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TERNARY RINGS WITH ZERO ASSOCIATED TO TRANSLATION PLANES

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The study of projective or affine planes (i.e., the study of a special case of structures with relations) is intimately connected with the study of planar ternary rings (i.e., of a special case of structures with operations). It is the method of coordinatization (introduction and use of coordinate systems) that enables us to transfer problems from one of these given structures to the other one. The correspondence between the properties of the projective or affine planes and planar ternary rings coordinatizing them is not one to one. The fact that a projective plane may be coordinatized by a planar intermediate ring (G. E. MARTIN) as well as by a planar ternary ring with zero (L. A. SKORNJAKOV) or by a usual Hall planar ternary ring (with zero and unity) shows that some properties relative to planar ternary ring depend on the coordinatization either and even only. In this paper we are dealing with a characterization of planar ternary rings with zero (from now only planar ternary rings or shortly PTR's) coordinatizing the translation plane. It is well-known that an affine plane is a translation plane if and only if there exists a left quasifield coordinatizing it. We shall present PTR's of any given translation plane, which are without unity and without the left distributivity. However, our main purpose is to deduce a convenient necessary and sufficient condition that a given PTR coordinatizes a translation plane.

This research was suggested by VÁCLAV HAVEL and written under his direction.

1. AXIOMS OF A PLANAR TERNARY RING AND THEIR IMMEDIATE CONSEQUENCES

For the codification reasons we introduce the axioms of a planar ternary ring and add some simple consequences (without proof) which we shall use in the sequel.

Let $\mathbf{S}$ be a set containing at least two different elements and let a ternary operation $\mathbf{t}$ be given on it. An ordered pair $(\mathbf{S}, \mathbf{t})$ will be called a *planar ternary ring* (PTR) if it holds:

A 1. $\forall a, b, c \in \mathbf{S}\ \exists!\, x \in \mathbf{S}\ \mathbf{t}(a, b, x) = c.$

A 2. $\forall a, b, c, d \in \mathbf{S};\ a \neq c\ \exists!\, x \in \mathbf{S}\ \mathbf{t}(x, a, b) = \mathbf{t}(x, c, d).$

A 3. $\forall a, b, c, d \in \mathbf{S};\ a \neq c\ \exists\, (x, y) \in \mathbf{S}^2\ \mathbf{t}(a, x, y) = b\ \mathbf{t}(c, x, y) = d.$

A 4. $\exists 0 \in \mathbf{S}\ \forall x, y, z \in \mathbf{S}\ \mathbf{t}(0, y, z) = z\ \mathbf{t}(x, 0, z) = z.$

Such an element 0 is called *zero element* or *zero.*

Consequences.

(a) *The pair* (x, y) *from* A 3 *is determined uniquely.*

(b) *The zero element from* A 4 *is precisely one.*

(c) $\forall a, b, c \in \mathbf{S};\ a \neq 0\ \exists!\, x \in \mathbf{S}\ \mathbf{t}(x, a, b) = c.$

(d) $\forall a, b, c \in \mathbf{S};\ a \neq 0\ \exists!\, x \in \mathbf{S}\ \mathbf{t}(a, x, b) = c.$

For each $a \in \mathbf{S};\ a \neq 0$ we shall denote by e_a the solution of the equation

$$\mathbf{t}(a, x, 0) = a\,;$$

additionaly we define

$$e_0 = 0\,.$$

Thus for each $a \in \mathbf{S}\ \mathbf{t}(a, e_a, 0) = a.$

Now let us introduce in $\mathbf{S}$ two binary operations, addition $+$ and multiplication by virtue of

$$a + b = \mathbf{t}(a, e_a, b) \quad \forall a, b \in \mathbf{S}\,,$$

$$a\ .\ b = \mathbf{t}(a, b, 0) \quad \forall a, b \in \mathbf{S}\,.$$

Consequences.

(e) $a + 0 = 0 + a = a\ \forall a \in \mathbf{S}.$

(f) $\forall a, c \in \mathbf{S}\ \exists!\, x \in \mathbf{S}\ a + x = c.$

(g) $a + b = a + c \Rightarrow b = c.$

(h) $a\,.\,0 = 0\,.\,a = 0\ \forall a \in \mathbf{S}.$

(i) $\forall c, m \in \mathbf{S};\ m \neq 0\ \exists!\, x \in \mathbf{S}\ x\,.\,m = c;$
$\forall c, m \in \mathbf{S};\ m \neq 0\ \exists!\, y \in \mathbf{S}\ m\,.\,y = c.$

2. COORDINATIZATION OF AN AFFINE PLANE BY A PLANAR TERNARY RING

Let us investigate a projective plane $P = (\mathbf{P}, \mathbf{L})$*). Let us distinguish a line n. Then by an affine plane $P(n)$ we shall mean as usual the restriction of $(\mathbf{P}, \mathbf{L})$ to the incidence structure $(\mathbf{P} \setminus n, \{l \setminus (l \cap n)\}/l \in \mathbf{L} \setminus \{n\})$.

*) With lines as point sets.

The points from $\mathbf{A} := \mathbf{P} \setminus n$ will be called *proper*, the points of n *improper* (or directions). The lines from $\mathbf{L} \setminus \{n\}$ will be called *proper* whereas their restrictions to $\mathbf{A}$ will be called *affine lines*.

Now choose a certain direction V and call it *vertical*. Proper lines with this direction will be called *vertical* too (and this term will be used also for the corresponding affine lines). For any $N \in n$ denote by $\tilde{N}$ the set of all lines containing N and set

$$\mathbf{B} = \mathbf{L} \setminus \tilde{V}.$$

As it is known, card $\mathbf{A}$ = card $\mathbf{B}$ = m^2 where m is the order of $\boldsymbol{P}$.

Start from $\boldsymbol{P}(n)$ with m = card $\mathbf{S}$. Then card $\mathbf{S}^2$ = card $\mathbf{A}$ = card $\mathbf{B}$.

Each couple of bijections $\pi : \mathbf{S}^2 \to \mathbf{A}$, $\lambda : \mathbf{S}^2 \to \mathbf{B}$ will be called *a coordinate system for $\boldsymbol{P}(n)$*.

Let (π, λ) be such a coordinate system that

$$(1) \qquad (x, y)^\pi \in (a, b)^\lambda \Leftrightarrow y = \mathbf{t}(x, a, b)$$

for some PTR $(\mathbf{S}, t)$. Then axioms A 1 – A 3 imply:

(a) *Lines $(a, b)^\lambda$, $(a', b')^\lambda$ have an improper point of intersection* (different from V) *if and only if $a = a'$.*

(b) *Points $(x, y)^\pi$, $(x', y')^\pi$ lie on the same vertical line if and only if $x = x'$.*

Thus the coordinate system (π, λ) induces two bijections

$$\bar{\pi} : \mathbf{S} \to n \setminus \{V\} ; \quad \bar{\lambda} : \mathbf{S} \to \tilde{V} \setminus \{n\}$$

in the following natural way:

$$\widetilde{a^{\bar{\pi}}} = \{(a, b)^\lambda \mid b \in \mathbf{S}\} \cup \{n\} ; \quad a^{\bar{\lambda}} = \{(a, b)^\pi \mid b \in \mathbf{S}\} \cup \{V\} .$$

Let H be a direction for which $H = 0^{\bar{\pi}}$ (the so-called *horizontal direction*) and let v denote a vertical line $0^{\bar{\lambda}}$ (the so-called vertical axis). Every proper line containing H will be called *horizontal* as well. Axioms A 1 – A 4 imply:

(c) *Points $(x, y)^\pi$, $(x', y')^\pi$ lie on the same horizontal line if and only if $y = y'$.*

(d) *Lines $(a, b)^\lambda$, $(a', b')^\lambda$ intersect in a proper point lying on the vertical axis if and only if $b = b'$.*

In this way we obtain further two bijections:

$$\pi' : \mathbf{S} \to v \setminus \{V\} ; \quad \lambda' : \mathbf{S} \to \tilde{H} \setminus \{n\}$$

as follows:

$$\widetilde{b^{\pi'}} = \{(a, b)^\lambda \mid a \in \mathbf{S}\} \cup \{v\} , \quad b^{\lambda'} = \{(a, b)^\pi \mid a \in \mathbf{S}\} \cup \{H\} .$$

For any $a \in \mathbf{S}$ we have $b = \mathbf{t}(0, a, b)$ (equivalently $(0, b)^\pi \in (a, b)^\lambda$) and therefore

$$b^{\pi'} = (0, b)^\pi .$$

Further

$$(x, y)^\pi = x^{\bar{\lambda}} \cap y^{\lambda'}, \quad (a, b)^\lambda \ni a^\pi, b^{\pi'} \quad \forall x, y, a, b \in \mathbf{S} .$$

Finally note that the affine plane $\boldsymbol{P}(n)$ may be coordinatized by a PTR.

3. VERTICALLY TRANSITIVE PLANES

An affine plane $\boldsymbol{P}(n)$ is said to be a *translation plane* if the group $\boldsymbol{G}$ of all translations of $\boldsymbol{P}(n)$ operates transitively on the set $\mathbf{A}$ of all proper points.

Let us recall without proof some well-known statements about affine planes:

(a) *A set of all translations with a given direction U is a subgroup* (denoted by $\boldsymbol{G}_U$) *of the group $\boldsymbol{G}$ of all translations.*

(b) *If the group $\boldsymbol{G}$ contains translations with two different directions then it is Abelian.*

(c) *The group $\boldsymbol{G}_U$ operates transitively on each affine line with the direction U if and only if it operates transitively on one of them.*

(d) *Let u, v be affine lines with different directions U, V. Then $\boldsymbol{G}$ operates transitively on the set $\mathbf{A}$ of all proper points if and only if the groups $\boldsymbol{G}_U$ and $\boldsymbol{G}_V$ operate so on the lines u, v. In this case*

$$\boldsymbol{G} = \boldsymbol{G}_U \oplus \boldsymbol{G}_V .$$

If U denotes any direction of an affine plane $\boldsymbol{P}(n)$, then $\boldsymbol{P}(n)$ will be called *U-transitive* (more exactly *(U, n)-transitive*) plane*) if $\boldsymbol{G}_U$ operates transitively on every affine line with the direction U.

Let $\boldsymbol{P}(n)$ be an affine plane coordinatized by a PTR $(\mathbf{S}, \mathbf{t})$, let V be the vertical direction and H the horizontal direction relative to the given PTR. The V-transitive plane or H-transitive plane $\boldsymbol{P}(n)$ are said to be *vertically* or *horizontally transitive,* respectively.

The planar ternary ring $(\mathbf{S}, \mathbf{t})$ will be called *a generalized Cartesian group***) if it has the following properties:

(α) Its addition is associative or equivalently $(\mathbf{S}, +)$ is a group.

(β) $(\mathbf{S}, \mathbf{t}, +, .)$ satisfies the linearity property, i.e.,

$$\mathbf{t}(a, b, c) = a \,.\, b + c \quad \forall a, b, c \in \mathbf{S} \text{ ***)} \tag{2}$$

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*) [1], p. 140, [4], p. 101.

**) For the definition of Cartesian groups cf. [4], p. 90.

***) cf. [2], p. 10.

The generalized Cartesian group is said to be *commutative*, if $(\mathbf{S}, +)$ is commutative.

Throughout this and the next chapter we assume that (π, λ) is a fixed coordinate system with the property (1) of any given affine plane $\boldsymbol{P}(n)$.

Proposition 1. *The group G_V operates transitively on each vertical affine line if and only if for any $a \in \mathbf{S}$*

$$(x, y)^\pi \mapsto (x, y + a)^\pi \tag{3}$$

is a translation from G_V.

Proof. I. Let (3) present a translation from G_V $\forall a \in \mathbf{S}$. By (c) it suffices to prove that G_V operates transitively on proper points of the vertical axis v. For this it suffices to show that for each point $(0, a)^\pi$ there exists a translation $f \in G_V$ so that $f((0, 0)^\pi) = (0, a)^\pi$. However, it is just the translation (3) which has this property.

II. Let G_V operates transitively on every vertical affine line. Then $\forall a \in \mathbf{S}$ there is a $f \in G_V$ carrying $(0, 0)^\pi$ into $(0, a)^\pi$. Now we have to prove that $f((x, y)^\pi) = (x, y + a)^\pi \, \forall (x, y)^\pi \in \mathbf{A}$. This is evident for $y = 0$ because the image of the horizontal line through $(0, 0)^\pi$ is a horizontal line containing the point $(0, a)^\pi$ and the points $(x, 0)^\pi$ and $(x, a)^\pi$ lie on the vertical line x^λ fixed under the translation considered.

Thus assume $y \neq 0$. Then obviously $f((y, y)^\pi) = (y, z)^\pi$ for some $z \in \mathbf{S}$. As $(0, 0)^\pi$, $(y, y)^\pi \in (e_y, 0)^\lambda$ so $(y, z)^\pi$ lies on the line going through $(0, a)^\pi$ parallelly to $(e_y, 0)^\lambda$. Thus $(y, z)^\pi \in (e_y, a)^\lambda \Leftrightarrow z = \mathbf{t}(y, e_y, a) = y + a$. The image of the horizontal line $(0, y)^\lambda$ under f is the horizontal line $(0, y + a)^\lambda$. Therefore $f((x, y)^\pi) = (x, y + a)^\pi$ as required.

Proposition 2. *G_V operates transitively on each vertical affine line if and only if the associative law for addition and the linearity property are valid.*

Proof. I. Let the associative law for addition and the linearity property hold. Then in consequence of Proposition 1 it suffices to show that (3) is a translation in the direction V for $\forall a \in \mathbf{S}$. Since $(\mathbf{S}, +)$ is a group, (3) is a bijective mapping fixing every vertical line. Now let $(m, b)^\lambda$ be an arbitrary (not vertical) line; this line has the equation $y = x \,.\, m + b$. Denote by f the mapping (3). Then the images of proper points $(x, y)^\pi$ of $(m, b)^\lambda$ are the points $(x, y + a)^\pi = (x, (x \,.\, m + b) + a)^\pi = (x, x \,.\, m + (b + a))^\pi$. So $f((m, b)^\lambda) = (m, b + a)^\lambda$. Thus f is a dilatation with the centre V i.e., it belongs to G_V.

II. Let G_V operate transitively on each vertical affine line. Then by Proposition 1 (3) belongs to G_V $\forall a \in \mathbf{S}$. If $a, b, c \in \mathbf{S}$, then denote by $f, g \in G_V$ the translations for which $f((0, 0)^\pi) = (0, b)^\pi$ and $g((0, 0)^\pi) = (0, c)^\pi$. It follows directly from Proposition 1 that $g((0, b)^\pi) = (0, b + c)^\pi$, so that $(g \circ f)((0, 0)^\pi) = (0, b + c)^\pi$. From this it is clear that $(g \circ f)((0, a)^\pi) = (0, a + (b + c))^\pi$ and on the other hand $(g \circ f)((0, a)^\pi) = g((0, a + b)^\pi) = (0, (a + b) + c)^\pi$, so that

$$a + (b + c) = (a + b) + c\,.$$

If again $a, b, c \in \mathbf{S}$ then denote by f such translation from G_V for which $f((0, 0)^\pi) = (0, c)^\pi$. By our assumption such a translation must exist. Let us consider a line $(b, 0)^\lambda$. It is $(0, 0)^\pi \in (b, 0)^\lambda$, therefore $(0, c)^\pi \in f((b, 0)^\lambda)$. As $(b, 0)^\lambda$, $f((b, 0)^\lambda)$ are parallel, we have $f((b, 0)^\lambda) = (b, c)^\lambda$. Further $(a, a\,.\,b)^\pi \in (b, 0)^\lambda$ so that $(a, a\,.\,b + c)^\pi \in f((b, 0)^\lambda)$. This implies $a\,.\,b + c = \mathbf{t}(a, b, c)$ as desired.

Theorem 1. *An affine plane $P(n)$ is vertically transitive if and only if one (and consequently each) planar ternary ring $(\mathbf{S}, \mathbf{t})$ of $P(n)$ is a generalized Cartesian group. Furthermore, G_V is Abelian if and only if $(\mathbf{S}, \mathbf{t})$ is commutative.*

Proof. I. Let $(\mathbf{S}, \mathbf{t})$ be a generalized Cartesian group. By Proposition 2 G operates transitively on each vertical affine line. It remains to prove that G_V is commutative. For this purpose let $f, g \in G_V$; $f((0, 0)^\pi) = (0, b)^\pi$, $g((0, 0)^\pi) = (0, c)^\pi$. According to the second part of the proof of Proposition 2 $(g \circ f)((0, 0)^\pi) = (0, b + c)^\pi = (0, c + b)^\pi = (f \circ g)((0, 0)^\pi)$ so that $g \circ f = f \circ g$.

II. Let $P(n)$ be a vertically transitive plane. By Proposition 2 the addition is associative and the linearity property holds. It remains to prove the commutative law for addition. Let $b, c \in \mathbf{S}$ and $f, g \in G_V$ be translations for which $f((0, 0)^\pi) = (0, b)^\pi$, $g((0, 0)^\pi) = (0, c)^\pi$. Since G_V operates transitively on each vertical affine line, such translation does exist. Then $(0, b + c)^\pi = (g \circ f)((0, 0)^\pi) = (f \circ g)((0, 0)^\pi) = (0, c + b)^\pi$ and consequently $b + c = c + b$.

4. TRANSLATION PLANES

In this chapter we shall consider a fixed vertically transitive plane $P(n)$ and its arbitrary coordinate system (π, λ) with the property (1) expressing a generalized Cartesian group $(\mathbf{S}, \mathbf{t})$. By the property (d) in § 3 $P(n)$ is a translation plane if and only if G_H (H is the horizontal direction) operates transitively on each horizontal affine line. To formulate the next results in a convenient form, let us introduce the following expressions:

(4_1) $\exists m \in \mathbf{S} \setminus \{0\}$ $(a + b)\,.\,m = a\,.\,m + b\,.\,m$

(4_2) $\forall m \in \mathbf{S}$ $(a + b)\,.\,m = a\,.\,m + b\,.\,m$

(5_1) $\exists m \in \mathbf{S} \setminus \{0\}$ $a\,.\,m + b\,.\,m = c\,.\,m$

(5_2) $\forall m \in \mathbf{S}$ $a\,.\,m + b\,.\,m = c\,.\,m$

In [5], p. 454,

$$(4_1) \Rightarrow (4_2) \quad \text{for all} \quad a, b \in \mathbf{S}$$

(($\mathbf{S}$, +) is Abelian of course) is stated as a necessary condition for $\boldsymbol{P}(n)$ to be a translation plane, while (4_2) is given as a sufficient condition for $\boldsymbol{P}(n)$ to be a translation plane.

If a PTR ($\mathbf{S}$, $\mathbf{t}$) is Hall, i.e., if there exists an element $e \in \mathbf{S}$ such that $e \,.\, a = a \,.\, e = a$ $\forall a \in \mathbf{S}$, then (4_1) holds for $m = e$. Thus for the vertically transitive plane $\boldsymbol{P}(n)$ coordinated by Hall PTR (4_2) is a necessary and sufficient condition for $\boldsymbol{P}(n)$ to be a translation plane.

The main result of this paper is the following theorem which has been motivated by the passages of [5] just quoted.

Theorem 2. *A vertically transitive affine plane $\boldsymbol{P}(n)$ is a translation plane if one of its PTR ($\mathbf{S}$, $\mathbf{t}$) satisfies for all a, b $c \in \mathbf{S}$*

$$(5_1) \Rightarrow (5_2)\,.$$

Proof. I. Let ($\mathbf{S}$, $\mathbf{t}$) be a generalized Cartesian group of a given vertically transitive plane $\boldsymbol{P}(n)$. It suffices to prove that the group $\boldsymbol{G}_H$ operates transitively on proper points of the line $(0, 0)^\lambda$ (horizontal axis). It remains only to show: $\forall b \in \mathbf{S}$ $\exists f \in \boldsymbol{G}_H$ $f((0, 0)^\pi) = (b, 0)^\pi$. Thus let $b \in \mathbf{S}$. Define a mapping

$$f : (x, y)^\pi \mapsto (x', y)^\pi$$

with $x' \in \mathbf{S}$ uniquely determined by

$$x \,.\, m + b \,.\, m = x' \,.\, m\,. \tag{6}$$

Thus (5_1) is satisfied for $(a, b, c) = (x, b, x')$. Clearly f is bijective and fixes every horizontal line. Further it is obvious that the image of the vertical line $x^{\pi'}$ is the vertical line $x'^{\pi'}$. Let us consider a line $(u, v)^\lambda$ where $u \neq 0$. If $(x, y)^\pi \in (u, v)^\lambda$, then $y = x \,.\, u + v$. For $f((x, y)^\pi) = (x', y')^\pi$ we have $x' \,.\, u = x \,.\, u + b \,.\, u$. Hence it is $y = (x' \,.\, u - b \,.\, u) + v = x' \,.\, u + (-(b \,.\, u) + v)$ or equivalently $(x', y)^\pi \in (u, -(b \,.\, u) + v)^\lambda$. This implies $f((u, v)^\lambda) = (u, -(b \,.\, u) + v)^\lambda$ and consequently f is a translation from $\boldsymbol{G}_H$.

II. Let $\boldsymbol{P}(n)$ be a translation plane. Let $a, b, c \in \mathbf{S}$. Further let f be the translation for which $f((0, 0)^\pi) = (b, 0)^\pi$. We shall prove:

(a) If $f((a, 0)^\pi) = (c, 0)^\pi$, then ($5_2$) holds.

(b) If (5_1) holds, then $f((a, 0)^\pi) = (c, 0)^\pi$.

Ad (a): First of all, evidently $a \,.\, 0 + b \,.\, 0 = c \,.\, 0$. Thus suppose $m \in \mathbf{S} \setminus \{0\}$ and $f((a, 0)^\pi) = (c, 0)^\pi$. Let us construct the last point in the following way: Denote by Q the point of intersection of $a^{\pi'}$, $(m, 0)^\lambda$ (Fig. 1). Thus $Q = (a, a \,.\, m)^\pi$. Let h be the horizontal line through Q. It is obviously $f(Q) \in h$. Furthemore the line r joining $(b, 0)^\pi = f((0, 0)^\pi)$, $f(Q)$ is parallel to $(m, 0)^\lambda$. Thus $r = (m, v)^\lambda$. But $(b, 0)^\pi \in (m, v)^\lambda$ means $0 = b \,.\, m + v$, $v = -(b \,.\, m)$. The line joining $(c, 0)^\pi$, $f(Q)$ must be vertical so that $f(Q) = (c, a \,.\, m)^\pi$. But $f(Q) \in r$ can be written as $c \,.\, m - (b \,.\, m) = a \,.\, m$.

Ad (b): Let $a \,.\, m_0 + b \,.\, m_0 = c \,.\, m_0$ hold for some $m_0 \in \mathbf{S} \setminus \{0\}$. Further let $f((a, 0)^\pi) = (\bar{c}, 0)^\pi$ for some $\bar{c} \in \mathbf{S}$. Then by (a) $a \,.\, m_0 + b \,.\, m_0 = \bar{c} \,.\, m_0$. Comparing both equations with the same left hand side $a \,.\, m_0 + b \,.\, m_0$ we obtain $c \,.\, m_0 = \bar{c} \,.\, m_0$ and therefore $c = \bar{c}$. Thus (b) is proved.

Finally $a \,.\, m_0 + b \,.\, m_0 = c \,.\, m_0$ for some $m_0 \in \mathbf{S} \setminus \{0\}$ implies $f((a, 0)^\pi) = (c, 0)^\pi$ by (b) and further (5_2) by (a).

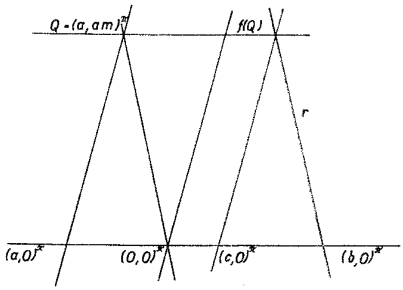

Fig. 1.

Corollary 1. *It follows immediately from Theorem 2 and property* (b) *of* § 3: *If any generalized Gartesian group* $(\mathbf{S}, \mathbf{t})$ *satisfies the condition* $(5_1) \Rightarrow (5_2)$ *for all* $a, b, c \in \mathbf{S}$ *then the group* $(\mathbf{S}, +)$ *is Abelian.*

Corollary 2. *The additional validity of the left distributivity*

$$(7) \qquad a \,.\, m + b \,.\, m = (a + b) \,.\, m \quad \forall a, b, m \in \mathbf{S}$$

implies that $\boldsymbol{P}(n)$ *is a translation plane.*

Proof. From $a \,.\, m_0 + b \,.\, m_0 = c \,.\, m_0$ for some $m_0 \in \mathbf{S} \setminus \{0\}$ it follows that $(a + b) \,.\, m_0 = c \,.\, m_0$ and thus $c = a + b$. So (7) coincides with (5_2) and $\boldsymbol{P}(n)$ must be a translation plane.

Corollary 3. *Let* $\boldsymbol{P}(n)$ *be a translation plane and let* $(\mathbf{S}, \mathbf{t})$ *be one of its PTR's. If there exists an element* $m_0 \in \mathbf{S} \setminus \{0\}$ *such that* $a \,.\, m_0 + b \,.\, m_0 = (a + b)' \,.\, m_0$ $\forall a, b \in \mathbf{S}$ *then* (7) *holds.*

Proof. Immediate, from Theorem 2 for $c = a + b$.

Corollary 4. *Let* $\boldsymbol{P}(n)$ *be an affine plane and* $(\mathbf{S}, \mathbf{t})$ *one of its PTR's. Then* $\boldsymbol{P}(n)$ *is a translation plane if and only if*:

(a) *Addition of* $(\mathbf{S}, \mathbf{t})$ *is associative.*

(b) $(\mathbf{S}, \mathbf{t})$ *satisfies the linearity property.*

(c) *For all* $a, b, c \in \mathbf{S}$ $(5_1) \Rightarrow (5_2)$.

Proof. Let (a), (b), (c) be valid. Then by (a) and (b) $(\mathbf{S}, \mathbf{t})$ is a generalized Cartesian group and by Theorem 1 $\boldsymbol{P}(n)$ is a vertically transitive plane. Finally Theorem 2 and (c) imply that $\boldsymbol{P}(n)$ is a translation plane.

If $\boldsymbol{P}(n)$ is a translation plane then it is of course also vertically transitive. Thus Theorem 2 gives (c) and Theorem 1 gives the remaining conditions (a) and (b).

Proposition 3. *Let $\boldsymbol{P}(n)$ be a translation plane and $(\mathbf{S}, \mathbf{t})$ one of its PTR's. Then the left distributivity (7) holds if and only if for every $a \in \mathbf{S}$*

$$f : (x, y)^\pi \mapsto (x + a, y)^\pi \tag{8}$$

is a translation from $\boldsymbol{G}_H$.

Proof. First let (8) be a translation from $\boldsymbol{G}_H$ for any $a \in \mathbf{S}$. Then $f((0, 0)^\pi) = (a, 0)^\pi$ and $\forall b \in \mathbf{S}$ $f((0, 0)^\pi) = (b + a, 0)^\pi = (a + b, 0)^\pi$ (by Corollary 1). By part II of the proof of Theorem 2 $a \,.\, m + b \,.\, m = (a + b) \,.\, m$ holds $\forall m \in \mathbf{S}$.

Let (7) hold. Note that f is evidently a bijection fixing every horizontal line and carrying every vertical line onto a vertical line again. Consider any line $(m, v)^\lambda$ for $v \in \mathbf{S}$, $m \in \mathbf{S} \setminus \{0\}$. If $(x, y)^\pi \in (m, v)^\lambda$ then $y = x \,.\, m + v$. So for $(x', y)^\pi := f((x, y)^\pi)$ we have $x' = x + a$ and consequently $y = (x' + (-a)) \,.\, m + v = x' \,.\, m + ((-a) \,.\, m + v)$. Thus f maps $(m, v)^\lambda$ onto $(m, (-a) \,.\, m + v)^\lambda$ and f is a translation of $\boldsymbol{G}_H$.

5. EXAMPLES

Let us have a PTR $(\mathbf{S}, \mathbf{t})$. An element $e \in \mathbf{S}$ $(j \in \mathbf{S})$ will be called its right (left) unity if for all $a \in \mathbf{S}$ $a \,.\, e = a$ $(j \,.\, a = a)$. If $(\mathbf{S}, \mathbf{t})$ has the right and left unity e and j, respectively, evidently $j = j \,.\, e = e$.

Let us consider an affine plane $\boldsymbol{P}(n)$ with a given coordinate system (π, λ) and let $(\mathbf{S}, \mathbf{t})$ be the PTR associated to it.

Proposition 4a. $(\mathbf{S}, \mathbf{t})$ *has a right unity if and only if the set*

$$\mathbf{D} = \{(x, x)^\pi \mid x \in \mathbf{S}\} \tag{9}$$

(the so-called diagonal) is an affine line.

Proof. Let $(\mathbf{S}, \mathbf{t})$ have a right unity e. Then $\mathbf{D} = \{(x, y)^\pi \mid y = x \,.\, e,\ x \in \mathbf{S}\} = \{(x, y)^\pi \mid y = t(x, e, 0)\} = (e, 0)^\lambda$.

II. Conversely, if $\mathbf{D}$ is an affine line, then $\mathbf{D} = (e, 0)^\lambda$ as $(0, 0)^\pi \in \mathbf{D}$. Then $a \in \mathbf{S} \Rightarrow (a, a)^\pi \in \mathbf{D} \Rightarrow a = a \,.\, e \Rightarrow e$ is a right unity of $(\mathbf{S}, \mathbf{t})$.

Proposition 4b. $(\mathbf{S}, \mathbf{t})$ *has a left unity if and only if the mapping* $f : \tilde{P} \setminus \{v\} \to \to \tilde{H} \setminus \{n\}$

$$f : Pa^{\bar{\pi}} \mapsto a^{\lambda'}$$

where $P = (0, 0)^{\pi}$, *is a perspectivity with vertical axis.*

Proof. I. Let $(\mathbf{S}, \mathbf{t})$ have a left unity j and $p = j^{\lambda} \Rightarrow Pa^{\bar{\pi}} \cap p = (a, 0)^{\lambda} \cap p = = (j, j \,.\, a)^{\pi} = (j, a)^{\pi} \in a^{\lambda'} \Rightarrow Pa^{\bar{\pi}} \cap a^{\lambda'} \in p \Rightarrow f$ is a perspectivity with axis p.

II. Let f be a perspectivity with a vertical axis $p = j^{\lambda}$. Let $a \in \mathbf{S} \Rightarrow (j, a)^{\pi} \in p$. Denote by q the line joining P and $(j, a)^{\pi}$. As p is the axis of perspectivity $f, q = = (a, 0)^{\lambda}$, $(j, a)^{\pi} \in (a, 0)^{\lambda} \Rightarrow a = j \,.\, a \Rightarrow j$ is left unity.

Now let us show shortly how to introduce a coordinate system in $\boldsymbol{P}(n)$. At first take a set $\mathbf{S}$ so that card $\mathbf{S}$ is equal to the order of the plane $\boldsymbol{P}(n)$, further select a direction $V \in n$ and a proper line $v \in \tilde{V}$. Denote, as usual, $\mathbf{A} = \mathbf{P} \setminus \{n\}$, $\mathbf{B} = \mathbf{L} \setminus \tilde{V}$.

Let us choose bijections

$$\bar{\pi} : \mathbf{S} \to n \setminus \{V\} \quad \bar{\lambda} : \mathbf{S} \to \tilde{V} \setminus \{n\} \tag{10}$$

and denote by 0 the element from $\mathbf{S}$ for which $0^{\bar{\lambda}} = v$ and by H the direction $0^{\bar{\pi}}$. Furthermore choose an arbitrary direction

$$\lambda' : \mathbf{S} \to \tilde{H} \setminus \{n\} \,. \tag{11}$$

This determines uniquely the bijection

$$\pi' : \mathbf{S} \to v \setminus \{V\} \tag{12}$$

so that

$$\forall b \in \mathbf{S} : b^{\pi'} = v \cap b^{\lambda'} \tag{13}$$

(equivalently the mapping $b^{\pi'} \mapsto b^{\lambda'}$ is the perspectivity from v onto $\tilde{H} \setminus \{n\}$). The mappings

$$\pi : (x, y) \mapsto x^{\bar{\lambda}} \cap y^{\lambda'} \quad \lambda : (a, b) \mapsto p \tag{14}$$

are bijections $\pi : \mathbf{S}^2 \to \mathbf{A}$, $\lambda : \mathbf{S}^2 \to \mathbf{B}$. Now intorducing in $\mathbf{S}$ a ternary operation by

$$y = \mathbf{t}(x, a, b) \Leftrightarrow (x, y)^{\pi} \in (a, b)^{\lambda} \; \forall x, y, a, b \in \mathbf{S} \,, \tag{15}$$

we can easily find that $(\mathbf{S}, \mathbf{t})$ is a PTR with the zero element 0. Thus (π, λ) is a coordinate system (which has property (1)). The coordinate system (π, λ) will be denoted by $(\bar{\pi}, \bar{\lambda}, \pi', \lambda')$ or only by $(\bar{\pi}, \bar{\lambda}, \lambda')$.

The diagonal $\mathbf{D}$ of the coordinate system (π, λ) is characterized by

$$\mathbf{D} = \{x^{\bar{\lambda}} \cap x^{\lambda'} \mid x \in \mathbf{S}\} \,. \tag{16}$$

As the diagonal intersects every vertical or horizontal line at a unique point, the mapping λ' is uniquely determined by λ and by the diagonal. The introduction of the coordinate system of $P(n)$ can be transformed as follows: we replace λ' by the diagonal $\mathbf{D}$ expressed by a set which is intersected by every proper line of pencils V and H at one point. If $P(n)$ is coordinatized by PTR $(\mathbf{S}, \mathbf{t})$, we obtain its other coordinatizations by „deforming" the original diagonal. By this we also transform $\mathbf{t}$ into a new ternary operation $\mathbf{t}'$. $(\mathbf{S}, \mathbf{t}')$ will be of course again a PTR determined by the new coordinatization.

a) Example 1. Let $P(n)$ be an affine plane*) coordinatized by the field Z_5**) and (π, λ) its coordinate system. As a new diagonal we take $\mathbf{D} = \{(0, 0)^\pi, (1, 4)^\pi, (2, 3)^\pi, (3, 1)^\pi, (4, 2)^\pi\}$. $\mathbf{D}$ is not a line so that the new PTR $(\mathbf{S}, \mathbf{t})$ (where $\mathbf{S} = \{0, 1, 2, 3, 4\}$), will not have a right unity. We can establish new multiplication and addition tables:

addition:

	0	1	2	3	4
0	0	1	2	3	4
1	1	2	4	0	3
2	2	4	3	1	0
3	3	0	1	4	2
4	4	3	0	2	1

multiplication:

	0	1	2	3	4
0	0	0	0	0	0
1	0	3	4	2	1
2	0	4	1	3	2
3	0	2	3	1	4
4	0	1	2	4	3

It follows from the tables that the left distributive law does not hold, for instance

$$(4 + 2) \,.\, 3 = 0 \cdot 3 = 0\,; \quad 4 \cdot 3 + 2 \cdot 3 = 4 + 3 = 2\,.$$

It follows from the existence of the original coordinate system that $P(n)$ is even Pappian and so $(5_1) \Rightarrow (5_2)$.

Example 2. Let $P(n)$ be an arbitrary Euclidean plane; let us choose an origin P and an orthonormal basis $(\mathbf{e}_1, \mathbf{e}_2)$. This induces naturally a coordinate system (π_1, λ_1) in the sense of § 2; its PTR is of course the field of real numbers. The vertical direction is the direction determined by $\mathbf{e}_2$. As a new diagonal $\mathbf{D}$ choose the curve with the equation

$$y = x^r\,, \quad r > 1\,, \quad \text{odd.}$$

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*) Every PTR coordinatizes for instance its canonical plane, see [3].

**) Let $(\mathbf{S}, \mathbf{t})$ be given, where $(S, +)$ is a Abelian group (with zero element 0), $(\mathbf{S} \setminus \{0\}, .)$ is a quasigroup, $\forall x \in \mathbf{S}\ 0 \,.\, x = x \,.\, 0 = 0$, and both equations $a \,.\, x = b \,.\, x + c$, $y \,.\, a = y \,.\, b + c$ are uniquely solvable for $a \neq b$. Then we can introduce the structure of a ternary ring by $\mathbf{t}(a, b, c) = a \,.\, b + c$. Then $(\mathbf{S}, \mathbf{t})$ is a generalized Cartesian group with commutative addition.

The new PTR $(\mathbf{S}, \mathbf{t})$ ($\mathbf{S}$ is now the set of all real number) has no right unity and it is easy to verify that the addition $\oplus$ and the multiplication $\odot$ of $(\mathbf{S}, \mathbf{t})$ fulfil

$$c = a \oplus b \Leftrightarrow c^r = a^r + b^r ,$$
$$d = a \odot b \Leftrightarrow d^r = a \,.\, b \,.$$

Thus in $(\mathbf{S}, \mathbf{t})$ the left distributive law does not hold. However, the implication $(5_1) \Rightarrow$ $\Rightarrow (5_2)$ holds as $\boldsymbol{P}(n)$ is a translation plane.

b) Example 3. Let $\mathbf{S} = \{0, 1, 2, 3, 4\}$ and let $+$ be the addition mod 5. The multiplication $\cdot$ will be defined by the following table:

	0	1	2	3	4
0	0	0	0	0	0
1	0	2	4	3	1
2	0	1	2	4	3
3	0	4	3	1	2
4	0	3	1	2	4

It is trivial to verify that $(\mathbf{S}, \mathbf{t})$ ($\mathbf{t}$ is a ternary operation in S as in the footnote**) to Example 1) is a generalized Cartezian group with commutative addition. It is evident that $(\mathbf{S}, \mathbf{t})$ possesses neither the right nor the left unitary element. In $(\mathbf{S}, \mathbf{t})$ the multiplication is not associative and the left distributive law does not hold:

$$2 \,.\, (3 \cdot 3) = 2 \cdot 1 = 1 \,; \qquad (2 \cdot 3) \,.\, 3 = 4 \cdot 3 = 2$$
$$(3 + 1) \,.\, 3 = 4 \cdot 3 = 2 \,; \qquad 3 \cdot 3 + 1 \cdot 3 = 1 + 3 = 4$$

We can verify that $(5_1) \Rightarrow (5_2)$ and so every affine plane coordinatized by $(\mathbf{S}, \mathbf{t})$ is a translation plane.

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