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# Ордена Ленина ИНСТИТУТ ПРИКЛАДНОЙ МАТЕМАТИКИ <br> имени М.В.Келдыша <br> Российской академии наук 

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# Testbench calibration technique for testing attitude determination algorithms by video processing 

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Методика калибровки стенда для отработки алгоритмов определения ориентации по видеоизображению

Работа посвящена задаче определения ориентации объекта с помощью обработки изображений. Для ее решения реализован двухэтапный подход. На первом этапе определяется матрица поворота через модель измерения, которая адаптирована для использования кватернионов. На втором этапе выполняется коррекция кватерниона ориентации с помощью метода наименьших квадратов.

Результаты экспериментальных исследований показали, что использование модели измерения камеры и итеративного процесса позволяет определить кватернион ориентации с хорошей точностью.

Ключевые слова: определение углового движения, обработка видеоизображения

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The paper is devoted to the problem of estimating the orientation of an object using image processing. A two-stage approach has been implemented to solve this problem. At the first stage the rotation matrix is determined using a measurement model adapted for using quaternions. At the second stage the orientation quaternion is corrected using the least squares method.

The results of experimental studies have shown that using a camera measurement model and an iterative process it is possible to determine the orientation quaternion with good accuracy.

Key words: angular motion determination, image processing

## Introduction

Determination of angular motion is of great importance because it allows to know and predict the attitude of the bodies with respect to a reference system, such information is vital for missions where maneuvers and interactions of two or more bodies are performed.

Studies on objects attitude and angular velocity estimation have been performed using different sensors such as photoelectric encoders, tachometers, inertial sensors, and even laser. However, their implementation can be expensive. The use of digital images as low-cost sources of information for evaluating the angular motion is actively used in the field of robotics, control system, augmented reality, and are also widely used in the field of satellite systems.

Over the last four decades, a variety studies have been done on measuring motion parameters of objects using cameras, where a considerable importance had the development of methods for camera calibration, i.e. the determination of internal parameters of the camera. In [4-7] the calibration methods with analytical solutions are presented, where in addition to determine internal parameters of the camera, the 3-D object attitude in space are determined as a part of the calibration process. R . Tsai [6] used the Euler angles, while Z. Zhang [5] used Rodrigues' rotation formula.

Researches have been proposed to investigate the measurement of object pose estimation. M. Dhome [8] proposed method to find the analytical solutions to the problem of the determination of the 3-D object attitude in space from a single perspective image. H. Kim [9] proposed a simple and fast stereo matching algorithm for real-time robotic applications using 3D information of vertexes on the outline of an object in image plane. Z. Zhong [10] presents a feature point pair based technique for object pose estimation and 3D structure recovery from a single view, where it is defined strategies for small rotational and large rotational motion, X. Zhang [11] presents algorithms for recovering the camera pose and the 3D-to-2D line correspondences simultaneously.

Determination of angular velocity by image processing is furthermore studied. Y. Zhang [14] by means blurred images processing proposed the estimation of motion parameters by measuring and comparing global geometric properties. S. Wang [13] proposed parameter measurement of rotation through analyzing the information of visual rotation motion blur based on a single blurred image. By using event cameras, which have independent pixels that respond asynchronously to brightness changes, G. Gallego and D. Scaramuzza [12] proposed algorithm to estimate the angular velocity of the camera by analyzing the spatio-temporal coordinates of the brightness change.

Several years ago, it is increased the interest in parameters movement estimation by image processing for space applications. A. Boguslavsky [17] presented a software package that by means of video signal received from the TV-camera, mounted on the spacecraft board, allows the automatic visual monitoring of a spacecraft "Progress" docking to International Space Station. D. Ivanov [18],
proposed a satellite relative position and orientation determination algorithm by performing image processing of the sunlit spacecraft. This algorithm was used to determine the relative movement of the Chibis-M microsatellite developed by IKI RAS.
M. Koptev [19] proposed a method for the translational and rotational motion determination of mock-ups suspended on an aerodynamic testbed. The algorithm was based on the detection of installed special marks on model's body to evaluate the location of model's center of mass, angular position and angular velocity in the coordinate system associated with the aerodynamic testbed.

The difference between the determination algorithm developed and the one described above is that it does not require the installation of an additional special objective or photodiodes on the satellite to shoot; it is enough to know the geometry of the object being shot. The algorithm does not require the transfer of any data from the satellite being taken, therefore a piece of space debris can act as the second device. So, the algorithm is suitable for the tasks of removing space debris from orbit: the satellite companion flies towards the debris, determines its movement, captures it and takes it to the dense atmosphere.

Most of researches on angular motion parameters estimation mentioned above are focused on parameter 3-D object attitude determination or measurement angular velocity, but not both, if 3-D object attitude determination and angular velocity are considered to be estimated at the same time, usually it is considered to add one more sensor in addition to camera.

While in [5] and [6] used Rodrigues' rotation formula and the Euler angles respectively, the purpose of this work is the estimation of the 3-D object attitude by using quaternions, and in addition the angular velocity estimation at the same time by means of a conventional low-cost camera without any additional sensor. The section 1 elaborates the problem statement of this research, and it is explained the importance of measurement model determination, which is the mathematical model for the camera. In the Section 2 it is explained step by step the mathematical expression for the measurement model and its gradient. The linear measurement model derived in the Section 3 allows to determine some parameters of the measurement model by linear methods. Due to the fact that the measurement model $h$ is nonlinear and depends on unknown parameters of the camera, a calibration process is needed to be performed. In the Section 4 the algorithm for calibration is explained and applied for 3-D object attitude determination.

## 1. Problem statement

The problem of the angular motion determination by the image processing is considered. The source of measurements is the camera which captures the object's movement by taking photographs at a certain frequency (see Figure 1), These pictures are processed to estimate the angular motion of the object using the following information:

- $X_{i}\left(x_{i}, y_{i}, z_{i}\right)$ : coordinates of the object points relative to the body-fixed frame OXYZ.
- $X_{i}{ }^{\prime}\left(x_{i}{ }^{\prime}, y_{i}{ }^{\prime}\right)$ : coordinates of the same points $X_{i}$, which are visualized and located in the image coordinate system (image).


Figure 1. Diagram of the problem statement
The basic structure of a camera is shown in the Figure 2, where two main components are involved:

- Lens: has the function of gathering and focus the light reflected from an object or scene. As the reflected light rays enter the camera lens, they are directed to the image sensor.
- Image sensor: it is a rectangular plane where the points are projected to, representing in that way the image of the object.
The image sensor is located parallel to the lens in the focal plane of the lens. The distance between the lens and the focal plane is called focal length $f$.

The locations of specific points of an object in an image varies according to the rotation matrix $R$, and its translation vector $\boldsymbol{T}_{\boldsymbol{c}}$ with respect to the camera. Thus, the


Figure 2. Basic structure of a camera
estimation of the rigid body rotation matrix is possible when function $h\left(R, \boldsymbol{T}_{c}\right)$, called
measurement model, that performs the projection of point $X_{i}$ into the Image coordinate system is found. In addition, an instantaneous angular velocity can be calculated from two consecutive rotation matrices, thus, the first stage on this work is focused on the rotation matrix determination.

The Cartesian coordinates systems used in this work are shown in the Figure 3:

- $O X Y Z$ - Body-fixed Frame (BF) is placed in any location on the object in such a way that the points $X_{i}$ are known with respect to BF.
- $O_{c} X_{c} Y_{c} Z_{c}$ - Camera Coordinate System (CCS) is based on the pinhole model, where its origin $O_{c}$ is located at camera center (center of the lens), $O_{c} Z_{c}$ is defined by the line from the camera center perpendicular to the image sensor, $O_{c} X_{c}$ is parallel to the horizontal side of the image sensor, $O_{c} Y_{c}$ is parallel to the vertical side of the image sensor.
- $O_{i m g} U V$ - Image Coordinate System (ICS), also known as the Image plane, is a space of 2D pixel coordinates, where each 2D pixel coordinate is the result of the conversion of points which are located on the image sensor plane in CCS, to 2D coordinate pixel. Its origin $O_{i m g}$ is located on the top-left corner of the image, $O_{i m g} U$ extends from left to right and $O_{i m g} V$ extends downward.
The following notation of points in the different coordinate systems is used:
- $X_{i}\left(x_{i}, y_{i}, z_{i}\right)-i$-th point with respect to BF .
- $X_{c_{i}}\left(x_{c_{i}}, y_{c_{i}}, z_{c_{i}}\right)-i$-th point with respect to CCS.
- $X_{i}^{\prime}\left(x_{i}^{\prime}, y_{i}^{\prime}\right)-i$-th point with respect to ICS.OXYZ - Body-fixed Frame (BF).

With regard to $X_{i}$, it is important to notice this point remain fixed with respect to the BF .


Figure 3. Pinhole camera model
Most of the elements mentioned in this section are considered for measurement model $h\left(R, \boldsymbol{T}_{\boldsymbol{c}}\right)$ definition because of its importance and relevance in the success of
this work, for that reason is given in details the process to define the measurement model.

## 2. Measurement model

In order to define the measurement model $h$ which is a function that performs the projection of point $X_{i}$ into the ICS from BF, it is required to consider the following:

- transformation from BF to CCS,
- projection of the points from CCS into the sensor plane,
- lens distortion,
- transformation from sensor plane to ICS
which are going to be explained in details in this section.


### 2.1.Transformation from BF to CCS

Let $X_{i}=\left[x_{i}, y_{i}, z_{i}\right]^{T}$ be any point in the BF, where its transformation to the CCS is defined as follows:

$$
X_{c_{i}}=\left[\begin{array}{l}
x_{c_{i}}  \tag{2.1}\\
y_{c_{i}} \\
z_{c_{i}}
\end{array}\right]=R\left[\begin{array}{l}
x_{i} \\
y_{i} \\
z_{i}
\end{array}\right]+\boldsymbol{T}_{\boldsymbol{c}},
$$

where the rotation matrix R can be expressed as

$$
R=\left[\begin{array}{lll}
r_{11} & r_{12} & r_{13}  \tag{2.2}\\
r_{21} & r_{22} & r_{23} \\
r_{31} & r_{32} & r_{31}
\end{array}\right]
$$

and $\boldsymbol{T}_{\boldsymbol{c}}$ is translation vector with respect to CCS:

$$
\boldsymbol{T}_{\boldsymbol{c}}=\left[\begin{array}{lll}
t_{x c} & t_{y c} & t_{z c} \tag{2.3}
\end{array}\right]^{T} .
$$

From equations (2.1), (2.2) and (2.3) the next expression

$$
X_{c_{i}}=\left[\begin{array}{l}
x_{c_{i}} \\
y_{c_{i}} \\
z_{c_{i}}
\end{array}\right]=\left[\begin{array}{l}
r_{11} x_{i}+r_{12} y_{i}+r_{13} z_{i}+t_{x c} \\
r_{21} x_{i}+r_{22} y_{i}+r_{23} z_{i}+t_{y c} \\
r_{31} x_{i}+r_{32} y_{i}+r_{33} z_{i}+t_{z c}
\end{array}\right]
$$

performs the transition from points from BF to the CCS.

### 2.2. Projection of the points from CCS into the image plane

The point $X_{c p}$ represents the projection of the points from CCS into the image plane, and it is expressed as

$$
X_{c p_{i}}=\left[\begin{array}{l}
x_{c p_{i}} \\
y_{c p_{i}}
\end{array}\right]=\left[\begin{array}{l}
x_{c_{i}} / z_{c_{i}} \\
y_{c_{i}} / z_{c_{i}}
\end{array}\right],
$$

where

$$
\begin{equation*}
x_{c p_{i}}=\frac{r_{11} x_{i}+r_{12} y_{i}+r_{13} z_{i}+t_{x c}}{r_{31} x_{i}+r_{32} y_{i}+r_{33} z_{i}+t_{z c}} \tag{2.4}
\end{equation*}
$$

and

$$
\begin{equation*}
y_{c p_{i}}=\frac{r_{21} x_{i}+r_{22} y_{i}+r_{23} z_{31} i_{31} x_{i c}+r_{32} y_{i}+r_{33} z_{i}+t_{z c}}{} . \tag{2.5}
\end{equation*}
$$

It is important to mention that $X_{c p}$ is still located in the CCS.

### 2.3. Lens distortion

It is necessary to take into account that the lens distortion affects the image during the projection of the point $X_{c}$ into the sensor plane. The usual types of distortion are radial and tangential ones. Radial distortion can be defined as a function which depends on the distance from the Principal point, this point is formed by the intersection point between the $O_{c} Z_{c}$-axis and the sensor plane. Tangential distortion is caused by an unperfected parallel alignment between the lens and the image sensor [3]. These distortions can be defined by the following expression

$$
X_{d_{i}}=\left[\begin{array}{l}
x_{c p_{i}}\left(1+k_{1} r_{i}{ }^{2}+k_{2} r_{i}{ }^{4}+k_{3} r_{i}{ }^{6}\right)+2 p_{1} x_{c p_{i}} y_{c p_{i}}+p_{2}\left(r_{i}{ }^{2}+2 x_{c p_{i}}{ }^{2}\right) \\
y_{c p_{i}}(1+\underbrace{k_{1} r_{i}{ }^{2}+k_{2} r_{i}^{4}+k_{3} r_{i} r^{6}}_{\text {Radial distortion }})+\underbrace{2 p_{2} x_{c p_{i}} y_{c p_{i}}+p_{1}\left(r_{i}{ }^{2}+2 y_{c p_{i}}{ }^{2}\right.}_{\text {Tangential distortion }})
\end{array}\right],
$$

where $X_{d_{i}}=\left[x_{d_{i}}, y_{d_{i}}\right]^{T}, r_{i}{ }^{2}=x_{c p_{i}}{ }^{2}+y_{c p_{i}}{ }^{2}$ and $k_{1}, k_{2}, k_{3}, p_{1}, p_{2}$ are the distortion coefficients, $X_{d_{i}}$ is the point coordinates when the lens distortion is taken into account. In case when there is no lens distortion (ideal lens), $X_{d_{i}}$ and $X_{c p_{i}}$ are equal.

### 2.4. Transformation from sensor plane to ICS

Due to the fact that the ICS and sensor plane are parallel, and both located at the same plane, this transformation is based on scaling and translation of the points located in the sensor plane as follows:

$$
X_{p_{i}}=\left[\begin{array}{c}
\left(x_{d_{i}}+s y_{d_{i}}\right) f_{x}+u_{0}  \tag{2.6}\\
y_{d_{i}} f_{y}+v_{0}
\end{array}\right],
$$

where:

- $P=\left(u_{0}, v_{0}\right)$ : Principal point expressed in pixels with respect to ICS.
- $f_{x}=\alpha_{x} f$ : focal length axis-x (pixel).
- $f_{y}=\alpha_{y} f$ : focal length axis-y (pixel).
- $\alpha_{x}, \alpha_{y}$ : number of pixels per unit distance.
- $s$ : skew coefficient, which usually is equal to zero.
- $X_{p_{i}}$ : mapped point in the ICS from the BF.
- $X_{i}$ : point in the BF.

The expression (2.6) can be rewritten as follows:

$$
X_{p_{i}}=\left[\begin{array}{l}
x_{p_{i}}  \tag{2.7}\\
y_{p_{i}}
\end{array}\right]=h_{X_{i}}\left(f_{x}, f_{y}, u_{0}, v_{0}, S, k_{1}, k_{2}, k_{3}, p_{1}, p_{2}, R, \boldsymbol{T}_{c}\right)
$$

where $h_{X_{i}}$ is called measurement model, which performs the projection of a point $X_{i}$ into the ICS from the BF. However, taking into account that the parameters $f_{x}, f_{y}, u_{0}, v_{0}, S, k_{1}, k_{2}, k_{3}, p_{1}, p_{2}$ are fixed values and specific for each camera, the expression (2.7) can be simplified to $h_{X_{i}}\left(R, \boldsymbol{T}_{\boldsymbol{c}}\right)$ once those parameters are determined.

### 2.5. Measurement model based on Quaternions

Due to the fact that the rotation matrix has 9 scalar elements, it is convenient to express the rotation matrix with less scalar elements during optimization processes. While in [5] and [6] the rotation matrix is expressed using Rodrigues' rotation formula and the Euler angles respectively, in this work the rotation matrix is expressed by means of quaternions.

Taking into account basic quaternion theory, let the multiplication of quaternions $\boldsymbol{Q}=\left[q_{0}, q_{1}, q_{2}, q_{3}\right]^{T}$ and $\boldsymbol{P}=\left[p_{0}, p_{1}, p_{2}, p_{3}\right]^{T}$ be defined as follow:

$$
\boldsymbol{P} \circ \boldsymbol{Q}=\left[\begin{array}{cc}
p_{0} q_{0}-\boldsymbol{p} \boldsymbol{q} \\
p_{0} \boldsymbol{q}+q_{0} \boldsymbol{p}+\boldsymbol{p} & \times \boldsymbol{q}
\end{array}\right]
$$

where $\boldsymbol{q}=\left[q_{1}, q_{2}, q_{3}\right]^{T}$ and $\boldsymbol{p}=\left[p_{1}, p_{2}, p_{3}\right]^{T}$ are vector parts of the quaternions $\boldsymbol{Q}$ and $\boldsymbol{P}$ respectively. The previous equation also can be rewritten in a matrix form:

$$
\boldsymbol{P} \circ \boldsymbol{Q}=\left[\begin{array}{cc}
p_{0} & -\boldsymbol{p}^{T}  \tag{2.8}\\
\boldsymbol{p} & p_{0} I_{3}+[\boldsymbol{p}]_{x}
\end{array}\right]\left[\begin{array}{c}
q_{0} \\
\boldsymbol{q}
\end{array}\right]=\left[\begin{array}{cc}
q_{0} & -\boldsymbol{q}^{T} \\
\boldsymbol{q} & q_{0} I_{3}-[\boldsymbol{q}]_{x}
\end{array}\right]\left[\begin{array}{c}
p_{0} \\
\boldsymbol{p}
\end{array}\right]
$$

The rotation of points by means of quaternions is defined as follows:

$$
\left[\begin{array}{c}
0  \tag{2.9}\\
X_{c}
\end{array}\right]=\Lambda \circ\left[\begin{array}{l}
0 \\
X
\end{array}\right] \circ \widetilde{\Lambda}
$$

where $X=[x, y, z]^{T}$ is a tri-dimensional point, and $\Lambda=\left[\lambda_{0}, \lambda_{1}, \lambda_{2}, \lambda_{3}\right]^{T}$ is an unit quaternion, which modulus $|\Lambda|$ is defined as follows:

$$
|\Lambda|=\sqrt{\lambda_{0}^{2}+\lambda_{1}^{2}+\lambda_{2}^{2}+\lambda_{3}^{2}}=1
$$

and the conjugated of $\Lambda$ is represented by

$$
\tilde{\Lambda}=\left[\begin{array}{c}
\lambda_{0} \\
-\lambda
\end{array}\right] .
$$

From the equations (2.9) and (2.8) is obtained the next matrix form for rotation of points:

$$
\left[\begin{array}{c}
0 \\
X_{c}
\end{array}\right]=\left[\begin{array}{cc}
1 & \mathbf{0}^{T} \\
\mathbf{0} & R(\Lambda)
\end{array}\right]\left[\begin{array}{l}
0 \\
X
\end{array}\right],
$$

where $R(\Lambda)$ represents the rotation matrix as a function of the quaternion $\Lambda$. However, due to the fact that $\Lambda$ is a unit quaternion, the rotation matrix $R$ can be expressed as a function of the vector part $\lambda=\left[\lambda_{1}, \lambda_{2}, \lambda_{3}\right]^{T}$ of the $\Lambda$ as follows:

$$
R(\lambda)=\left[\begin{array}{ccc}
1-2\left(\lambda_{2}^{2}+\lambda_{3}^{2}\right) & 2\left(\lambda_{1} \lambda_{2}-\lambda_{0} \lambda_{3}\right) & 2\left(\lambda_{1} \lambda_{3}+\lambda_{0} \lambda_{2}\right)  \tag{2.10}\\
2\left(\lambda_{1} \lambda_{2}+\lambda_{0} \lambda_{3}\right) & 1-2\left(\lambda_{1}^{2}+\lambda_{3}^{2}\right) & 2\left(\lambda_{2} \lambda_{3}-\lambda_{0} \lambda_{1}\right) \\
2\left(\lambda_{1} \lambda_{3}-\lambda_{0} \lambda_{2}\right) & 2\left(\lambda_{2} \lambda_{3}+\lambda_{0} \lambda_{1}\right) & 1-2\left(\lambda_{1}^{2}+\lambda_{2}^{2}\right)
\end{array}\right] .
$$

Thus, the equation (2.7) can be rewritten as follows:

$$
X_{p_{i}}=\left[\begin{array}{l}
x_{p_{i}}  \tag{2.11}\\
y_{p_{i}}
\end{array}\right]=h_{X_{i}}\left(f_{x}, f_{y}, u_{0}, v_{0}, S, k_{1}, k_{2}, k_{3}, p_{1}, p_{2}, \lambda, \boldsymbol{T}_{c}\right) .
$$

The measurement model based on quaternion is obtained in the equation (2.11), which is important to determine its gradient for the optimization process.

### 2.6. Measurement model gradient

Due to the fact that measurement model based on quaternions in the equation (2.11) is a composed function of several transformations, which involves vector and matrices, it is convenient to determine the gradient by means of matrix calculus. Let the measurement model be rewritten as follow:

$$
X_{p_{i}}=\left[\begin{array}{l}
x_{p_{i}}  \tag{2.12}\\
y_{p_{i}}
\end{array}\right]=h_{X_{i}}\left(\boldsymbol{F}, \boldsymbol{C}, S, \boldsymbol{K}_{\boldsymbol{D}}, \boldsymbol{\lambda}, \boldsymbol{T}_{\boldsymbol{c}}\right),
$$

where $\boldsymbol{F}=\left[\begin{array}{ll}f_{x} & f_{y}\end{array}\right], \boldsymbol{C}=\left[\begin{array}{ll}u_{0} & v_{0}\end{array}\right]$, and $\boldsymbol{K}_{\boldsymbol{D}}=\left[\begin{array}{lllll}k_{1} & k_{2} & p_{1} & p_{2} & k_{3}\end{array}\right]$, and let the measurement model gradient be defined as follows:

$$
D h_{X_{i}}=\left[\begin{array}{llllll}
\frac{\partial h_{X_{i}}}{\partial \boldsymbol{F}} & \frac{\partial h_{X_{i}}}{\partial C} & \frac{\partial h_{X_{i}}}{\partial S} & \frac{\partial h_{X_{i}}}{\partial K_{D}} & \frac{\partial h_{X_{i}}}{\partial \lambda} & \frac{\partial h_{X_{i}}}{\partial T_{c}} \tag{2.13}
\end{array}\right],
$$

where $\partial h_{X_{i}} / \partial \boldsymbol{F}, \partial h_{X_{i}} / \partial \boldsymbol{C}, \partial h_{X_{i}} / \partial S$, and $\partial h_{X_{i}} / \partial \boldsymbol{K}_{\boldsymbol{D}}$ are defined in a matrix form:

$$
\frac{\partial h_{x_{i}}}{\partial \boldsymbol{F}}=\left[\begin{array}{c}
\frac{\partial x_{p_{i}}}{\partial \boldsymbol{F}}  \tag{2.14}\\
\frac{\partial y_{p_{i}}}{\partial \boldsymbol{F}}
\end{array}\right]=\left[\begin{array}{cc}
x_{d_{i}}+s y_{d_{i}} & 0 \\
0 & y_{d_{i}}
\end{array}\right]
$$

$$
\begin{gathered}
\frac{\partial h_{X_{i}}}{\partial \boldsymbol{C}}=\left[\begin{array}{c}
\frac{\partial x_{p_{i}}}{\partial \boldsymbol{C}} \\
\frac{\partial y_{p_{i}}}{\partial \boldsymbol{C}}
\end{array}\right]=\left[\begin{array}{ll}
1 & 0 \\
0 & 1
\end{array}\right] \\
\frac{\partial h_{X_{i}}}{\partial s}=\left[\begin{array}{c}
\frac{\partial x_{p_{i}}}{\partial s} \\
\frac{\partial y_{p_{i}}}{\partial s}
\end{array}\right]=\left[\begin{array}{c}
y_{p_{i}} f_{x} \\
0
\end{array}\right]
\end{gathered}
$$

$$
\frac{\partial h_{X_{i}}}{\partial K_{D}}=
$$

$$
\left[\begin{array}{ccccc}
x_{c p_{i}}\left(x_{c p_{i}}{ }^{2}+y_{c p_{i}}{ }^{2}\right) & x_{c p_{i}}\left(x_{c p_{i}}{ }^{2}+y_{c p_{i}}{ }^{2}\right)^{2} & 2 x_{d_{i}} y_{d_{i}} & y_{c p_{i}}{ }^{2}+3 x_{c p_{i}}{ }^{2} & x_{c p_{i}}\left(x_{c p_{i}}{ }^{2}+y_{c p_{i}}{ }^{2}\right)^{3} \\
y_{c p_{i}}\left(x_{c p_{i}}{ }^{2}+y_{c p_{i}}{ }^{2}\right) & y_{c p_{i}}\left(x_{c p_{i}}{ }^{2}+y_{c p_{i}}{ }^{2}\right)^{2} & x_{c p_{i}}{ }^{2}+3 y_{c p_{i}}{ }^{2} & 2 x_{d_{i}} y_{d_{i}} & y_{c p_{i}}\left(x_{c p_{i}}{ }^{2}+y_{c p_{i}}{ }^{2}\right)^{3}
\end{array}\right]
$$

With regard to $\partial h_{X_{i}} / \partial \lambda$ and $\partial h_{X_{i}} / \partial \boldsymbol{T}_{\boldsymbol{c}}$, these values have remarkably complicated expressions due to the fact that the measurement model in the equation (2.11) is a composed function of several transformations.

As the rotation matrix $R$ is a function of $\lambda$, see equation (2.10), the derivative $\partial h_{X_{i}} / \partial \lambda$ is

$$
\begin{gather*}
\frac{\partial h_{X_{i}}}{\partial \lambda}=\frac{\partial h_{X_{i}}}{\partial R} \frac{\partial R(\lambda)}{\partial \lambda},  \tag{2.15}\\
\frac{\partial h_{X_{i}}}{\partial R}=\left[\begin{array}{c}
\frac{\partial x_{p_{i}}}{\partial R} \\
\frac{\partial y_{p_{i}}}{\partial R}
\end{array}\right]
\end{gather*}
$$

where $\partial y_{p_{i}} / \partial R$ and $\partial x_{p_{i}} / \partial R$ are

$$
\begin{gathered}
\frac{\partial y_{p_{i}}}{\partial R}=f_{y}\left(D_{r_{i}} \frac{\partial y_{c p_{i}}}{\partial R}+y_{c p_{i}}\left[k 1 \frac{\partial\left(r_{i}^{2}\right)}{\partial R}+k 2 \frac{\partial\left(r_{i}^{4}\right)}{\partial R}+k 3 \frac{\partial\left(r_{i}{ }^{6}\right)}{\partial R}\right]+\left(2 p_{2} y_{c p_{i}}+\right.\right. \\
\left.\left.2 p_{1} x_{c p_{i}}\right) \frac{\partial x_{c p_{i}}}{\partial R}+\left(2 p_{2} x_{c p_{i}}+6 p_{1} y_{c p_{i}}\right) \frac{\partial y_{c p_{i}}}{\partial R}\right), \\
\frac{\partial x_{p_{i}}}{\partial R}=f_{x}\left(D_{r_{i}} \frac{\partial x_{c p_{i}}}{\partial R}+x_{c p_{i}}\left[k 1 \frac{\partial\left(r_{i}^{2}\right)}{\partial R}+k 2 \frac{\partial\left(r_{i}^{4}\right)}{\partial R}+k 3 \frac{\partial\left(r_{i}^{6}\right)}{\partial R}\right]+\left(2 p_{1} y_{c p_{i}}+\right.\right. \\
\left.\left.6 p_{2} x_{c p_{i}}\right) \frac{\partial x_{c p_{i}}}{\partial R}+\left(2 p_{1} x_{c p_{i}}+2 p_{2} y_{c p_{i}}\right) \frac{\partial y_{c p_{i}}}{\partial R}\right)+s f_{x}\left(D_{r_{i}} \frac{\partial y_{c p_{i}}}{\partial R}+\right. \\
y_{c p_{i}}\left[k 1 \frac{\partial\left(r_{i}^{2}\right)}{\partial R}+k 2 \frac{\partial\left(r_{i}^{4}\right)}{\partial R}+k 3 \frac{\partial\left(r_{i}{ }^{6}\right)}{\partial R}\right]+\left(2 p_{2} y_{c p_{i}}+2 p_{1} x_{c p_{i}}\right) \frac{\partial x_{c p_{i}}}{\partial R}+ \\
\left.\left(2 p_{2} x_{c p_{i}}+6 p_{1} y_{c p_{i}}\right) \frac{\partial y_{c p_{i}}}{\partial R}\right), \\
\frac{\partial\left(r_{i}^{2}\right)}{\partial R}=2\left[\begin{array}{ll}
x_{c p_{i}} & \left.y_{c p_{i}}\right] \frac{\partial x_{c p_{i}}}{\partial R},
\end{array}\right.
\end{gathered}
$$

$$
\frac{\partial\left(r_{i}{ }^{6}\right)}{\partial R}=3 r_{i}^{4} \frac{\partial\left(r_{i}^{2}\right)}{\partial R}=6\left(x_{c p_{i}}{ }^{2}+y_{c p_{i}}{ }^{2}\right)^{2}\left[x_{c p_{i}} \quad y_{c p_{i}}\right] \frac{\partial x_{c p p_{i}}}{\partial R} .
$$

The expression for $\partial R / \partial \lambda$ is

$$
\frac{\partial R}{\partial \lambda}=\left[\begin{array}{ccc}
0 & -4 \lambda_{2} & -4 \lambda_{3} \\
2 \lambda_{2}-2 \lambda_{3} \lambda_{1} / \lambda_{0} & 2 \lambda_{1}-2 \lambda_{3} \lambda_{2} / \lambda_{0} & 2 \lambda_{0}-2 \lambda_{3} \lambda_{3} / \lambda_{0} \\
2 \lambda_{3}+2 \lambda_{2} \lambda_{1} / \lambda_{0} & -2 \lambda_{0}+\lambda_{2} \lambda_{2} / \lambda_{0} & 2 \lambda_{1}+2 \lambda_{2} \lambda_{3} / \lambda_{0} \\
2 \lambda_{2}+2 \lambda_{3} \lambda_{1} / \lambda_{0} & 2 \lambda_{1}+2 \lambda_{3} \lambda_{2} / \lambda_{0} & -2 \lambda_{0}+2 \lambda_{3} \lambda_{3} / \lambda_{0} \\
-4 \lambda_{1} & 0 & -4 \lambda_{3} \\
2 \lambda_{0}-2 \lambda_{1} \lambda_{1} / \lambda_{0} & 2 \lambda_{3}-2 \lambda_{1} \lambda_{2} / \lambda_{0} & 2 \lambda_{2}-2 \lambda_{1} \lambda_{3} / \lambda_{0} \\
2 \lambda_{3}-2 \lambda_{2} \lambda_{1} / \lambda_{0} & 2 \lambda_{0}-2 \lambda_{2} \lambda_{2} / \lambda_{0} & 2 \lambda_{1}-2 \lambda_{2} \lambda_{3} / \lambda_{0} \\
-2 \lambda_{0}+2 \lambda_{1} \lambda_{1} / \lambda_{0} & 2 \lambda_{3}+2 \lambda_{1} \lambda_{2} / \lambda_{0} & 2 \lambda_{2}+2 \lambda_{1} \lambda_{3} / \lambda_{0} \\
-4 \lambda_{1} & -4 \lambda_{2} & 0
\end{array}\right] .
$$

The derivative $\partial h_{X_{i}} / \partial \boldsymbol{T}_{\boldsymbol{c}}$ is

$$
\begin{align*}
& \frac{\partial h_{X_{i}}}{\partial \boldsymbol{T}_{c}}=\left[\begin{array}{l}
\frac{\partial x_{p_{i}}}{\partial \boldsymbol{T}_{c}} \\
\frac{\partial y_{p_{i}}}{\partial \boldsymbol{T}_{c}}
\end{array}\right],  \tag{2.16}\\
& \frac{\partial x_{p_{i}}}{\partial \boldsymbol{T}_{c}}=f_{x}\left(D_{r_{i}} \frac{\partial x_{c p_{i}}}{\partial \boldsymbol{T}_{c}}+x_{c p_{i}}\left[k 1 \frac{\partial\left(r_{i}^{2}\right)}{\partial \boldsymbol{T}_{c}}+k 2 \frac{\partial\left(r_{i}{ }^{4}\right)}{\partial \boldsymbol{T}_{c}}+k 3 \frac{\partial\left(r_{i}{ }^{6}\right)}{\partial \boldsymbol{T}_{c}}\right]+\left(2 p_{1} y_{c p_{i}}+6 p_{2} x_{c p_{i}}\right) \frac{\partial x_{c p_{i}}}{\partial \boldsymbol{T}_{c}}+\right. \\
& \left.\left(2 p_{1} x_{c p_{i}}+2 p_{2} y_{c p_{i}}\right) \frac{\partial y_{c p_{i}}}{\partial \boldsymbol{T}_{c}}\right)+s f_{x}\left(D_{r_{i}} \frac{\partial y_{c p_{i}}}{\partial \boldsymbol{T}_{c}}+y_{c p_{i}}\left[k 1 \frac{\partial\left(r_{i}^{2}\right)}{\partial \boldsymbol{T}_{c}}+k 2 \frac{\partial\left(r_{i}{ }^{4}\right)}{\partial \boldsymbol{T}_{c}}+k 3 \frac{\partial\left(r_{i}{ }^{6}\right.}{\partial \boldsymbol{T}_{c}}\right]+\right. \\
& \left.\left(2 p_{2} y_{c p_{i}}+2 p_{1} x_{c p_{i}}\right) \frac{\partial x_{c p_{i}}}{\partial \boldsymbol{T}_{c}}+\left(2 p_{2} x_{c p_{i}}+6 p_{1} y_{c p_{i}}\right) \frac{\partial y_{c p_{i}}}{\partial \boldsymbol{T}_{c}}\right), \\
& \frac{\partial y_{p_{i}}}{\partial \boldsymbol{T}_{c}}=f_{y}\left(D_{r_{i}} \frac{\partial y_{c p_{i}}}{\partial \boldsymbol{T}_{c}}+y_{c p_{i}}\left[k 1 \frac{\partial\left(r_{i}^{2}\right)}{\partial \boldsymbol{T}_{c}}+k 2 \frac{\partial\left(r_{i}{ }^{4}\right)}{\partial \boldsymbol{T}_{c}}+k 3 \frac{\partial\left(r_{i}{ }^{6}\right)}{\partial \boldsymbol{T}_{c}}\right]+\left(2 p_{2} y_{c p_{i}}+2 p_{1} x_{c p_{i}}\right) \frac{\partial x_{c p_{i}}}{\partial \boldsymbol{T}_{c}}+\right. \\
& \left.\left(2 p_{2} x_{c p_{i}}+6 p_{1} y_{c p_{i}}\right) \frac{\partial y_{c p_{i}}}{\partial \boldsymbol{T}_{c}}\right), \\
& \frac{\partial\left(r_{i}^{2}\right)}{\partial \boldsymbol{T}_{c}}=2\left[\begin{array}{ll}
x_{c p_{i}} & y_{c p_{i}}
\end{array}\right] \frac{\partial X_{c p_{i}}}{\partial \boldsymbol{T}_{c}}, \\
& \frac{\partial\left(r_{i}^{4}\right)}{\partial \boldsymbol{T}_{c}}=2 r_{i}{ }^{2} \frac{\partial\left(r_{i}^{2}\right)}{\partial \boldsymbol{T}_{c}}=4\left(x_{c p_{i}}{ }^{2}+y_{c p_{i}}{ }^{2}\right)\left[\begin{array}{ll}
x_{c p_{i}} & y_{c p_{i}}
\end{array}\right] \frac{\partial X_{c p_{i}}}{\partial \boldsymbol{T}_{c}}, \\
& \frac{\partial\left(r_{i}{ }^{6}\right)}{\partial \boldsymbol{T}_{c}}=3 r_{i}^{4} \frac{\partial\left(r_{i}^{2}\right)}{\partial \boldsymbol{T}_{c}}=6\left(x_{c p_{i}}{ }^{2}+y_{c p_{i}}{ }^{2}\right)^{2}\left[\begin{array}{ll}
x_{c p_{i}} & y_{c p_{i}}
\end{array}\right] \frac{\partial X_{c p_{i}}}{\partial \boldsymbol{T}_{c}}, \\
& \frac{\partial X_{c p_{i}}}{\partial \boldsymbol{T}_{c}}=\left[\begin{array}{l}
\frac{\partial x_{c p_{i}}}{\partial \boldsymbol{T}_{c}} \\
\frac{\partial y_{c p_{i}}}{\partial \boldsymbol{T}_{c}}
\end{array}\right]=\left[\begin{array}{ccc}
\frac{1}{z_{c_{i}}} & 0 & -\frac{x_{c_{i}}}{z_{c_{i}}{ }^{2}} \\
0 & \frac{1}{z_{c_{i}}} & -\frac{y_{c_{i}}}{z_{c_{i}}{ }^{2}}
\end{array}\right] .
\end{align*}
$$

It is important to mention that the measurement model gradient is obtained for any point $X_{i}$ located in BF and projected to ICS. And as a result, it is obtained a $2 \times 16$ matrix for each point $X_{i}$, which is taken into account in the optimization process.

## 3. Linearization of the measurement model

The linearization of the measurement model allows to solve the nonlinear calibration problem for cameras by linear method, in which the nonlinear radial and tangential distortion components are ignored.

The expressions (2.4) and (2.5) are nonlinear functions, which perform the projection to the sensor plane, and can be linearized by means of Homogeneous coordinates provided that the vector $X_{c_{i}}$ and $X_{c p_{i}}$ are expressed homogeneous vectors [1], obtaining the equation (3.1) as a linear expression, where the symbol $\sim$ means that the two homogeneous vectors are not equal, but they have the same direction.

$$
\left[\begin{array}{c}
x_{c p_{i}}  \tag{3.1}\\
y_{c p_{i}} \\
z_{c p_{i}}
\end{array}\right] \sim\left[\begin{array}{c}
x_{c_{i}} f \\
y_{c_{i}} f \\
z_{c_{i}}
\end{array}\right]=\left[\begin{array}{llll}
f & 0 & 0 & 0 \\
0 & f & 0 & 0 \\
0 & 0 & 1 & 0
\end{array}\right]\left[\begin{array}{c}
x_{c_{i}} \\
y_{c_{i}} \\
z_{c_{i}} \\
1
\end{array}\right]
$$

With regard to the effect of the lens distortion, it is convenient to consider it to be equal to zero during the linearization process [2]. Therefore, considering this particular case it is possible to obtain a linear expression, see equation (3.2), from the nonlinear measurement model (2.6) by means of the homogeneous coordinates which is usually done in order to determine initial values of internal and external parameter.

$$
\left[\begin{array}{c}
\tilde{u}  \tag{3.2}\\
\tilde{v} \\
\widetilde{w}
\end{array}\right] \sim\left[\begin{array}{ccc}
\alpha_{x} & s & u_{0} \\
0 & \alpha_{y} & v_{0} \\
0 & 0 & 1
\end{array}\right]\left[\begin{array}{llll}
f & 0 & 0 & 0 \\
0 & f & 0 & 0 \\
0 & 0 & 1 & 0
\end{array}\right]\left[\begin{array}{cc}
R & \boldsymbol{T}_{c} \\
0_{1 x 3} & 1
\end{array}\right]\left[\begin{array}{c}
x_{i} \\
y_{i} \\
z_{i} \\
1
\end{array}\right]
$$

In the equation (3.2), $\tilde{u}, \tilde{v}, \widetilde{w}$ are homogeneous coordinates for the points in the ICS, and can be expressed as follows:

$$
\left[\begin{array}{c}
\tilde{u}  \tag{3.3}\\
\tilde{v} \\
\widetilde{w}
\end{array}\right] \sim\left[H_{3 x 4}\right]\left[\begin{array}{c}
x_{i} \\
y_{i} \\
z_{i} \\
1
\end{array}\right],
$$

where matrix H is the transition matrix, or linear measurement model. The BF is chosen in such a way that the points $X_{i}$ are located on the $X Y$-plane, in consequence, the component $z_{i}$ is zero, it means that the equation (3.3) can be reduced to equation (3.4).

$$
\left[\begin{array}{c}
\tilde{u}_{i}  \tag{3.4}\\
\tilde{v}_{i} \\
\widetilde{w}_{i}
\end{array}\right] \sim\left[\begin{array}{lll}
a_{1} & a_{2} & a_{3} \\
a_{4} & a_{5} & a_{6} \\
a_{7} & a_{8} & a_{9}
\end{array}\right]\left[\begin{array}{c}
x_{i} \\
y_{i} \\
1
\end{array}\right]=H\left[\begin{array}{c}
x_{i} \\
y_{i} \\
1
\end{array}\right]
$$

Due to the fact that the vectors $\left[\tilde{u}_{i}, \tilde{v}_{i}, \widetilde{w}_{i}\right]^{T}$ and $H\left[x_{i}, y_{i}, 1\right]^{T}$ have the same direction, their cross product is zero and based on the Direct Linear Transformation (DLT) algorithm [1] the equation is

$$
\begin{gather*}
{\left[\begin{array}{c}
\tilde{u}_{i} \\
\tilde{v}_{i} \\
\widetilde{w}_{i}
\end{array}\right] \times H\left[\begin{array}{c}
x_{i} \\
y_{i} \\
1
\end{array}\right]=\left[\begin{array}{l}
0 \\
0 \\
0
\end{array}\right],} \\
{\left[\begin{array}{ccc}
0^{T} & -\widetilde{w}_{i} X_{i}^{T} & \tilde{v}_{i} X_{i}^{T} \\
\widetilde{w}_{i} X_{i}^{T} & 0^{T} & -\tilde{u}_{i} X_{i}^{T}
\end{array}\right] \boldsymbol{L}=\left[\begin{array}{l}
0 \\
0
\end{array}\right],} \tag{3.5}
\end{gather*}
$$

where $\boldsymbol{L}=\left[\begin{array}{lllllll}a_{1} & a_{2} & a_{3} & a_{4} & a_{5} & a_{6} & a_{8}\end{array} a_{9}\right]^{T}$ and $X_{i}=\left[\begin{array}{lll}x_{i} & y_{i} & 1\end{array}\right]^{T}$.
As it can be seen, the equation (3.5) has the form of a homogeneous system, where $L$ can be determined by the Single Values Decomposition (SVD). This DLT algorithm is widely used to calculate the transition matrix H where is needed a set of four points as minimum. However, because matrix H is a projective transformation, it has a non-linear nature, therefore, an iterative method can be applied in order to optimize the components of the matrix H by means of reduction of the error projection [2]. Thus, it is necessary to work in inhomogeneous coordinates.

Let the matrix $H$ already be determined by means of DLT, then

$$
\left[\begin{array}{c}
\tilde{x}_{i}  \tag{3.6}\\
\tilde{y}_{i} \\
\widetilde{\omega}_{i}
\end{array}\right]=\left[\begin{array}{lll}
a_{1} & a_{2} & a_{3} \\
a_{4} & a_{5} & a_{6} \\
a_{7} & a_{8} & a_{9}
\end{array}\right]\left[\begin{array}{c}
x_{i} \\
y_{i} \\
1
\end{array}\right]=H\left[\begin{array}{c}
x_{i} \\
y_{i} \\
1
\end{array}\right]
$$

where $\left(\tilde{x}_{i}, \tilde{y}_{i}, \widetilde{\omega}_{i}\right)$ is the homogeneous coordinate representation of a point $\left(u_{i}, v_{i}\right)$ located in the ICS, then the projective transformation in the equation (3.6) can be written in inhomogeneous form as

$$
\begin{align*}
& u_{i}=\frac{\tilde{x}_{i}}{\widetilde{\omega}_{i}}=\frac{a_{1} x_{i}+a_{2} y_{i}+a_{3}}{a_{7} x_{i}+a_{8} y_{i}+a_{9}}  \tag{3.7}\\
& v_{i}=\frac{\tilde{y}_{i}}{\widetilde{\omega}_{i}}=\frac{a_{4} x_{i}+a_{5} y_{i}+a_{6}}{a_{7} x_{i}+a_{8} y_{i}+a_{9}}, \tag{3.8}
\end{align*}
$$

where $\left(u_{i}, v_{i}\right)$ finally represents the mapped point in the ICS from the BF. The Jacobian matrix for projective transformation is shown below, which is widely used by the most of the iterative methods.

$$
J=\frac{\partial\left[\begin{array}{l}
u_{i}  \tag{3.9}\\
v_{i}
\end{array}\right]}{\partial L}=\frac{1}{\widetilde{w}_{i}}\left[\begin{array}{ccccccccc}
x_{i} & y_{i} & 1 & 0 & 0 & 0 & -u_{i} x_{i} & -u_{i} y_{i} & -u_{i} \\
0 & 0 & 0 & x_{i} & y_{i} & 1 & -v_{i} x_{i} & -v_{i} y_{i} & -v_{i}
\end{array}\right]
$$

Below it is described an algorithm to compute the transition matrix H .

## I. Initialize data:

- Let $i=1,2, \ldots, n$, where $n \geq 4$ is the number of mapped points.
- Let $X_{i}=\left[\begin{array}{lll}x_{i} & y_{i} & 1\end{array}\right]^{T}$ be a homogeneous coordinate representation of a $i$-th point from the BF, where the component $z_{i}$ is zero.
- Let ( $\tilde{u}_{i}, \tilde{v}_{i}, \widetilde{w}_{i}$ ) be a homogeneous coordinate representation of a $i$-th point located in the ICS.
- Let $\widetilde{w}_{i}$ to be one, in order to make $\left(\tilde{u}_{i}, \tilde{v}_{i}\right)$ points measured in the ICS.
$\checkmark$ Apply the preconditioning matrix to each point as follow:

$$
\left[\begin{array}{l}
\tilde{u}_{i}^{\prime} \\
\tilde{v}_{i}{ }^{\prime} \\
\widetilde{w}_{i}^{\prime}
\end{array}\right]=H_{\text {prec }}\left[\begin{array}{c}
\tilde{u}_{i} \\
\tilde{v}_{i} \\
\widetilde{w}_{i}
\end{array}\right]
$$

- Write the homogeneous system according to the equation (3.5) for $n$ points.
- Solve the homogenous system in order to obtain $L$, and obtain the transition matrix $H$ from $L$.
- Update the transition matrix as follow:

$$
\begin{gathered}
H \leftarrow H / a_{9} \\
H \leftarrow H_{\text {prec }}{ }^{-1} H
\end{gathered}
$$

## II. Optimize the matrix $\boldsymbol{H}$ :

- Let $X_{i}{ }^{\prime}=\left[x_{i}{ }^{\prime}, y_{i}{ }^{\prime}\right]^{T}$ be measured point in the ICS.
- Let $P_{i}=\left[u_{i}, v_{i}\right]^{T}$ mapped point in the image coordinate obtained from the equations (3.7) and (3.8).
- By means of iterative process and using the Jacobian matrix of the equation (3.8), minimize:

$$
\epsilon=\sum\left\|X_{i}^{\prime}-P_{i}\right\|
$$

Algorithm 1. Computing Transition matrix H
Before determining the transition matrix, a normalization of the data is recommended to avoid bad results because of noisy data. In [1] is recommended a normalization data so that the centroid of the new set of points is the origin of coordinates $(0,0)$ and the average distance from the origin equals to $\sqrt{2}$.

In the next section it is shown how the Computing transition matrix algorithm can be used to estimate the rotation matrix and translation vector of the BF with respect to CCS by a linear method rather than use the linearized measurement model.

## 4. Calibration Algorithm

In this section the calibration algorithm of the widely known tool for camera calibration developed in [2], based on Rodrigues' rotation formula, is adapted to quaternion rotation representation.

In the Figure 4, it is shown that a chessboard is photographed with different orientations and translation vectors in order to obtain considerable amount of points for calibration process. Additionally, intrinsic parameters are shown, which are internal fixed parameters of the camera itself. They have to be determined in the calibration process and then will remain fixed. On the other hand, extrinsic parameters, rotation matrix and translation vector, are determined for each image, and they are not fixed parameters because the location and orientation of the object can


Figure 4. Intrinsic and Extrinsic parameters
change.
Calibration process is based on two main steps: initialization of the parameters and optimization of the parameters by the gradient method.

### 4.1. Initialization of parameters

The initial value of the principal point can be initialized as the center point of the image, for example, if the resolution of the camera is $640 \times 480$ pixels, then the principal point $P=\left(u_{0}, v_{0}\right)=(320,240)$. The Skew parameter can be initialized as zero as well as the distortions coefficients $k_{1}, k_{2}, k_{3}, p_{1}, p_{2}$.

With regards to the focal distance ( $f_{x}, f_{y}$ ), it can be initialized using vanishing points as in [1] and different methods as in [2] and [4], which make use of transition matrices from BF to the ICS by using the Algorithm 1 .

Considering initial values for skew factor ' S ' and distortion coefficients $\boldsymbol{K}_{\boldsymbol{D}}$ equal to zero, the points in the ICS $\left(x_{i}^{\prime}, y_{i}^{\prime}\right)$ can be transformed into CCS as follows:

$$
\begin{aligned}
& x_{c p_{i}}=\left(x_{i}^{\prime}-u_{0}\right) / f_{x}, \\
& y_{c p_{i}}=\left(y_{i}^{\prime}-v_{0}\right) / f_{y} .
\end{aligned}
$$

The equations above show that the point $\left(x_{i}^{\prime}, y_{i}^{\prime}\right)$ is located in the sensor plane in the CCS, and it is related to the BF by the next equation, where the points in the ICS and BF are expressed by homogeneous coordinate.

$$
\left[\begin{array}{c}
x_{c}  \tag{4.1}\\
y_{c} \\
1
\end{array}\right] \sim\left[\begin{array}{cc}
R & \boldsymbol{T}_{\boldsymbol{c}} \\
\mathbf{0}_{\mathbf{1} x 3} & 1
\end{array}\right]\left[\begin{array}{c}
x_{i} \\
y_{i} \\
z_{i} \\
1
\end{array}\right]
$$

Because of component $z_{i}$ is zero for flat objects, equation (4.1) can be rewritten as

$$
\left[\begin{array}{c}
x_{c} \\
y_{c} \\
1
\end{array}\right] \sim\left[\begin{array}{lll}
\boldsymbol{r}_{1} & \boldsymbol{r}_{2} & \boldsymbol{T}_{\boldsymbol{c}}
\end{array}\right]\left[\begin{array}{c}
x_{i} \\
y_{i} \\
1
\end{array}\right]=H_{r}\left[\begin{array}{c}
x_{i} \\
y_{i} \\
1
\end{array}\right],
$$

where $\boldsymbol{T}_{\boldsymbol{c}}$ is translation vector and $\boldsymbol{r}_{\boldsymbol{i}}$ are the columns of rotation matrix R , and $\left\|\boldsymbol{r}_{i}\right\|=1$. The matrix $H_{r}$ can be computed by means of the Algorithm 1, and additionally it is necessary to perform a normalization so that the vectors $\boldsymbol{r}_{\boldsymbol{i}}$ have modulus equal to one, then to use the QR decomposition to obtain a better result in the orthogonality of the vector $\boldsymbol{r}_{\boldsymbol{i}}$.

### 4.2. Optimization process

Due to the non-linearity of the measurement model, an optimization process is required to be performed in order to tune up the parameters which have been initialized previously. The essential step is the definition of the equations.

Let us consider a scheme where it is available just one image as it is shown in the Figure 5 . Let $i=1,2, \ldots, n$, where $n$ is the number of mapped points to the ICS. Let $X^{\prime}{ }_{i}=\left[x^{\prime}{ }_{i}, y^{\prime}{ }_{i}\right]^{T}$ and $X_{i}=\left[x_{i}, y_{i}, 0\right]^{T}$ be the known vector representations of a point in the ICS and BF.

Image coordinate system


Figure 5. Projection from Body-fixed frame to image coordinate system

Let $X p_{i}=\left[x p_{i}, y p_{i}\right]^{T}$ be $i$-th point already mapped to the ICS from the BF by using the measurement model $h_{X_{i}}$ from the equation (2.12), where the rotation matrix can be expressed by using the vector part $\lambda$ of a unit quaternion, below the equations for one image with $n$ points:

$$
\begin{gathered}
X p_{1}=h_{X_{1}}\left(\boldsymbol{F}, \boldsymbol{C}, S, \boldsymbol{K}_{\boldsymbol{D}}, \boldsymbol{\lambda}, \boldsymbol{T}_{\boldsymbol{c}}\right) \\
X p_{2}=h_{X_{2}}\left(\boldsymbol{F}, \boldsymbol{C}, S, \boldsymbol{K}_{\boldsymbol{D}}, \boldsymbol{\lambda}, \boldsymbol{T}_{\boldsymbol{c}}\right) \\
\vdots \\
X p_{i}=h_{X_{i}}\left(\boldsymbol{F}, \boldsymbol{C}, S, \boldsymbol{K}_{\boldsymbol{D}}, \boldsymbol{\lambda}, \boldsymbol{T}_{\boldsymbol{c}}\right) \\
\vdots \\
X p_{n}=h_{X_{n}}\left(\boldsymbol{F}, \boldsymbol{C}, S, \boldsymbol{K}_{\boldsymbol{D}}, \boldsymbol{\lambda}, \boldsymbol{T}_{\boldsymbol{c}}\right)
\end{gathered}
$$

Let $\Delta X$ be the error vector which is defined as the difference between the points $X p_{i}$ and $X_{i}^{\prime}$ as follow:

$$
\Delta X=\left[\begin{array}{c}
X p_{1}-X_{1}^{\prime}  \tag{4.2}\\
X p_{2}-X_{2}^{\prime} \\
\vdots \\
X p_{n}-X_{n}^{\prime}
\end{array}\right]=\left[\begin{array}{c}
x p_{1}-x_{1}^{\prime} \\
y p_{1}-y_{1}^{\prime} \\
x p_{2}-x_{2}^{\prime} \\
y p_{2}-y_{2}^{\prime} \\
\vdots \\
x p_{n}-x^{\prime}{ }_{n} \\
y p_{n}-y^{\prime}
\end{array}\right]
$$

Now let us assume that $m$ images are available with $n$ points in each image. Let $i=1,2, \ldots, n$, and $k=1,2, \ldots, m$ where $n$ is the number of mapped points to the ICS and $m$ is the number images. It is important to mention that the location of points in each image depends on the translation vector and orientation of the BF with respect to the CCS.

Let $X_{i}^{\prime k}=\left[x_{i}^{\prime k}, y_{i}^{\prime k}\right]^{T}$ be the vector representation of the $i$-th point in the $k$-th image (ICS).

Let $X_{i}=\left[x_{i}, y_{i}, 0\right]^{T}$ be the vector representation of the $i$-th point in the BF.
Let $X p_{i}^{k}=\left[x p_{i}^{k}, y p_{i}^{k}\right]^{T}$ be the point $X_{i}$ already mapped to the $k$-th image (ICS) from the BF by using the nonlinear model $h_{X_{i}}^{k}$, which represent the projection of the point $X_{i}$ to the $k$-th image. Below the equations for $m$ images with $n$ points in each image are given:

$$
\begin{align*}
& X p_{1}^{1}=h_{X_{1}}^{1}\left(\boldsymbol{F}, \boldsymbol{C}, S, \boldsymbol{K}_{\boldsymbol{D}}, \boldsymbol{\lambda}_{\mathbf{1}}, \boldsymbol{T}_{\boldsymbol{c}_{\mathbf{1}}}\right) \\
& X p_{2}^{1}=h_{X_{2}}^{1}\left(\boldsymbol{F}, \boldsymbol{C}, S, \boldsymbol{K}_{\boldsymbol{D}}, \boldsymbol{\lambda}_{\mathbf{1}}, \boldsymbol{T}_{\boldsymbol{c}_{\boldsymbol{1}}}\right) \\
& X p_{n}^{1}=h_{X_{n}}^{1}\left(\boldsymbol{F}, \boldsymbol{C}, S, \boldsymbol{K}_{\boldsymbol{D}}, \boldsymbol{\lambda}_{\mathbf{1}}, \boldsymbol{T}_{\boldsymbol{c}_{\mathbf{1}}}\right) \\
& X p_{1}^{2}=h_{X_{1}}^{2}\left(\boldsymbol{F}, \boldsymbol{C}, S, \boldsymbol{K}_{\boldsymbol{D}}, \boldsymbol{\lambda}_{\mathbf{2}}, \boldsymbol{T}_{\boldsymbol{c}_{\mathbf{2}}}\right) \\
& X p_{2}^{2}=h_{X_{2}}^{2}\left(\boldsymbol{F}, \boldsymbol{C}, S, \boldsymbol{K}_{\boldsymbol{D}}, \boldsymbol{\lambda}_{\mathbf{2}}, \boldsymbol{T}_{\boldsymbol{c}_{\boldsymbol{2}}}\right) \\
& X p_{n}^{2}=h_{X_{n}}^{2}\left(\boldsymbol{F}, \boldsymbol{C}, S, \boldsymbol{K}_{\boldsymbol{D}}, \boldsymbol{\lambda}_{\mathbf{2}}, \boldsymbol{T}_{\boldsymbol{c}_{\mathbf{2}}}\right)  \tag{4.3}\\
& X p_{i}^{k}=h_{X_{i}}^{k}\left(\boldsymbol{F}, \boldsymbol{C}, S, \boldsymbol{K}_{\boldsymbol{D}}, \boldsymbol{\lambda}_{\boldsymbol{k}}, \boldsymbol{T}_{\boldsymbol{c}_{\boldsymbol{k}}}\right) \\
& X p_{1}^{m}=h_{X_{1}}^{m}\left(\boldsymbol{F}, \boldsymbol{C}, S, \boldsymbol{K}_{\boldsymbol{D}}, \boldsymbol{\lambda}_{\boldsymbol{m}}, \boldsymbol{T}_{\boldsymbol{c}_{\boldsymbol{m}}}\right) \\
& X p_{2}^{m}=h_{X_{2}}^{m}\left(\boldsymbol{F}, \boldsymbol{C}, S, \boldsymbol{K}_{\boldsymbol{D}}, \boldsymbol{\lambda}_{\boldsymbol{m}}, \boldsymbol{T}_{\boldsymbol{c}_{\boldsymbol{m}}}\right) \\
& X p_{n}^{m}=h_{X_{n}}^{m}\left(\boldsymbol{F}, \boldsymbol{C}, S, \boldsymbol{K}_{\boldsymbol{D}}, \boldsymbol{\lambda}_{\boldsymbol{m}}, \boldsymbol{T}_{\boldsymbol{c}_{\boldsymbol{m}}}\right)
\end{align*}
$$

Let $h^{k}$ be defined as

$$
h^{k}\left(\boldsymbol{F}, \boldsymbol{C}, S, \boldsymbol{K}_{\boldsymbol{D}}, \boldsymbol{\lambda}_{\boldsymbol{k}}, \boldsymbol{T}_{\boldsymbol{c}_{\boldsymbol{k}}}\right)=\left[\begin{array}{c}
h_{X_{1}}^{k}\left(\boldsymbol{F}, \boldsymbol{C}, S, \boldsymbol{K}_{\boldsymbol{D}}, \boldsymbol{\lambda}_{\boldsymbol{k}}, \boldsymbol{T}_{\boldsymbol{c}_{\boldsymbol{k}}}\right) \\
h_{X_{2}}^{k}\left(\boldsymbol{F}, \boldsymbol{C}, S, \boldsymbol{K}_{\boldsymbol{D}}, \boldsymbol{\lambda}_{\boldsymbol{k}}, \boldsymbol{T}_{\boldsymbol{c}_{\boldsymbol{k}}}\right) \\
\vdots \\
h_{X_{n}}^{k}\left(\boldsymbol{F}, \boldsymbol{C}, S, \boldsymbol{K}_{\boldsymbol{D}}, \boldsymbol{\lambda}_{\boldsymbol{k}}, \boldsymbol{T}_{\boldsymbol{c}_{\boldsymbol{k}}}\right)
\end{array}\right] .
$$

The partial derivatives of $h^{k}$ from the measurement model gradient in the equation (2.13) are

$$
\left[\frac{\partial h^{k}}{\partial F} \frac{\partial h^{k}}{\partial C} \frac{\partial h^{k}}{\partial S} \frac{\partial h^{k}}{\partial K_{D}} \frac{\partial h^{k}}{\partial \lambda_{k}} \frac{\partial h^{k}}{\partial \boldsymbol{c}_{c_{k}}}\right]=\left[\begin{array}{cccccc}
\frac{\partial h_{X_{1}}^{k}}{\partial F} & \frac{\partial h_{X_{1}}^{k}}{\partial C} & \frac{\partial h_{X_{1}}^{k}}{\partial S} & \frac{\partial h_{X_{1}}^{k}}{\partial K_{D}} & \frac{\partial h_{X_{1}}^{k}}{\partial \lambda_{k}} & \frac{\partial h_{X_{1}}^{k}}{\partial T_{c_{k}}} \\
\frac{\partial h_{X_{2}}^{k}}{\partial F} & \frac{\partial h_{X_{2}}^{k}}{\partial C} & \frac{\partial h_{X_{2}}^{k}}{\partial S} & \frac{\partial h_{X_{2}}^{k}}{\partial K_{D}} & \frac{\partial h_{X_{2}}^{k}}{\partial \lambda_{k}} & \frac{\partial h_{X_{2}}^{k}}{\partial \boldsymbol{T}_{c_{k}}} \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
\frac{\partial h_{n}^{k}}{\partial F} & \frac{\partial h_{n}^{k}}{\partial C} & \frac{\partial h_{n}^{k}}{\partial S} & \frac{\partial h_{n}^{k}}{\partial K_{D}} & \frac{\partial h_{n}^{k}}{\partial \lambda_{k}} & \frac{\partial h_{n}^{k}}{\partial T_{c_{k}}}
\end{array}\right],
$$

then the equations (4.3) can be expressed in a shorter form as follows:

$$
\begin{gathered}
X p^{1}=h^{1}\left(\boldsymbol{F}, \boldsymbol{C}, S, \boldsymbol{K}_{\boldsymbol{D}}, \lambda_{\mathbf{1}}, \boldsymbol{T}_{\boldsymbol{c}_{\mathbf{1}}}\right) \rightarrow \Delta X^{1}=X p^{1}-X^{\prime 1}, \\
X p^{2}=h^{2}\left(\boldsymbol{F}, \boldsymbol{C}, S, \boldsymbol{K}_{\boldsymbol{D}}, \boldsymbol{\lambda}_{\mathbf{2}}, \boldsymbol{T}_{\boldsymbol{c}_{\mathbf{2}}}\right) \rightarrow \Delta X^{2}=X p^{2}-X^{\prime 2}, \\
\vdots \\
X p^{k}=h^{k}\left(\boldsymbol{F}, \boldsymbol{C}, S, \boldsymbol{K}_{\boldsymbol{D}}, \boldsymbol{\lambda}_{\boldsymbol{k}}, \boldsymbol{T}_{\boldsymbol{c}_{\boldsymbol{k}}}\right) \rightarrow \Delta X^{k}=X p^{k}-X^{\prime k}, \\
\vdots \\
X p^{m}=h^{m}\left(\boldsymbol{F}, \boldsymbol{C}, S, \boldsymbol{K}_{\boldsymbol{D}}, \boldsymbol{\lambda}_{\boldsymbol{m}}, \boldsymbol{T}_{\boldsymbol{c}_{\boldsymbol{m}}}\right) \rightarrow \Delta X^{m}=X p^{m}-X^{\prime m}
\end{gathered}
$$

where $X p^{k}=\left[x p_{1}^{k}, y p_{1}^{k}, x p_{2}^{k}, y p_{2}^{k}, \ldots, x p_{n}^{k}, y p_{n}^{k}\right]^{T}$ is column vector representation of the points mapped to the $k$-th image (ICS) from the BF, and $X^{\prime k}=\left[x_{1}^{\prime k}, y_{1}^{\prime k}, x_{2}^{\prime k}, y_{2}^{\prime k}, \ldots, x_{n}^{\prime k}, y_{n}^{\prime k}\right]^{T}$ is vector representation of all the points in the $k$-th image.

In the equation (4.2) the error vector $\Delta X$ can be express as $\Delta X^{k}$ where $k$ indicates the error vector for the corresponding $k$-th image and nonlinear model $h^{k}$.

Finally, the equations can be express as a column of functions

$$
\hbar\left(\boldsymbol{F}, \boldsymbol{C}, S, \boldsymbol{K}_{\boldsymbol{D}}, \boldsymbol{\lambda}_{\mathbf{1}}, \boldsymbol{T}_{\boldsymbol{c}_{\mathbf{1}}}, \ldots, \boldsymbol{\lambda}_{\boldsymbol{m}}, \boldsymbol{T}_{\boldsymbol{c}_{\boldsymbol{m}}}\right)=\left[\begin{array}{c}
X p^{1} \\
X p^{2} \\
\vdots \\
X p^{m}
\end{array}\right]=\left[\begin{array}{c}
h^{1}\left(\boldsymbol{F}, \boldsymbol{C}, S, \boldsymbol{K}_{\boldsymbol{D}}, \boldsymbol{\lambda}_{\mathbf{1}}, \boldsymbol{T}_{\boldsymbol{c}_{\boldsymbol{1}}}\right) \\
h^{2}\left(\boldsymbol{F}, \boldsymbol{C}, S, \boldsymbol{K}_{\boldsymbol{D}}, \boldsymbol{\lambda}_{\mathbf{2}}, \boldsymbol{T}_{\boldsymbol{c}_{\boldsymbol{2}}}\right) \\
\vdots \\
h^{m}\left(\boldsymbol{F}, \boldsymbol{C}, S, \boldsymbol{K}_{\boldsymbol{D}}, \boldsymbol{\lambda}_{\boldsymbol{m}}, \boldsymbol{T}_{\boldsymbol{c}_{\boldsymbol{m}}}\right)
\end{array}\right]
$$

Let $X^{\prime}$ be the column vector $\left[X^{\prime 1^{T}}, X^{\prime 2^{T}}, \ldots, X^{\prime m}\right]^{T}$, and the global error vector can be defined as follows:

$$
\boldsymbol{\epsilon}=\hbar(M)-X^{\prime}=\left[\begin{array}{c}
\Delta X^{1} \\
\Delta X^{2} \\
\vdots \\
\Delta X^{m}
\end{array}\right]
$$

The Gauss-Newton Method is used to solve the optimization problem, which is based on the minimization of the global error vector $\boldsymbol{\epsilon}$; Let $\boldsymbol{M}=\left[\boldsymbol{F}, \boldsymbol{C}, S, \boldsymbol{K}_{\boldsymbol{D}}, \boldsymbol{\lambda}_{\mathbf{1}}, \boldsymbol{T}_{\boldsymbol{c}_{\boldsymbol{1}}}, \ldots, \boldsymbol{\lambda}_{\boldsymbol{m}}, \boldsymbol{T}_{\boldsymbol{c}_{\boldsymbol{m}}}\right]$ be the vector of parameters, let $\boldsymbol{M}_{\boldsymbol{0}}$ be the initial values for the vector of parameters $\boldsymbol{M}$, and let $\boldsymbol{\epsilon}_{\mathbf{0}}$ be the initial error vector.

$$
\boldsymbol{\epsilon}_{\mathbf{0}}=\hbar\left(\boldsymbol{M}_{\mathbf{0}}\right)-X^{\prime}
$$

Let $\boldsymbol{\epsilon}_{\boldsymbol{l}}$ be the error vector and $\boldsymbol{M}_{\boldsymbol{l}}$ be the vector of parameters which are updated for each iteration as follows:

$$
\begin{gathered}
\boldsymbol{\epsilon}_{\boldsymbol{l}}=\hbar\left(\boldsymbol{M}_{\boldsymbol{l}}\right)-X^{\prime} \\
\Delta \boldsymbol{M}=\left(J^{T} J\right)^{-1} J^{T} \boldsymbol{\epsilon}_{\boldsymbol{l}}
\end{gathered}
$$

$$
M_{l+1}=M_{l}+\Delta M
$$

where the Jacobian matrix $J$ is defined as follows:
where $\boldsymbol{F}=\left[\begin{array}{ll}f_{x} & f_{y}\end{array}\right], \boldsymbol{C}=\left[\begin{array}{ll}u_{0} & v_{0}\end{array}\right]$ and $\boldsymbol{K}_{\boldsymbol{D}}=\left[\begin{array}{lllll}k_{1} & k_{2} & p_{1} & p_{2} & k_{3}\end{array}\right]$.

### 4.3. Algorithm for camera calibration

Assume $m$ images with $n$ points in each image are given and let $i=1,2, \ldots, n$, and $k=1,2, \ldots, m$ where $n$ is the number of mapped points to the ICS and $m$ is the number images. The calibration algorithm is shown below.
I. Initialize parameters: Use the Algorithm 1 to initialize the vector parameters $\boldsymbol{M}=\left[\boldsymbol{F}, \boldsymbol{C}, S, \boldsymbol{K}_{D}, \boldsymbol{\lambda}_{1}, \boldsymbol{T}_{\boldsymbol{c}_{1}}, \ldots, \boldsymbol{\lambda}_{m}, \boldsymbol{T}_{\boldsymbol{c}_{m}}\right]$.
II. Initialize global error vector $\epsilon_{0}: \epsilon_{0}=\hbar\left(M_{0}\right)-X^{\prime}$
III. Iterative process:
a. $\Delta \boldsymbol{M}_{0}=\left(J_{0}{ }^{T} J_{0}\right)^{-1} J_{0}{ }^{T} \boldsymbol{\epsilon}_{0}$, where $J_{0}$ is $J$ jacobian matrix evaluated at $\boldsymbol{M}_{\mathbf{0}}$
b. $M_{1}=M_{0}+\Delta M_{0}$
c. Change $\leftarrow\left|\Delta M_{0}\right| /\left|M_{1}\right|$
d. Iteration $\leftarrow 0$
e. While ((Change > le-10) \& (Iterarion < MaxIteration ))
i. $\boldsymbol{\epsilon}_{\mathbf{1}}=\hbar\left(\boldsymbol{M}_{1}\right)-X^{\prime}$
ii. $\Delta \boldsymbol{M}_{1}=\left(J_{1}{ }^{T} J_{1}\right)^{-1} J_{1}{ }^{T} \epsilon_{1}$
iii. $M_{2}=M_{1}+\Delta M_{1}$
iv. The quaternion part of $\boldsymbol{M}_{\mathbf{2}}$ must be normalized for each iteration, and then $\boldsymbol{M}_{2}$ must be updated.
v. Change $\leftarrow\left|\Delta M_{1}\right| /\left|M_{2}\right|$
vi. Iterarion $\leftarrow$ Iterarion +1

The vector $M_{2}$ is the optimal vector $\boldsymbol{M}$.

Algorithm 2. Camera calibration
After the calibration the Algorithm 2 can be used to determine the matrix rotation and $\boldsymbol{T}_{\boldsymbol{c}}$ without considering the other parameters in the vector $\boldsymbol{M}$.

### 4.4. Application algorithm for camera calibration

Using 50 images and 70 points per image. The images were taken using the camera Model FI8918W with resolution $480 \times 640$ pixels.


Figure 6. Image used for calibration
As the result of the calibration process using the Algorithm 2 the values of the intrinsic parameters are obtained:

- Principal point $P=(318.85122,255.46648)$ (pixel)
- Focal length axis-x $f_{x}=633.54607$ (pixel)
- Focal length axis-y $f_{y}=634.02213$ (pixel)
- Skew $s=0.0$
- Distortion coefficients $k_{1}=-0.46378, k_{2}=0.28011, k_{3}=0.0, p_{1}=0.00083$, $p_{2}=0.00269$
- The total error is expressed in pixels $\sigma_{x}=0.20879, \sigma_{y}=0.24828$

It is necessary to keep in mind that only intrinsic parameters remain fixed because they are fixed values which depend on the camera assembly. On the other hand, the external parameters, rotation matrix and translation vector change as the BF or the camera move.

In the Figure 7 the extrinsic parameters by mean of the locations and orientations of the chessboard with respect to the CCS are shown which has been obtained during the calibration process.


Figure 7. Visualization of the extrinsic parameters with a fixed camera
As a result of the calibration process, the equation (2.11) for the measurement model based on quaternions can be rewritten as follows:

$$
X_{p_{i}}=\left[\begin{array}{l}
x_{p_{i}}  \tag{4.4}\\
y_{p_{i}}
\end{array}\right]=h_{X_{i}}\left(\lambda, \boldsymbol{T}_{c}\right) .
$$

Thus, if the points $X_{p_{i}}$ and points $X_{i}$ are known, it is possible to determine the orientation and the translation vector by means of Algorithm 2 as in the next section is shown.

### 4.5. Testing measurement model

In this part, the results of the camera calibration by means of a rotating table are presented. The facilities for testing are shown in the Figure 8. It is used to determine how accurate the measurement model is. The kinematic equation of the schema is analyzed for initial time $t_{0}$ and for the time $t_{1}$ when a rotation angle $\alpha$ around the axis- $Y_{T}$ is performed.

Kinematic equation in $t_{0}$ :

$$
\begin{equation*}
R_{C W}{ }_{t 0} X_{i}+\boldsymbol{T}_{c_{t 0}}=R_{C T} R_{T W} X_{i}+R_{C T} \boldsymbol{T}_{\boldsymbol{W}}+\boldsymbol{T}_{\boldsymbol{T}} \tag{4.5}
\end{equation*}
$$

Kinematic equation in $\mathrm{t}_{1}$ :

$$
\begin{equation*}
R_{C W} X_{1} X_{i}+\boldsymbol{T}_{c_{t 1}}=R_{C T} R_{T W} R_{\alpha} X_{i}+R_{C T} \boldsymbol{T}_{\boldsymbol{W}}+\boldsymbol{T}_{\boldsymbol{T}} \tag{4.6}
\end{equation*}
$$

where

- $X_{i}$ : Points with respect to the coordinate system $O_{w} X_{w} Y_{w} Z_{w}$.
- $R_{T W}$ : Transformation matrix from the coordinate system $O_{w} X_{w} Y_{w} Z_{w}$ to the rotating table coordinate system $O_{T} X_{T} Y_{T} Z_{T}$.
- $R_{C T}$ : Transformation matrix from the rotating table coordinate system $O_{T} X_{T} Y_{T} Z_{T}$ to the CCS.
- $R_{C W_{t 0}}, R_{C W_{t 1}}$ : Transformation matrix at time $t_{0}$ and $t_{1}$ from the coordinate system $O_{w} X_{w} Y_{w} Z_{w}$ to the CCS obtained by the Algorithm 2.
- $\boldsymbol{T}_{c_{t 0}}, \boldsymbol{T}_{c_{t 1}}$ : Transformation vectors at time $t_{0}$ and $t_{1}$ with respect to the camera coordinate system obtained by the Algorithm 2.
- $\boldsymbol{T}_{W}$ : Translation vectors with respect to the rotating table coordinate system $O_{T} X_{T} Y_{T} Z_{T}$.
- $\boldsymbol{T}_{\boldsymbol{T}}$ : Translation vectors with respect to the CCS.


Figure 8. Testing schema using rotation table
From the equation (4.5) and (4.6) it is seen that

$$
\begin{gathered}
R_{C W} 0 \\
R_{C W_{t 1}}=R_{C T} R_{T W} R_{T W} R_{\alpha}
\end{gathered}
$$

From the previous equations it is possible to obtain a direct formula to estimate the rotation matrix $R_{\alpha}$ (intrinsic rotation) with respect to $O_{w} X_{w} Y_{w} Z_{w}$, see the next equation

$$
\begin{equation*}
R_{\alpha}=R_{C W_{t 0}}{ }^{-1} R_{C W_{t 1}} \tag{4.7}
\end{equation*}
$$

The equation (4.7) can be rewritten using quaternions:

$$
\begin{equation*}
\Lambda_{\alpha}=\Lambda_{C W}{ }^{-1} \circ \Lambda_{C W_{t 1}} \tag{4.8}
\end{equation*}
$$

As it can be noticed in the previous equation, $R_{\alpha}$ depends on two consecutives rotations of coordinate system $O_{w} X_{w} Y_{w} Z_{w}$ which can be expressed as a function of
the unit quaternion $\Lambda_{C W}$. The rotation quaternion $\Lambda_{C W}$ can be obtained by means of Algorithm 2 considering extrinsic parameters, orientation and the translation vector only. The intrinsic parameters, on the other hand, are not included in the parameters because they already have been determined during the calibration and they remain fixed.

It is important to mention that $\Lambda_{\alpha}$ represents the local rotation with respect to BF for period of time $\Delta t=t 1-t 0$, and this rotation quaternion $\Lambda_{\alpha}$ is composed by the rotation angle $\alpha$ and its rotation axis $\boldsymbol{n}$ as shown below:

$$
\Lambda_{\alpha}=\left[\begin{array}{c}
\lambda_{0 \alpha}  \tag{4.9}\\
\lambda_{\alpha}
\end{array}\right]=\left[\begin{array}{c}
\cos \frac{\alpha}{2} \\
\boldsymbol{n} \sin \frac{\alpha}{2}
\end{array}\right]
$$

From the previous equation $\alpha$ and $\boldsymbol{n}$ are obtained by means of the next expressions

$$
\begin{gather*}
\alpha=2 \operatorname{arcos}\left(\lambda_{0 \alpha}\right)  \tag{4.10}\\
\boldsymbol{n}=\lambda_{\alpha} \sin (\alpha / 2)
\end{gather*}
$$

which allow the instantaneous angular velocity to be calculated from two consecutive rotation for period of time $\Delta t$ as follows:

$$
\begin{equation*}
\boldsymbol{w}_{\boldsymbol{r e l}}=\frac{\alpha \boldsymbol{n}}{\Delta t} \tag{4.11}
\end{equation*}
$$

where $\boldsymbol{w}_{\text {rel }}$ represents the angular velocity of the rigid body with respect to the BF .
A testing with the rotating table which consists of three rotations of $1^{\circ}, 2^{\circ}$ and $3^{\circ}$ around axis $-Y_{T}$ is performed, then by using the equation (4.10) the measurement model accuracy is shown. In the Figure 9 it is noticed that the mean $(\mu)$ of consecutively rotations is very close to the true angle $\alpha$ with small standard deviation $(\sigma)$.


Figure 9. Detections for three rotations of $1^{\circ}, 2^{\circ}$ and $3^{\circ}$ using the intrinsic parameters
Another testing is performed in order to know if it is possible to detect very small rotation angles such as $1 \operatorname{arcmin}\left(0.0167^{\circ}\right), 5 \operatorname{arcmin}\left(0.0833^{\circ}\right)$ and $15 \operatorname{arcmin}$ $\left(0.25^{\circ}\right)$ using low resolution camera. As it can be seen in the Figure 10, the accuracy it is less as the rotation angle is smaller. On the other hand, the precision $(\sigma)$ is still maintained.


Figure 10. Detection of three rotations $1 \operatorname{arcmin}\left(0.0167^{\circ}\right), 5 \operatorname{arcmin}\left(0.0833^{\circ}\right)$ and $15 \operatorname{arcmin}\left(0.25^{\circ}\right)$ using the intrinsic parameters

Until this moment the testing has been performed using chessboard where a remarkable amount of points is provided. However, it is not possible to establish the correspondence between the point from the BF and the ICS automatically, this required the user support. It is very important that the program for image processing detects and localizes automatically and accurately the points of correspondence between the BF and the ICS, since the accuracy of the rotation matrix and translation vector depends on it.

It is shown in the Figure 11 that once the four points are detected and their position in the image is evaluated, it is impossible to determine which point is $\mathrm{P} 1, \mathrm{P} 2$, P3 or P4. Therefore, the correspondences are not possible to be determined.


Figure 11. Example where correspondences are not possible determined
To solve this problem a pattern between each point can be used in order to determine the correspondences. In order to do that the utilization of the Aruco pattern is considered [15], [16]. It helps to establish the correspondence between the point from the BF and the ICS as it can be seen below.


Figure 12. Correspondences determined by using Aruco patterns
In this experiment the correspondences are established automatically using the Aruco library. As it is understood, the measurement model's error is inherent, and in addition to that error, another source of errors appears such as: the error produced by change of brightness in the enviroment, by the digitization of the image, and by the algorithm for corner detection.

In the Figure 13 it is shown how the used Aruco pattern is installed on the rotating table


Figure 13. Aruco pattern and rotating table
In the Figure 14 it is shown how the location of a detected corner change for each image with the rotating table being static. The located corner present in coordinates x (pixel) and y (pixel) maximum standard deviation 0.11 and 0.12 respectively, the effects of this deviation are reflected in the precision of the rotation angle.


Figure 14. Mean and standard deviation (STD) of the point X1

In the Figure $15(\mathrm{a})$ it is shown that the rotation angle has a mean value of $158.77^{\circ}$, and the standard deviaton $(\sigma)$ equals to $0.156^{\circ}$. The distance showed in the Figure $15(\mathrm{~b})$ represent the modulus of the translation vector $\boldsymbol{T}_{\boldsymbol{C}}$.


Figure 15. (a) Estimation of initial angle position with the rotating table being static. (b) Estimated distance with the rotating table being static.
Another experiment has been peformed where the rotating table rotates $90^{\circ}$ around the axis-Z. In the Figure 16, it is seen that, as it is expected, the estimated rotated angle is close to $90^{\circ}$. Additionally, the featuring of some peaks are seen, which appers due to the corner detecter's errors.


Figure 16. Estimated angle position with a rotation of $90^{\circ}$
The next experiment is focused on the local angular velocity calculation by using the camera FI8918W, previously calibrated in sub-section 4.4 while the rotating table rotates around the axis-Z.


Figure 17. Angular velocity ( $\%$ s)
The camera captures the object's movement by taking photographs every period of time $\Delta t$ where $\Delta t=1 / 15$ seconds, and at every object's movement sample the rotation quaternion is obtained by means of Algorithm 2, then by using the equation (4.11) the local angular velocity can be calculated as shown in the Figure 17.

As it can be observed in the Figure 17 the angular velocity calculation is strongly imprecise, its standard deviation $\sigma$ can reach $4.47^{\circ} / \mathrm{s}$, therefore to apply adavanced technique for improving the angular velocity precision is to be recommended.

## Conclusion

This work is dedicated to the problem of estimating the orientation of an object and its angular velocity by image processing. Two different approches were considered: the orientation determination by means of the measurement model adapted for the use of quaternions, in addition the angular velocity calculation.

As result of quaternions use, the simulations showed that there is no difference with respect to the precision with its analog adapted measurement model for the Rodrigues rotation formula. However, using measurement model based on quaternions is a slight advantage in computing time.

Experiments for rotation matrix determination by means of quaternions showed high precision. However, the estimation of the angular velocity from consecutives rotation matrix has low precision. Thus, in order to improve the precision of the angular velocity measurement additional technique has to be implemented.

The good results obtained for orientation quaternions determination allow that this work can be easily integrated to another systems based on quaterions.

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