K_I}$.

Now we can define under which conditions an $\mathcal{ALC}_\mathcal{M}$-formula F is *satisfied in a Kripke structure* $K = (\mathcal{W}, \Gamma, K_I)$ *and a world* $w \in \mathcal{W}$, written as $K, w \models F$, by induction on the length of the modal prefix:

$$
\begin{aligned}
K, w \models C = D &\quad \text{iff} \quad (C, w)^{K_I} = (D, w)^{K_I}, \\
K, w \models x : C &\quad \text{iff} \quad x^{K_I} \in (C, w)^{K_I}, \\
K, w \models xRy &\quad \text{iff} \quad (x^{K_I}, y^{K_I}) \in (R, w)^{K_I}, \\
K, w \models [o]\, G &\quad \text{iff} \quad K, w' \models G \\
&\qquad \text{for each world } w' \text{ with } (w, w') \in \gamma_o, \\
K, w \models \langle o \rangle\, G &\quad \text{iff} \quad K, w' \models G \\
&\qquad \text{for some world } w' \text{ with } (w, w') \in \gamma_o.
\end{aligned}
$$

Here G is an $\mathcal{ALC}_\mathcal{M}$-formula, C, D are concepts, x, y are object names, R is a role name, and o is a modality. A set $\{F_1, \ldots, F_n\}$ of $\mathcal{ALC}_\mathcal{M}$-formulas is *satisfiable* iff there exists a Kripke structure $K = (\mathcal{W}, \Gamma, K_I)$ and a world $w_0 \in \mathcal{W}$ such that $K, w_0 \models F_i$ for $i = 1, \ldots, n$. In this case we write $K \models F_1, \ldots, F_n$.

Even though we have introduced a domain $\Delta^{K_I}(w)$ for each world w, we have not yet said anything about the relationship between the domains of different worlds. In the simplest approach, the domains $\Delta^{K_I}(w_1)$ and $\Delta^{K_I}(w_2)$ of each pair w_1, w_2 of worlds are independent of each other. This approach is known as *varying domain assumption*. In most cases, however, it is more reasonable to assume certain relationships between the domains of different worlds. The most commonly used approach is the so-called *increasing domain assumption*, where $\Delta^{K_I}(w) \subseteq \Delta^{K_I}(w')$ if the world w' is accessible from the world w, i.e., there exists a modality o such that $(w, w') \in \gamma_o$. The advantage of this approach is that domain elements that have been introduced in w can also be referred to in all worlds that are accessible from w, i.e., domain elements do not "vanish" when we move from one world to another. As a special case, the *constant domain assumption* is sometimes used, where the domains $\Delta^{K_I}(w_1)$ and $\Delta^{K_I}(w_2)$ are identical for all worlds W_1 and w_2. Finally, the *decreasing domain assumption* can be used to express that new domain elements cannot arise when moving from one world to another one. In Section 5 we shall see that changing the requirements on the relationship between domains of worlds considerably changes the set of satisflable formulas.

With the exception of Section 5, however, we shall restrict our attention to increasing domains in the following. Furthermore, we assume that all terminological axioms are of the form m $(C = \top)$, where C is a concept and m is a (possibly empty) sequence of modal operators. As in the case of *ACC* without modalities, it is easy to verify that this can be done without loss of generality.

4 Testing Satisfiability of *ACCM* -formulas

We present an algorithm for testing satisfiability of a finite set $\{F_1, \ldots, F_n\}$ of *ACCM*-formulas.[1] To keep notation simple we assume (without loss of generality) that concepts are in *negation normal form,* i.e., negation signs occur immediately in front of concept names only. Our calculus for testing satisfiability of ACC_M-formulas is based on the notions of labeled *ACCM*-formulas and of world constraint systems. A *labeled ACCM -formula* consists of an *ACCM*-formula F together with a label /, written as $F \,||\, l$. The label / is a syntactic representation of a world in which F is required to hold. A *world constraint* is either a labeled *ACCM*-formula or a term $l \bowtie_o l'$, where l, l' are labels and $\bowtie_o$ is a syntactic representation of the accessibility relation of modality *o*. A *world constraint system* is a finite, non-empty set of world constraints.

A Kripke structure $K = (W, \Gamma, K_I)$ satisfies a world constraint system W iff there is a mapping α that maps labels in W to worlds in W such that (i) $K, \alpha(l) \models F$ for each world constraint $F \,||\, l$ in W, and *(ii)* $(\alpha(l), \alpha(l')) \in \gamma_o$ for each world constraint $l \bowtie_o l'$ in W. A world constraint system W is *satisfiable* iff there exists a Kripke structure satisfying W. In order to test satisfiability of a set $\{F_1, \ldots, F_n\}$ of *ACCM*-formulas we translate this set into the world constraint system $W_0 = \{x_0 : \top \,||\, l_0, F_1 \,||\, l_0, \ldots, F_n \,||\, l_0\}$, where x_0 is a new object name not occurring in $\{F_1, \ldots, F_n\}$, and l_0 is an arbitrary label (which is intended to represent the real world). We say the world constraint system W_0 is *induced by* $\{F_1, \ldots, F_n\}$. It is easy to verify that $\{F_1, \ldots, F_n\}$ is satisfiable iff W_0 is satisfiable. The world constraint $x_0 : \top \,||\, l_0$ can obviously be satisfied by any Kripke structure. This constraint is necessary to guarantee that the domains $\Delta^{K_I}(w)$ of the canonical Kripke structure constructed in the proof of completeness are non-empty (see the full paper [Baader and Laux, 1994]).

The *ACCM*-satisfiability algorithm takes as input a world constraint system W_0 that is induced by a finite set of AICCM-formulas. It successively adds new world constraints to W_0 by applying several propagation rules, which will be defined later. A world constraint system that is induced by a finite set of $\mathcal{ALC_M}$-formulas, or that is obtained by a finite sequence of applications of propagation rules to an induced system, will be called *derived system*.

In the following, we use the letters x, y, z to denote object names, l to denote labels, A, B to denote concept names, C, D to denote concepts, and R to denote role names. If necessary, these letters will have an appropriate subscript. Before introducing the propagation rules in a formal way (in Figure 1), let us first describe the underlying ideas on an intuitive level. The rules that handle the usual $\mathcal{ALC}$ concept forming operators are well-known and rather straightforward (see, e.g., [Baader and Hollunder, 1991]). In order to illustrate the rules that handle modalities and world constraints of the form $C = \top \,||\, l$, suppose that the $\mathcal{ALC_M}$-formula $\langle o \rangle\,(B = \top)$ is given, where o is a modality of some dimension. In order to test satisfiability of this $\mathcal{ALC_M}$-formula, we start with the induced world constraint system $W_0 = \{x_0 : \top \,||\, l_0, \langle o \rangle\,(B = \top) \,||\, l_0\}$. By definition, W_0 is satisfiable iff there is a Kripke structure $K = (W, \Gamma, K_I)$, a mapping α, and a world $w_0 = \alpha(l_o) \in W$ such that $x_0^{K_I} \in \Delta^{K_I}(w_0)$ and $K, w_0 \models \langle o \rangle\,(B = \top)$. Since $K, w_0 \models \langle o \rangle\,(B = \top)$ iff $K, w_1 \models B = \top$ for some world w_1 with $(w_0, w_1) \in \gamma_o$, the $\rightarrow_{\diamond}$ rule adds the world constraints $l_0 \bowtie_o l_1$ and $B = \top \,||\, l_1$ to W_0, where l_1 is a new label. This yields the new world constraint system $W_1 = W_0 \cup \{l_0 \bowtie_o l_1, B = \top \,||\, l_1\}$. Because of the semantics of $\mathcal{ALC_M}$-formulas, we furthermore know that $K, w_1 \models B = \top$ iff $\delta \in (B, w_1)^{K_I}$ for all $\delta \in \Delta^{K_I}(w_1)$. By the increasing domain assumption, $x_0^{K_I} \in \Delta^{K_I}(w_0)$ implies $x_0^{K_I} \in \Delta^{K_I}(w_1)$. Summing up, we must guarantee that $x_0^{K_I} \in (B, w_1)^{K_I}$ and therefore must add the world constraint $x_0 : B \,||\, l_1$ to W.

More generally, we say that an object name x is *relevant for label* l (in a world constraint system W) iff there is a label l' occurring in W such that (i) W contains a world constraint of the form $x : C \,||\, l'$, $xRy \,||\, l'$, or $yRx \,||\, l'$, and (ii) l is *accessible from* l', i.e., either l is l' or there are world constraints $l' \bowtie_{o_1} l_1, \ldots, l_{n-1} \bowtie_{o_n} l$ in W for some modalities $o_1, \ldots, o_n$. Now, if x is relevant for l and there is a world constraint $C = \top \,||\, l$ in W for some concept C, then the $\rightarrow_=$ rule adds $x : C \,||\, l$ to W (unless this world constraint is already contained in W). In our example, this rule applies to W_1, and it yields the world constraint system $W_2 = W_1 \cup \{x_0 : B \,||\, l_1\}$. To W_2 no more propagation rules are applicable, and—as shown in [Baader and Laux, 1994]—we can use this system to construct a Kripke structure that satisfies the $\mathcal{ALC_M}$-formula $\langle o \rangle\,(B = \top)$. A world constraint system to which no more propagation rules are applicable will be called *complete*.

Termination of the propagation rule applications can only be guaranteed if applicability of the usual rule for handling exists restrictions is restricted in an appropriate way. This is due to the presence of axioms of the form $C = \top$. To illustrate this problem, consider the world constraint system $W = \{x : A \,||\, l, \exists R.C = \top \,||\, l\}$. Since x is relevant for l, the $\rightarrow_=$ rule adds $x : \exists R.C \,||\, l$. Now, the

[1] It is easy to see that all the other interesting inference problems (like the subsumption or the instance problem) can be reduced to this problem.

usual propagation rule $\rightarrow_\exists$ that treats exists restrictions would add $xRy \,||\, l$ and $y : C \,||\, l$ to W, where y is a new object. However, y is again relevant for l, and thus we must add $y : \exists R.C \,||\, l$. The $\rightarrow_\exists$ rule would thus be applicable to $y : \exists R.C \,||\, l$, generating new world constraints $yRz \,||\, l$ and $z : C \,||\, l$, etc.

In order to avoid such infinite chains of rule application, we introduce the notion of blocked objects.[2] Intuitively, an object x is blocked w.r.t. label l if we need not introduce a new object in order to be sure that the exists restrictions on x can be satisfied. Consider, for instance, the world constraint system $W = \{x : \exists R.C||l, x : D||l, xRy||l, y : \exists R.C||l\}$. In this case, it is sufficient to apply the $\rightarrow_\exists$ rule just to x. In fact, since all constraints for y are also constraints for x, any contradiction that could be obtained by applying this propagation rule to y can already be obtained by applying it to x. The idea is thus to say that y is blocked by x with respect to a label l if $\{C \mid y : C||l \in W\} \subseteq \{D \mid x : D||l \in W\}$. In the above example, y would thus be blocked by x, and the $\rightarrow_\exists$ rule would only be applied to x. In general, this notion of blocking is too strong, though. In fact, consider the system W' that is obtained from W by deleting the constraint $x : D||l$. In this system, x would be blocked by y and vice versa. Such cyclic blocking is clearly not appropriate since contradictions that are possibly hidden in C would never be detected.

In order to avoid cyclic blocking, we assume that the (countably infinite) set of all object names is given by an enumeration $y_1, y_2, y_3, \ldots$ We write $y < x$ if y comes before x in this enumeration. This ordering is used as follows. Whenever a new object y is introduced by applying the $\rightarrow_\exists$ rule to a world constraint system W, y is chosen such that all objects in W are smaller than y w.r.t. this ordering. In addition, only smaller objects can block a given object.

Definition 4.1 *An object y is blocked by an object x w.r.t. label l in a world constraint system W iff* $\{C \mid y : C \,||\, l \in W\} \subseteq \{D \mid x : D \,||\, l \in W\}$ and $x < y$.

Now, the $\rightarrow_\exists$ rule is applicable to a world constraint $x : \exists R.C \,||\, l$ in a world constraint system W only if x is not blocked by some object z w.r.t. l in W. A formal description of the propagation rules is given in Figure 1. Given a set $\{F_1, \ldots, F_n\}$ of $\mathcal{ALC_M}$-formulas the *$\mathcal{ALC_M}$-satisfiability algorithm* proceeds as follows. Starting with the world constraint system W_0 that is induced by $\{F_1, \ldots, F_n\}$, propagation rules are applied as long as possible.

The transformation rules are *sound* in the sense that, if W is a satisfiable world constraint system, each applicable propagation rule can be applied in such a way that the obtained derived system is satisfiable (see [Baader and Laux, 1994] for a proof). For the "don't-know" non-deterministic $\rightarrow_\sqcup$ rule there are two alternative successor systems, and soundness means that one of them is satisfiable if the original system is satisfiable.[3] For the other rules (which are deterministic), soundness just means that application of the rule transforms a satisfiable system into a new satisfiable system. Furthermore, given an arbitrary induced world constraint system W_0, only a finite number of propagation rules can successively be applied, starting with *Wo* (see also [Baader and Laux, 1994] for a proof). This *termination property* means that, after a finite number of propagation rule applications to W_O we obtain a complete world constraint system (i.e., a system to which no more rules apply), say *W*. If *W* is satisfiable we can conclude that Wo is satisfiable (since *WO* is a subset of *W').* Otherwise, if *W'* is unsatisfiable, we can possibly derive another complete world constraint system from *Wo* by another choice for the non-deterministic $\rightarrow_\sqcup$ rule. If all the (finitely many) choices lead to an unsatisfiable complete system then soundness of the rules implies that the original system *Wo* was unsatisfiable.

Thus, it remains to be explained how satisfiability of a complete world constraint system can be decided. For this purpose, we say that a world constraint system *W* contains an *obvious contradiction* (or *clash* for short) if it contains either a pair of labeled *ACCM*-formulas of the form $x : A \,||\, l$ and $x : \neg A \,||\, l$ or a labeled *ACCM*-formula $x : \bot \,||\, l$ (for some object x, concept name A) and label l). Obviously, a world constraint system containing a clash is unsatisfiable. On the other hand, if a system is clash-free and complete then it is satisfiable (see [Baader and Laux, 1994] for a proof of this property, which shows *completeness* of the propagation rules). Summing up, we obtain the following theorem.

Theorem 4.2 *Satisfiability of a finite set of ACCM-formulas is decidable if we assume increasing domains.*

5 The Constant Domain Assumption

Up to now we have investigated increasing domains only. In this section we will consider the algorithmic consequences of assuming that the domains of all worlds are identical. Since this constant domain assumption is a special case of assuming increasing domains, an appropriate extension of the presented $\mathcal{ALC_M}$-satisfiability algorithm might seem to be rather easy. The goal of this section is to point out why developing such an extended algorithm requires more than a straightforward modification of the existing approach. In fact, until now we did not succeed in finding an appropriate modification. Nevertheless we think that pointing out the problems we have observed can be useful for anyone trying to solve this problem.

In a first attempt one could try to use the presented $\mathcal{ALC_M}$-satisfiability algorithm for the case of constant domains as well. However, not surprisingly, this does not always yield the correct answers. For example, consider the $\mathcal{ALC_M}$-formulas $([o]\,\neg A) = \top$ and $\langle o\rangle\,(x : A)$ where o is a modality, x an object, and A a concept name. It is easy to see that an application of the $\mathcal{ALC_M}$-satisfiability algorithm to the induced system $\{x_0 : \top \,||\, l_0, ([o]\,\neg A) = \top \,||\, l_0, \langle o\rangle\,(x : A) \,||\, l_0\}$ yields

[2] This idea was already used in [Buchheit *et al.*, 1993; Baader *et al.*, 1994], with slightly differing definitions of blocked objects.

[3] Note that the choice of an applicable rule is "don't-care" non-deterministic, i.e., we need not try different orders of rule applications

$W \rightarrow_{\Diamond} \{l \bowtie_o l', \varphi' \,||\, l'\} \cup W$

if $\varphi \,||\, l$ is in W, where φ is $\langle o \rangle\, F$ (resp. $x : \langle o \rangle\, C$), φ' is F (resp. $x : C$), there is no label l'' in W such that the world constraints $l \bowtie_o l''$ and $\varphi' \,||\, l''$ are in W, and l' is a new label.

$W \rightarrow_{\Box} \{\varphi' \,||\, l'\} \cup W$

if $\varphi \,||\, l$ and $l \bowtie_o l'$ are in W, where φ is $[o]\, F$ (resp. $x : [o]\, C$), φ' is F (resp. $x : C$), and $\varphi' \,||\, l'$ is not in W.

$W \rightarrow_{\sqcap} \{x : C_1 \,||\, l, x : C_2 \,||\, l\} \cup W$

if $x : C_1 \sqcap C_2 \,||\, l$ is in W and W does not contain both world constraints $x : C_1 \,||\, l$ and $x : C_2 \,||\, l$.

$W \rightarrow_{\sqcup} \{x : D \,||\, l\} \cup W$

if $x : C_1 \sqcup C_2 \,||\, l$ is in W, neither $x : C_1 \,||\, l$ nor $x : C_2 \,||\, l$ is in W, and D is either C_1 or C_2.

$W \rightarrow_{\exists} \{xRy \,||\, l, y : C \,||\, l\} \cup W$

if $x : \exists R.C \,||\, l$ is in W, x is not blocked in W, and y is a new object such that $y > z$ for all objects z occurring in W.

$W \rightarrow_{\forall} \{y : C \,||\, l\} \cup W$

if $x : \forall R.C \,||\, l$ and $xRy \,||\, l$ are in W and W does not contain the world constraint $y : C \,||\, l$.

$W \rightarrow_{=} \{x : C \,||\, l\} \cup W$

if x is relevant for l, $C = \top \,||\, l$ is in W, and $x : C \,||\, l$ is not in W.

Figure 1: Propagation rules of the ACC_M-satisfiability algorithm.

a complete and clash-free derived system. The reason is that the object name x is not relevant for l_0. This shows that the above $\mathcal{ALC}_{\mathcal{M}}$-formulas are satisfiable if we assume increasing domains. On the other hand, assuming constant domains causes unsatisfiability. Suppose, to the contrary, that $K = (W, \Gamma, K_I)$ is a Kripke structure such that $K, w \models ([o]\neg A) = \top$ and $K, w \models \langle o \rangle (x : A)$ for some world w in W. Because of $K, w \models \langle o \rangle\, (x : A)$ there exists a world w' with $(w, w') \in \gamma_o$ and $K, w' \models x : A$, i.e. $x^{K_I} \in (A, w')^{K_I}$. On the other hand, however, we have $x^{K_I} \in (\neg A, w')^{K_I}$ because of (i) $x^{K_I} \in \Delta^{K_I}(w)$ (constant domain assumption) and (ii) $K, w \models ([o]\neg A) = \top$.

In the $\mathcal{ALC}_{\mathcal{M}}$-satisfiability algorithm we took the increasing domain assumption into consideration by an appropriate definition of the notion of "relevant objects," which was then used in the $\rightarrow_{=}$ rule: given a labeled $\mathcal{ALC}_{\mathcal{M}}$-formula $C = \top \,||\, l$ in a derived system W, the $\rightarrow_{=}$ rule adds the labeled $\mathcal{ALC}_{\mathcal{M}}$-formula $x : C \,||\, l$ to W whenever x is relevant for l. Recall that an object x is said to be relevant for label l if there is a label l' occurring in W such that (i) W contains a world constraint of the form $x : C \,||\, l'$, $xRy \,||\, l'$, or $yRx \,||\, l'$, and (ii) l is accessible from l'. Now, if we want to deal with constant domains, a promising approach seems to be a modification of the $\rightarrow_{=}$ rule according to the following idea. Suppose W to be a derived system and l, l' to be labels in W. Furthermore, let $K = (W, \Gamma, K_I)$ be a Kripke structure that satisfies W. Because of the constant domain assumption we know that $x^{K_I} \in \Delta^{K_I}(w)$ for each world w in W, whenever there is a world constraint of the form $x : D \,||\, l$, $xRy \,||\, l$, or $yRx \,||\, l$ in W. In this case we say that x is a *top-level object* in W (to distinguish it from objects occurring only inside assertional axioms with leading modal operators). If x is a top-level object in W, and if the world constraint $C = \top \,||\, l'$ occurs in W, then the $\rightarrow_{=}$ rule must add $x : C \,||\, l'$ to W—independently from the fact whether or not x is relevant for l' (where "relevant" is defined as in the increasing domain approach). This consideration leads us to a modified rule $\rightarrow_{='}$ to handle world constraints of the form $C = \top \,||\, l$, which is given by

$W \rightarrow_{='} \{x : C \,||\, l\} \cup W$

if x is a top-level object in W, $C = \top \,||\, l$ is in W, and $x : C \,||\, l$ is not in W.

This apparently "slight" modification of the $\rightarrow_{=}$ rule may, however, cause infinite chains of propagation rule applications. As an example, consider the world constraint system W that consists of the two labeled $\mathcal{ALC}_{\mathcal{M}}$-formulas $x_0 : \top \,||\, l_0$ and $(\langle o \rangle \exists R.C) = \top \,||\, l_0$, where o is an arbitrary modality. An application of the $\rightarrow_{='}$ rule yields the derived system $W_1 = W \cup \{x_0 : \langle o \rangle \exists R.C \,||\, l_0\}$, and, by one application of the $\rightarrow_{\Diamond}$ and of the $\rightarrow_{\exists}$ rule each, we obtain $W_2 = W_1 \cup \{l_0 \bowtie_o l_1, x_0 : \exists R.C \,||\, l_1, x_0 R x_1 \,||\, l_1, x_1 : C \,||\, l_1\}$ where x_1 is a new object and l_1 is a new label. Because of the newly introduced object x_1 and the world constraint $(\langle o \rangle\, \exists R.C) = \top \,||\, l_0$ in W_2, the $\rightarrow_{='}$ rule is again applicable, and yields $W_3 = W_2 \cup \{x_1 : \langle o \rangle\, \exists R.C \,||\, l_0\}$. However, to $x_1 : \langle o \rangle\, \exists R.C \,||\, l_0$ the same propagation rules are applicable as to $x_0 : \langle o \rangle\, \exists R.C \,||\, l_0$ before. This means, another new label and a new object are introduced, and so on. Note that none of the newly generated objects is ever blocked since they all have different world labels. In order to avoid such infinite chains of propagation rule applications, the definition of blocked objects must be modified such that assertions with other labels are taken into account as well.

To sum up, we have seen that the problem of how to avoid infinite chains of propagation rule applications is more complicated if we are dealing with constant domains. In particular, the above example shows that, for testing whether or not an object is blocked w.r.t. some label *l*, it is not sufficient to consider only *ACCM*-formulas that are labeled with *l*. A straightforward generalization of the notion of blocked objects (called *cd-blocked)* is obtained by allowing for different labels *l* and *l'* when considering the sets of concept assertions for the objects *x* and *y.*

Although this modification can handle the above example correctly, it is not sufficient in general. On the one hand, it can become necessary to consider *ACCM~* formulas with *more than two different labels* as well as *information about role-successors* in the current world constraint when testing whether or not an object should be blocked. On the other hand, the test whether or not the —»B rule must be applied in a world constraint system *W* may depend on the information *W* (implicitly) contains about *the accessibility relations* of Kripke structures satisfying *W.* The full paper [Baader and Laux, 1994] contains examples that illustrate these problems. Due to these rather complex interactions, we did not yet succeed in finding an appropriate definition of cd-blocked objects in world constraints. We thus leave this definition as an open problem for the moment.[4]

6 Conclusion

The framework for integrating modal operators into terminological knowledge representation languages presented in this paper should be seen as the starting point for developing more elaborate hybrid languages of this type. Extensions in at least two directions will be necessary.

First, for the adequate representation of notions like belief and time, the basic modal logic K is not sufficient. Instead, one must consider modalities that satisfy appropriate modal axioms. Second, the multi-dimensionality of our language has not really been made use of. In fact, it is easy to see that with respect to satisfiability there is no difference between the v-dimensional and the corresponding 1-dimensional case (see [Baader and Laux, 1994] for details). We have introduced a multi-dimensional framework since it is more flexible. In an extended language, different dimensions could satisfy different modal axioms (e.g., KD45 in the belief dimension, and at least S4 in the time dimension).[5] In addition, one might want to specify certain interactions between different dimensions such as independence of one dimension from certain other dimensions.

The reason for considering a simplified framework without any of these extensions in the present paper is that in this context it is possible to design a rather intuitive calculus for satisfiability. Also, the proof of soundness, termination and completeness of this calculus is still relatively short and comprehensible. For this reason, we claim that this calculus can serve as a basis for satisfiability algorithms for more complex languages.

[4] [Donini *et al.*, 1992] use constant domain assumption in their epistemic extension of *ACC.* However, since they consider a nonmonotonic version of S5, the algorithmic problems are quite different.

[5] In the prepositional case, the combination of different modal logics obtained this way corresponds to what Gabbay calls "dove-tailing of propositional modal logics" [Gabbay, 1994].

Another topic of future research will be investigating the constant domain assumption and its algorithmic ramifications. The answer to the question whether constant domain assumption or increasing domain assumption is more appropriate from the semantic point of view strongly depends on the intended interpretation of the modalities (belief, knowledge, time, etc.).

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