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# Ternary Quartics and 3-dimensional Commutative Algebras

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**Abstract.** We find a connection between 3-dimensional commutative algebras with trivial trace and plane quartics and their bitangents.

## 1. Introduction

In this paper a structure of a commutative algebra on $\mathbf{C}^3$ is called a 3-dimensional algebra. Let $\mathcal{A}$ be the set of 3-dimensional algebras. Consider $\mathcal{A}$ as a linear space. Let $\mathcal{A}_0 \subset \mathcal{A}$ be the linear subspace of algebras with trivial trace. By definition, $\eta \in \mathcal{A}_0$ if the contraction of the structure tensor of $\eta$ is equal to zero.

By $PV$ we denote the projectivization of a vector space $V$. For $v \in V$, $v \neq 0$ we denote by $\overline{v}$ the corresponding point of the projective space $PV$.

Let $\eta \in \mathcal{A}_0$ be an algebra with trivial trace. Recall that an element $a \in \mathbf{C}^3$ is called an idempotent if $a \neq 0$, $a^2 = a$. We say that an element $\overline{a} \in P\mathbf{C}^3 = \mathbf{P}^2$ is a generalized idempotent if $a^2 = \lambda a$, where $\lambda \in \mathbf{C}$. Every idempotent defines a generalized idempotent. Every generalized idempotent $\overline{a} \in \mathbf{P}^2$ such that $a^2 \neq 0$ defines uniquely an idempotent $a' \in \mathbf{C}^3$ such that $\overline{a} = \overline{a'}$. Define the subscheme $X(\eta) \subset \mathbf{P}^2$ of the generalized idempotents by the following equation:

$$a^2 \wedge a = 0. \tag{0.1}$$

Consider the open $\mathbf{SL}_3$-invariant subset

$$\mathcal{A}_0' = \{\eta \in \mathcal{A}_0 \mid \dim X(\eta) = 0\} \subset \mathcal{A}_0.$$

**Lemma 1.1.** *$\mathcal{A}_0'$ is nonempty*

Consider $\eta \in \mathcal{A}_0$. The algebra $\eta$ defines the quadratic mapping

$$\mathbf{C}^3 \longrightarrow \mathbf{C}^3, \quad a \mapsto a^2.$$

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This quadratic mapping defines the section $\tilde{\eta}$ of the vector bundle $T_{\mathbf{P}^2}(1)$ (see [3], Ch. 1). The scheme of zeros of the section $\tilde{\eta}$ is $X(\eta)$. We have

$$\deg X(\eta) = c_2(T_{\mathbf{P}^2}(1)) = 7,$$

for $\eta \in \mathcal{A}_0$ (see [3], Ch. 1).

Consider the open $\mathbf{SL}_3$-invariant subset

$$\mathcal{A}_0'' = \{\eta \in \mathcal{A}_0' \mid X(\eta) = \{\overline{a_1}, \ldots, \overline{a_7}\},\ a_i^2 \neq 0,\ 1 \leq i \leq 7, \\ \textit{every } 3 \textit{ points of } X(\eta) \textit{ do not lie on a line}, \\ \textit{every } 6 \textit{ points of } X(\eta) \textit{ do not lie on a quadric}\} \subset \mathcal{A}_0'.$$

**Lemma 1.2.** *$\mathcal{A}_0''$ is nonempty.*

Consider the rational $\mathbf{SL}_3$-morphism

$$\varphi : P\mathcal{A}_0 \longrightarrow (\mathbf{P}^2)^{(7)}, \quad \eta \mapsto X(\eta).$$

We use the standard notation $(\mathbf{P}^2)^{(7)}$ for the 7-th symmetric degree of $\mathbf{P}^2$.

**Proposition 1.3.** *$\varphi$ is a birational isomorphism.*

In other words, a 3-dimensional algebra in general position with a trivial trace is uniquely (up to a scalar factor) defined by its generalized idempotents.

Fix $\eta \in \mathcal{A}_0''$. Let $a_1, \ldots, a_7$ be the idempotents of the algebra $\eta$. Let

$$\pi = \pi(\eta) : Z = Z(\eta) \longrightarrow \mathbf{P}^2$$

be the blowing up of $X(\eta)$ in $\mathbf{P}^2$. It is well known that $Z$ is a Del Pezzo surface of degree 2. We use some facts on the Del Pezzo surfaces, see [2], Ch. 5, section 4 for details. Let

$$\beta = \beta(\eta) : Z \longrightarrow \mathbf{P}^{2*} = P\mathbf{C}^{3*}$$

be the canonical double covering with a nonsingular quartic $Y = Y(\eta) \subset \mathbf{P}^{2*}$ as the branch locus.

The $\mathbf{SL}_3$-module $S^2\mathcal{A}_0$ contains with multiplicity one a submodule isomorphic to $S^4\mathbf{C}^3$. Therefore, there exists a unique (up to a scalar factor) nontrivial quadratic $\mathbf{SL}_3$-mapping

$$\varepsilon : \mathcal{A}_0 \longrightarrow S^4\mathbf{C}^3.$$

**Lemma 1.4.** *$\varepsilon(\eta) \neq 0$.*

Consider the quaternary form $\varepsilon(\eta)$ on the space $\mathbf{C}^{3*}$. This quaternary form defines a quartic $Y' = Y'(\eta) \subset \mathbf{P}^{2*}$. Consider the 28 linear forms $a_1, \ldots, a_7$, $(a_i - a_j)^2$, $1 \leq i < j \leq 7$ on the space $\mathbf{C}^{3*}$. These linear forms define 28 lines $A_1, \ldots, A_7$, $A_{ij} \in \mathbf{P}^{2*}$.

**Theorem 1.5.** *1. $Y$ is isomorphic to $Y'$.*

*2. The 28 bitangents to $Y'$ are $A_1, \ldots, A_7$, $A_{ij}$, $1 \leq i < j \leq 7$.*

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## 2. Proofs

Let $e_1, e_2, e_3$ be the standard basis in $\mathbf{C}^3$, and $x_1, x_2, x_3$ the dual basis in $\mathbf{C}^{3*}$. The group $\mathbf{SL}_3$ acts canonically in the space $S^a\mathbf{C}^3 \otimes S^b\mathbf{C}^{3*}$, $a, b \geq 0$. For $a, b \geq 1$ consider the linear $\mathbf{SL}_3$-mapping

$$\Delta = \sum \frac{\partial}{\partial e_i} \otimes \frac{\partial}{\partial x_i} : S^a\mathbf{C}^3 \otimes S^b\mathbf{C}^{3*} \longrightarrow S^{a-1}\mathbf{C}^3 \otimes S^{b-1}\mathbf{C}^{3*}.$$

It is well known that the representation of the group $\mathbf{SL}_3$ in the space $V(a,b) = \ker \delta$ is irreducible (see [1], part III, section 13). Assume that $V(a,0) = S^a\mathbf{C}^3$, $V(0,b) = S^b\mathbf{C}^{3*}$.

A structure of a commutative algebra on $\mathbf{C}^3$ is a symmetric bilinear mapping

$$\mathbf{C}^3 \times \mathbf{C}^3 \longrightarrow \mathbf{C}^3.$$

The set of such symmetric bilinear mappings is $\mathbf{C}^3 \otimes S^2\mathbf{C}^{3*}$. Therefore, the linear space $\mathcal{A}$ of 3-dimensional algebras is $\mathbf{C}^3 \otimes S^2\mathbf{C}^{3*}$. The contraction of structure tensors of algebras is the mapping

$$\Delta : \mathbf{C}^3 \otimes S^2\mathbf{C}^{3*} \longrightarrow \mathbf{C}^{3*}.$$

Therefore, the linear space $\mathcal{A}_0$ of 3-dimensional algebras with trivial trace is $V(1,2)$.

¿From the Littelwood-Richardson rule we get the following $\mathbf{SL}_3$-module decomposition

$$V(1,2) \otimes \mathbf{C}^3 \otimes \mathbf{C}^3 \simeq \mathbf{C}^3 \oplus 2V(0,2) \oplus 2V(2,1) \oplus 2V(1,3) \oplus V(3,2)$$

(see [1], App. A). Thus there exists a unique (up to a scalar factor) nontrivial trilinear $\mathbf{SL}_3$-mapping

$$\mu : V(1,2) \times \mathbf{C}^3 \times \mathbf{C}^3 \longrightarrow \mathbf{C}^3.$$

Let us give the explicit form of $\mu$:

$$\mu(e_{i_1} \otimes x_{j_1}x_{j_2},\ e_{i_2}, e_{i_3}) = \Delta^2(e_{i_1}e_{i_2}e_{i_3} \otimes x_{j_1}x_{j_2}).$$

The algebraic structure corresponding to $\eta \in V(1,2)$ is the bilinear symmetric mapping

$$\mu(\eta, \cdot, \cdot) : \mathbf{C}^3 \times \mathbf{C}^3 \longrightarrow \mathbf{C}^3.$$

**Example 2.1.** Consider the algebra

$$\eta_0 = \frac{1}{4}(e_1 \otimes x_3^2 + e_2 \otimes x_1^2 + e_3 \otimes x_2^2).$$

The multiplication table of $\eta_0$ is as following:

$$e_1 * e_2 = e_2 * e_3 = e_3 * e_1 = 0, \quad e_1^2 = e_2, \quad e_2^2 = e_3, \quad e_3^2 = e_1.$$

It can be easily checked that the subscheme $X(\eta_0) \subset \mathbf{P}^2$ of the generalized idempotents of $\eta_0$ is

$$X(\eta_0) = \{\overline{\theta e_1 + \theta^2 e_2 + \theta^4 e_3} \mid \theta \in \mu_7\},$$

where $\mu_7 = \{\theta \in \mathbf{C} \mid \theta^7 = 1\}$.

We have $\eta_0 \in \mathcal{A}'_0$. It follows that $\mathcal{A}'_0$ is nonempty. It can easily be checked that $\eta_0 \in \mathcal{A}''_\prime$. Therefore, $\mathcal{A}''_0$ is nonempty. This proves of Lemmas 1.1 and 1.2.

It can easily be checked that $\varphi^{-1}(X(\eta_0)) = \{\overline{\eta_0}\}$. It follows from (0.1) that a fiber of $\varphi$ of general position is a point in $PV(1,2)$. Hence $\varphi$ is a birational isomorphism. This proves Proposition 1.3.

Fix $\eta \in \mathcal{A}''_0$ and let $a_1, \ldots, a_7$ be the idempotents of $\eta$.

Consider the cubic mapping

$$\psi = \psi(\eta) : \mathbf{C}^3 \longrightarrow \wedge^2\mathbf{C}^3 \simeq \mathbf{C}^{3*}, \quad a \mapsto a^2 \wedge a.$$

**Lemma 2.2.** *Consider $a_i$ as a linear form on $\mathbf{C}^{3*}$. Let $Q_i = Q_i(\eta)$ be the cubic corresponding to the cubic form $\psi^*(a_i)$. Then the cubic $Q_i$ contains each of $\overline{a_1}, \ldots, \overline{a_7}$ with multiplicity $\geq 2$.*

**Proof.** It is obvious that $Q_i \ni \overline{a_1}, \ldots, \overline{a_7}$. Let us prove that $Q_i$ contains $\overline{a_i}$ with multiplicity $\geq 2$. We have

$$\begin{gathered}\psi^*(a_i) : a \mapsto a^2 \wedge a \wedge a_i \in \wedge^3\mathbf{C}^3 \simeq \mathbf{C},\\ \psi^*(a_i)(a_i + tb) = (a_i + tb)^2 \wedge (a_i + tb) \wedge a_i = a_i^2 \wedge a_i \wedge a_i +\\ t((2a_i * b) \wedge a_i \wedge a_i + a_i^2 \wedge b \wedge a_i) + o(t) = 0 + t \cdot 0 + o(t)\end{gathered}$$

for any $b \in \mathbf{C}^3$. ■

**Lemma 2.3.** *Consider $(a_i - a_j)^2$, $i < j$ as linear forms on $\mathbf{C}^{3*}$. Let $Q_{ij} = Q_{ij}(\eta)$ be the cubic corresponding to the cubic form $\psi^*((a_i - a_j)^2)$. Then $Q_{ij}$ is the union of the line $\langle \overline{a_i}, \overline{a_j} \rangle$ and the quadric containing the points $\overline{a_1}, \ldots, \widehat{\overline{a_i}}, \ldots, \widehat{\overline{a_j}}, \ldots, \overline{a_7}$.*

**Proof.** It is obvious that $Q_{ij} \ni \overline{a_1}, \ldots, \overline{a_7}$. We have to prove that $Q_{ij}$ contains the line $\langle \overline{a_i}, \overline{a_j} \rangle$. We have

$$\psi^*((a_i - a_j)^2) : a \mapsto a^2 \wedge a \wedge (a_i - a_j)^2 \in \wedge^3\mathbf{C}^3 \simeq \mathbf{C},$$

$$\begin{gathered}\psi^*((a_i - a_j)^2)(t_i a_i + t_j a_j) = (t_i a_i + t_j a_j)^2 \wedge (t_i a_i + t_j a_j) \wedge (a_i - 2a_i * a_j + a_j) =\\ t_i^3 a_i^2 \wedge a_i \wedge (a_i - 2a_i * a_j + a_j) + t_i^2 t_j (a_i^2 \wedge a_j \wedge (a_i - 2a_i * a_j + a_j) +\\ (2a_i * a_j) \wedge a_i \wedge (a_i - 2a_i * a_j + a_j)) + t_i t_j^2 (a_j^2 \wedge a_i \wedge (a_i - 2a_i * a_j + a_j) +\\ (2a_j * a_i) \wedge a_j \wedge (a_i - 2a_i * a_j + a_j)) + t_j^3 a_j^2 \wedge a_j \wedge (a_i - 2a_i * a_j + a_j) = 0.\end{gathered}$$

■

Consider the rational morphism

$$\Psi = \Psi(\eta) : \mathbf{P}^2 \longrightarrow \mathbf{P}^{2*}, \quad \overline{a} \mapsto \overline{\psi(a)}.$$

It is not defined exactly on $X(\eta)$. Let

$$\mathbf{P}^2 \xleftarrow{\pi} Z = Z(\eta) \xrightarrow{\beta} \mathbf{P}^{2*}$$

be the regularization of $\Psi$, $\Psi = \beta \circ \pi^{-1}$. It is well known that $Z$ is a Del Pezzo surface of degree 2, $\pi$ is the blowing up of the seven points $\overline{a_1}, \ldots, \overline{a_7}$ in $\mathbf{P}^2$, $\beta$ :

$Z \longrightarrow \mathbf{P}^{2*}$ is a double covering with the nonsingular quartic $Y = Y(\eta) \subset \mathbf{P}^{2*}$ as the branch locus (see [2], Ch. 5).

Let us prove Lemma 1.4 and Theorem 1.5. Consider the nontrivial homogeneous $\mathbf{SL}_3$-equivariant mapping of degree 6

$$\gamma_i : V(1,2) \longrightarrow V(0,12) = S^{12}\mathbf{C}^{3*},$$
$$\gamma_1 : \eta \mapsto \psi^*(\varepsilon(\eta)), \qquad \gamma_2 : \eta \mapsto (\det(\tfrac{\partial\psi^*(e_i)}{\partial x_j}))^2.$$

It can be checked that the $\mathbf{SL}_3$-module $S^6V(1,2)$ contains $V(0,12)$ with multiplicity one. Thus $\gamma_1 = c\gamma_2$, where $c \neq 0$. This implies Lemma 1.4 and statement 1. of Theorem 1.5.

The second part of Theorem 1.5 is a corollary of Lemmas 2.2 and 2.3 (see [2], Ch. 5, Section 4)

## References

[1] Fulton W. and Harris J., "Representation Theory," Springer-Verlag, 1991.

[2] Hartshorne R., "Algebraic Geometry," Springer-Verlag, 1977.

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