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ABSTRACT

We concern ourselves with the hypothesis that two variables have a perfect disattenuated correlation, hence measure the same trait except for errors of measurement. This hypothesis is equivalent to saying, within the adopted model, that true scores of two psychological tests satisfy a linear relation. Statistical tests of this hypothesis are derived when the relation is specified with the exception of the additive constant. Two approaches are presented and various assumptions concerning the error parameters are used. Then the results are reinterpreted in terms of the possible existence of an unspecified linear relation between true scores of two psychological tests. A numerical example is appended by way of illustration. (Author)

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TESTING A LINEAR RELATION BETWEEN TRUE SCORES OF TWO MEASURES

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TESTING A LINEAR RELATION BETWEEN TRUE SCORES OF TWO MEASURES

Walter Kristof

Summary

We concern ourselves with the hypothesis that two variables have a perfect disattenuated correlation, hence measure the same trait except for errors of measurement. This hypothesis is equivalent to saying, within the adopted model, that true scores of two psychological tests satisfy a linear relation. Statistical tests of this hypothesis are derived when the relation is specified with the exception of the additive constant. Two approaches are presented and various assumptions concerning the error parameters are used. Then the results are reinterpreted in terms of the possible existence of an unspecified linear relation between true scores of two psychological tests. A numerical example is appended by way of illustration.

TESTING A LINEAR RELATION BETWEEN TRUE SCORES OF TWO MEASURES[1]

1. Introduction

Let $X = T + E$ and $Y = U + F$ be two random variables made up of true scores T, U and errors of measurement E, F. Suppose there is a linear relation between T and U. Then X and Y may be viewed as measuring the same dimension, but each individual measurement will be disturbed by an error. This error will cause X and Y to be less than perfectly correlated although the disattenuated correlation will still be perfect.

The statistical problem of deciding whether a disattenuated correlation may be assumed to be perfect is of obvious practical significance. However, earlier techniques proposed for this purpose (Forsyth & Feldt, 1969, 1970; Lord, 1957; McNemar, 1958) have been recently assessed by Lord (1971) as "cumbersome, approximate, or flawed."

Lord (1971) suggested instead a procedure based on the construction of a confidence interval for the coefficients of a linear relation between true scores. This procedure is an adaptation of a result by Villegas (1964).

However, simple techniques for testing whether a disattenuated correlation is perfect can be obtained even when the statistical assumptions underlying Lord's suggestion are considerably relaxed. The derivation of such techniques is the main concern of this paper.

In particular, we plan to proceed as follows. (i) Statistical tests will be derived for the hypothesis that there is a linear relation between true scores of two measures. This relation will be specified with the exception of the additive constant. Two approaches will be presented, and various assumptions concerning the error parameters will be used. (ii) The results will be

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reinterpreted in terms of the possible existence of an unspecified linear relation between true scores. (iii) A numerical example will be given by way of illustration.

2. First Approach to Testing a Linear Relation Between True Scores

Let X, Y be observed scores, T, U true scores and E, F errors of measurement on two psychological tests. Suppose that each test has been divided into two parts with observed scores X_1, X_2 and Y_1, Y_2 and errors E_1, E_2 and F_1, F_2 We will write $E' = E_1 - E_2$ and $F' = F_1 - F_2$. Let the division of the tests be such that true scores on the parts of a given test may differ only by a constant. We introduce the following four variables:

$$
\begin{aligned}
X &= X_1 + X_2 = T + E \\
X' &= X_1 - X_2 = a + E' \\
Y &= Y_1 + Y_2 = U + F \\
Y' &= Y_1 - Y_2 = b + F' \quad ,
\end{aligned}
\tag{1}
$$

a and b being constants.

The following assumptions concerning second moments of the errors will be made:

(i) $\sigma^2_{E_1} = \sigma^2_{E_2}$ and $\sigma^2_{F_1} = \sigma^2_{F_2}$

(ii) Variables E_1, E_2, F_1, F_2 are pairwise independent with the possible exception $\sigma_{E_1F_1} = \sigma_{E_2F_2} \neq 0$

(iii) Variables T, U are independent of variables E_1, E_2, F_1, F_2.

Assumptions (i), (ii), (iii) are equivalent to the following assumptions.

(i') $\sigma_E^2 = \sigma_{E'}^2$ and $\sigma_F^2 = \sigma_{F'}^2$

(ii') Variables E,E',F,F' are pairwise independent with the possible exception $\sigma_{EF} = \sigma_{E'F'} \neq 0$.

(iii') Variables T,U are independent of variables E,E',F,F' .

This set of assumptions is somewhat weaker than the usual set of assumptions about error second moments in classical test theory when repeated measurements have been obtained. For, $\sigma_{E_1F_1} = \sigma_{E_2F_2}$ and $\sigma_{EF} = \sigma_{E'F'}$ need not be zero.

The linear hypothesis we wish to test on the basis of a sample of observations is

$$(2) \qquad H_0 : \beta_1 T + \beta_2 U + \gamma = 0 \qquad (\beta_1 \neq 0)$$

with specified coefficients β_1 and β_2 . The additive constant γ remains unspecified.

Let us introduce the following new variables:

$$(3) \qquad \begin{aligned} Z &= \beta_1 X + \beta_2 Y = \beta_1 T + \beta_2 U + \beta_1 E + \beta_2 F \\ Z' &= \beta_1 X' + \beta_2 Y' = \beta_1 a + \beta_2 b + \beta_1 E' + \beta_2 F' \end{aligned} .$$

The variances become

$$(4) \qquad \begin{aligned} \sigma_Z^2 &= \sigma_{\beta_1 T+\beta_2 U}^2 + \sigma_{\beta_1 E+\beta_2 F}^2 \\ \sigma_{Z'}^2 &= \sigma_{\beta_1 E+\beta_2 F}^2 \end{aligned} .$$

In addition, Z and Z' are independent regardless of β_1, β_2 .

Hence we have

$$\frac{\sigma_Z^2}{\sigma_{Z'}^2} = 1 + \frac{\sigma^2_{\beta_1 T+\beta_2 U}}{\sigma^2_{\beta_1 E+\beta_2 F}} \quad . \tag{5}$$

Precisely when H_0 is correct this reduces to

$$\frac{\sigma_Z^2}{\sigma_{Z'}^2} = 1 \quad . \tag{6}$$

This result enables us at once to devise the desired statistical test.

The details of this test will depend on additional assumptions that we are willing to make. We will consider two cases.

Case I: Suppose that each test has been split such that true scores on the parts of a given test are equal. This implies $a = b = 0$. Suppose further that expected errors on the two parts of a given test are equal. This implies expectations zero for E' and F' . Hence Z' will have expectation zero.

Case II: Suppose that a and/or b need not be zero. Suppose further that expected errors on the two parts of a given test need not be equal. This implies possibly nonzero expectations for E' and F' . Hence Z' need not have expectation zero.

It will be assumed in each case that errors are multinormally distributed. However, no distributional assumptions concerning T and U need be made under H_0 . Then, considering (6) and the independence of Z and Z' under H_0 , we obtain at once the following results when a random sample of size N is given.

Case I: Let s_Z^2 be the observed sample variance of Z about the sample mean with df = N - 1 . Let $s_{Z'}^{*2}$ be the observed sample variance of Z' about zero with df = N . Then, under H_0 , the ratio

$$(7) \qquad F = \frac{s_Z^2}{s_{Z'}^{*2}}$$

follows an F distribution with $df_1 = N - 1$ and $df_2 = N$. This provides the desired test.

It will be convenient to express (7) in terms of the original observed second moments. Let $\underline{V}_s$ be the observed sample variance-covariance matrix of total scores $X = X_1 + X_2$ and $Y = Y_1 + Y_2$ (taken about means with df = N - 1). Let $\underline{V}_d^*$ be the observed sample variance-covariance matrix of difference scores $X' = X_1 - X_2$ and $Y' = Y_1 - Y_2$ (taken about zero with df = N). Introduce the row vector $\underline{\beta}' = (\beta_1, \beta_2)$. Then (7) becomes

$$(8) \qquad F = \frac{\underline{\beta}'\underline{V}_s\underline{\beta}}{\underline{\beta}'\underline{V}_d^*\underline{\beta}} \quad .$$

H_0 will be accepted if this ratio does not exceed the critical value F_α of the F distribution with $df_1 = N - 1$, $df_2 = N$ when the chosen level of significance is α . Otherwise H_0 will be rejected. We may also say that acceptance of H_0 is implied precisely when

$$(9) \qquad \underline{\beta}'(\underline{V}_s - F_\alpha \underline{V}_d^*)\underline{\beta} \leq 0 \quad .$$

Case II: The previous definitions of s_Z^2 and $\underline{V}_s$ are retained. However, let $s_{Z'}^2$ be the observed sample variance of Z' about the sample mean with df = N - 1 . Then, under H_0 , the ratio

(10) $$F = \frac{s_Z^2}{s_{Z'}^2}$$

follows an F distribution with $df_1 = df_2 = N - 1$. This provides the desired test.

Let $\underline{V}_d$ be the observed sample variance-covariance matrix of difference scores $X' = X_1 - X_2$ and $Y' = Y_1 - Y_2$ (taken about means with $df = N - 1$). Then (10) becomes

(11) $$F = \frac{\underline{\beta}'\underline{V}_s\underline{\beta}}{\underline{\beta}'\underline{V}_d\underline{\beta}} ,$$

vector $\underline{\beta}$ as defined before. H_0 will be accepted if this ratio does not exceed the critical value F_α of the F distribution with $df_1 = df_2 = N - 1$ when the chosen level of significance is α. Otherwise H_0 will be rejected. We may also say that acceptance of H_0 is implied precisely when

(12) $$\underline{\beta}'(\underline{V}_s - F_\alpha \underline{V}_d)\underline{\beta} \leq 0 .$$

Case II is obviously more general than case I. Any set of data satisfying the assumptions of case I may be subjected to the test developed for case II. The corresponding loss of power will be quite negligible unless N is very small.

Remarks: It has been assumed that each of two psychological tests can be split into two parts. It will be recognized that the statistical developments apply as well if two forms of each test are available. The assumptions of case I are met if two corresponding forms are parallel. We are dealing with case II if the equality of means requirement is dropped. Equality of variances of two corresponding forms is still postulated, however.

It is seen that the previous (and following) developments can be generalized. Tests of one linear relation between true scores of more than two psychological tests are obtained if the pattern of the previous derivations is followed. However, the existence of one linear relation alone would then not imply that each two tests have a perfect disattenuated correlation. In fact, true scores of p tests must obey $p - 1$ independent linear relations in order that this property may hold.

The testing procedure suggested by Lord (1971) and adapted from a result by Villegas (1964) presupposes that true scores on the two parts of a given test are equal and that errors on the two parts have equal variances and, expressly, expectations zero. These assumptions are stronger than those of our case I. We might speak of a case 0 if we were to include the Lord/Villegas approach into our classification of cases.

3. Second Approach to Testing a Linear Relation Between True Scores

The approach presented in this section is a correlational one. It may be somewhat less direct but it will allow us to dispense in a very natural way with the previous equality of variances requirement of errors on the two parts of a psychological test (or the two equivalent forms). We will again limit ourselves to two psychological tests; the extension to $p > 2$ such tests is straightforward.

Observed scores on the two parts of a psychological test will again be signified by X_1, X_2 and Y_1, Y_2 and the associated errors of measurement by E_1, E_2 and F_1, F_2. True scores on the parts will be written as T_1, T_2 and U_1, U_2 with sums $T = T_1 + T_2$ and $U = U_1 + U_2$. Differences $T_1 - T_2$ and $U_1 - U_2$ are supposed to be constants. The previous assumptions concerning second moments of errors along with their normal distribution are retained. The hypothesis H_0 to be tested is again (2).

Let us now introduce the following new variables:

$$W = \beta_1X_1 + \beta_2Y_1 = \beta_1T_1 + \beta_2U_1 + \beta_1E_1 + \beta_2F_1 \tag{13}$$

$$W' = \beta_1X_2 + \beta_2Y_2 = \beta_1T_2 + \beta_2U_2 + \beta_1E_2 + \beta_2F_2 \quad .$$

Under H_0 the sums $\beta_1T_1 + \beta_2U_1$ and $\beta_2T_2 + \beta_2U_2$ are constants. Variables W and W' are independent precisely when H_0 is true. However, if H_0 is false, then W and W' will be positively correlated since

$$\sigma_{WW'} = \sigma_{\beta_1T_1+\beta_2U_1,\beta_1T_2+\beta_2U_2} = \frac{1}{4}\,\sigma^2_{\beta_1T+\beta_2U} \quad . \tag{14}$$

Testing H_0 will amount to testing whether a population correlation coefficient may be assumed to be zero. This is the basic result of the present approach.

We are in a position to rederive the previous testing techniques. This is not surprising at all because the previous variables (3) and the present variables (13) are linear transforms of each other: $W = (Z + Z')/2$ and $W' = (Z - Z')/2$.

Case I: Under H_0 variables W and W' have equal expectations and equal variances. Hence the statistical problem consists in testing whether the intraclass correlation between W and W' may be assumed to be zero.

A test of this hypothesis can be written down at once when we follow Scheffé (1959, pp. 223-227). His SS_A is $(N - 1)/2$ times the observed sample variance of $W + W'$ taken about the sample mean with $df = N - 1$. Taking (13) into account we derive $SS_A = \underline{\beta}'\underline{V}_s\underline{\beta}(N - 1)/2$. Similarly, Scheffé's SS_e becomes $\underline{\beta}'\underline{V}_d^*\underline{\beta}N/2$. This yields exactly the F test as given in (8).

Case II: Under H_0 variables W and W' have equal variances but not necessarily equal expectations. Hence we have to test whether W and W' may be uncorrelated when we know that $\sigma_W^2 = \sigma_{W'}^2$.

To this end we introduce the quantity

$$(15) \qquad u = \frac{2s_{WW'}}{s_{WW} + s_{W'W'}}$$

where s_{WW} , $s_{WW'}$ and $s_{W'W'}$ denote the usual observed sample second moments of W and W' taken about sample means (N - 1 in denominators). Quantity u is the maximum-likelihood estimator of the correlation between W and W' when these variables are binormally distributed and $\sigma_W^2 = \sigma_{W'}^2$. Under H_0 the ratio

$$(16) \qquad F = \frac{1 + u}{1 - u}$$

follows an F distribution with $df_1 = df_2 = N - 1$ (Kristof, in press). Taking (13) into account it is seen that (16) is identical to the F test (11). One-sidedness of this F test reflects the fact that covariance (14) is non-negative when H_0 is false.

The present correlational approach enables us at once to drop the earlier assumptions $\sigma_{E_1}^2 = \sigma_{E_2}^2$, $\sigma_{F_1}^2 = \sigma_{F_2}^2$ and $\sigma_{E_1F_1} = \sigma_{E_2F_2}$. This new situation will be named

Case III: Under H_0 variables W and W' need not have equal expectations nor equal variances. The problem reduces to a very familiar one: We have to test whether W and W' may be uncorrelated when no restrictions are imposed on means and variances.

Obviously H_0 will be retained if the sample correlation coefficient $r_{WW'}$ does not exceed a certain upper limit. Such limits corresponding to various levels of significance are easily calculated from the t distribution with df = N - 2 . It should be borne in mind that (14) requires us to employ a one-sided statistical test.

Let the sample variance-covariance matrix $\underline{V}$ of the partitioned vector $(X_1, Y_1 \vdots X_2 . Y_2)$ be partitioned accordingly:

(17) $$\underline{V} = \begin{pmatrix} \underline{V}_{11} & \vdots & \underline{V}_{12} \\ \cdots & & \cdots \\ \underline{V}_{12}' & \vdots & \underline{V}_{22} \end{pmatrix}$$

$\underline{V}_{11}$ and $\underline{V}_{22}$ positive definite. We may now write

(18) $$r_{WW'} = \frac{\underline{\beta}'\underline{V}_{12}\underline{\beta}}{\sqrt{\underline{\beta}'\underline{V}_{11}\underline{\beta}\underline{\beta}'\underline{V}_{22}\underline{\beta}}} \quad .$$

Knowledge of this quantity enables us to carry out the test of H_0 .

$\underline{V}_{12}$ cannot be expected to be symmetric. However, in (18) we may replace $\underline{V}_{12}$ by the symmetric matrix $(\underline{V}_{12} + \underline{V}_{12}')/2$ without altering the numerical value of $r_{WW'}$. In fact, $2\underline{\beta}'\underline{V}_{12}\underline{\beta} \equiv \underline{\beta}'(\underline{V}_{12} + \underline{V}_{12}')\underline{\beta}$.

4. Testing an Unspecified Linear Relation Between True Scores

The previous results enable us to test hypotheses about vector $\underline{\beta}$ when the existence of a linear relation between true scores of two psychological tests is assumed. Depending upon additional assumptions, formulas (8), (11) or (18) may be used. Conversely, these formulas can be utilized in constructing "confidence intervals" for $\underline{\beta}$. It is obvious that $\underline{\beta}$ can be replaced by any nonzero multiple of it.

However, a linear relation between true scores may not exist at all. This is equivalent to saying that the disattenuated correlation between two psychological tests is less than perfect. In fact, we may be primarily interested in testing whether there is <u>any</u> linear relation between true scores. The assumption of such an unspecified linear relation will be denoted by $\overline{H}_0$. It is our aim now to give statistical tests of $\overline{H}_0$.

<u>Cases I and II</u>: If $\overline{H}_0$ is correct, then there is (up to multiplication) precisely <u>one</u> nontrivial linear relation between true scores, $\beta_1 T + \beta_2 U + \gamma = 0$, say, with unknown values β_1, β_2 and γ . There cannot be another independent linear relation if T and U are to be variables and not constants. Hence, in order to test $\overline{H}_0$, we would use (8) or (11), depending on the case, if $\underline{\beta}$ were known. Now, since $\underline{\beta}$ is unknown, let us find the quantity

$$\overline{F} = \min_{\underline{\beta} \neq \underline{0}} \frac{\underline{\beta}' \underline{V}_s \underline{\beta}}{\underline{\beta}' \underline{V}_d^* \underline{\beta}} \tag{19}$$

in case I and

$$\overline{F} = \min_{\underline{\beta} \neq \underline{0}} \frac{\underline{\beta}' \underline{V}_s \underline{\beta}}{\underline{\beta}' \underline{V}_d \underline{\beta}} \tag{20}$$

in case II when the minimum is taken over all vectors $\underline{\beta} \neq \underline{0}$. Let us agree to retain H_0 exactly when $\overline{F}$ does not exceed the upper critical value F_α of the F distribution with $df_1 = N - 1$, $df_2 = N$ in case I and $df_1 = df_2 = N - 1$ in case II. This will produce a conservative test of $\overline{H}_0$. If rejection of $\overline{H}_0$ occurs, then the true corresponding level $\overline{\alpha}$ will not exceed α , $\overline{\alpha} \leq \alpha$.

Calculation of $\overline{F}$ is not difficult. $\overline{F}$ is the smallest eigenvalue of $\underline{V}_s \underline{V}_d^{*-1}$ in case I and of $\underline{V}_s \underline{V}_d^{-1}$ in case II.

<u>Case III</u>: Considerations similar to those in cases I and II lead us to seek the minimum of $r_{WW'}$ when $\underline{\beta}$ varies over all nonzero vectors. This

minimum may be compared with the critical value (level α, df = N - 2) of a product-moment correlation under normality assumptions when the population correlation is zero. We have to use a one-sided test; only a sufficiently large positive value of the minimal $r_{WW'}$ will lead to rejection of $\bar{H}_0$. Again, the resulting test will be conservative.

According to (18) the minimum of $r_{WW'}$ may be written as

$$\bar{r} = \min_{\underline{\beta} \neq \underline{0}} \frac{\underline{\beta}'\underline{V}_{12}\underline{\beta}}{\sqrt{\underline{\beta}'\underline{V}_{11}\underline{\beta}\underline{\beta}'\underline{V}_{22}\underline{\beta}}} \quad . \tag{21}$$

We will think of $\underline{V}_{12}$ as a symmetric matrix; it has already been remarked that substitution of $(\underline{V}_{12} + \underline{V}'_{12})/2$ for $\underline{V}_{12}$ is allowable. We will further suppose that $\underline{V}_{12}$ is positive definite. Otherwise $\bar{r}$ would not be a positive quantity and rejection of $\bar{H}_0$ would not be possible at any level α.

In order to determine $\bar{r}$ we may proceed as follows.[2] Let $\underline{V}_{12} = \underline{P\Delta P}'$ with $\underline{P}$ orthogonal and $\underline{\Delta}$ positive diagonal and define

$$\begin{aligned} \underline{\xi} &= \underline{\Delta}^{\frac{1}{2}}\underline{P}'\underline{\beta}/\sqrt{\underline{\beta}'\underline{P\Delta P}'\underline{\beta}} \quad , \qquad |\underline{\xi}| = 1 \\ \underline{A} &= \underline{\Delta}^{-\frac{1}{2}}\underline{P}'\underline{V}_{11}\underline{P\Delta}^{-\frac{1}{2}} = \|a_{ij}\| \\ \underline{B} &= \underline{\Delta}^{-\frac{1}{2}}\underline{P}'\underline{V}_{22}\underline{P\Delta}^{-\frac{1}{2}} = \|b_{ij}\| \quad . \end{aligned} \tag{22}$$

We derive that

$$(\bar{r})^{-2} = \max_{|\underline{\xi}|=1} \underline{\xi}'\underline{A\xi}\underline{\xi}'\underline{B\xi} \quad . \tag{23}$$

[2]This procedure was developed in cooperation with Bary G. Wingersky of ETS. A program is available from him.

Writing $\underline{\xi}' = (\cos \phi , \sin \phi)$ we seek maximization of the function $\eta(\phi) = \underline{\xi}'\underline{A\xi}\underline{\xi}'\underline{B\xi}$ under variation of ϕ . The condition $d\eta/d\phi = 0$ becomes

(24) $g_4\cos^4\phi + g_3\cos^3\phi \sin \phi + g_2\cos^2\phi \sin^2\phi + g_1\cos \phi \sin^3\phi + g_0\sin^4\phi = 0$

with coefficients

(25)
$$
\begin{aligned}
g_4 &= a_{11}b_{12} + a_{12}b_{11} \\
g_3 &= a_{11}(b_{22} - b_{11}) + b_{11}(a_{22} - a_{11}) + 4a_{12}b_{12} \\
g_2 &= 3[a_{12}(b_{22} - b_{11}) + b_{12}(a_{22} - a_{11})] \\
g_1 &= a_{22}(b_{22} - b_{11}) + b_{22}(a_{22} - a_{11}) - 4a_{12}b_{12} \\
g_0 &= -a_{12}b_{22} - a_{22}b_{12} \quad .
\end{aligned}
$$

In (24) ϕ is no longer a variable; however, it is convenient to retain the same symbol.

We see that $\sin \phi = 0$ is a solution of (24) precisely when $g_4 = 0$. It suffices to solve the quartic

(26) $$g(\phi) \equiv g_4\text{ctn}^4\phi + g_3\text{ctn}^3\phi + g_2\text{ctn}^2\phi + g_1\text{ctn}\,\phi + g_0 = 0$$

which reduces to a cubic precisely when $g_4 = 0$. Hence there will always be four solutions of which at least two must be real because $\eta(\phi)$ certainly assumes a maximum and a minimum.

5. Application

Lord (1957) reported the following observed variance-covariance matrix for $N = 649$:

$$\begin{pmatrix} 86.3979 & 57.7751 & 56.8651 & 58.8986 \\ 57.7751 & 86.2632 & 59.3177 & 59.6683 \\ 56.8651 & 59.3177 & 97.2850 & 73.8201 \\ 58.8986 & 59.6683 & 73.8201 & 97.8192 \end{pmatrix}$$

The first two variables represent parallel halves of a vocabulary test administered under very liberal time limits. The last two variables represent parallel halves of a vocabulary test constructed so as to be as nearly equivalent as possible to the halves of the first test except that the time of administration was so short that only two per cent of the examinees completed each half.

We wish to test the hypothesis that the two tests measure the same trait except for errors of measurement regardless of speed conditions.

It appears that the data should satisfy the assumptions of case I. However, matrix $\underline{V}_d^*$ cannot be reconstructed from the available data. We will therefore proceed according to case II and, for reasons of comparison, also according to case III. We expect that the two methods should yield nearly identical results because of the parallelism of corresponding test halves.

<u>Case II</u>: From the given data we find $\underline{V}_s$ and $\underline{V}_d$ to be

$$\underline{V}_s = \begin{pmatrix} 288.2113 & 234.7497 \\ 234.7497 & 342.7444 \end{pmatrix}, \qquad \underline{V}_d = \begin{pmatrix} 57.1109 & -1.6829 \\ -1.6829 & 47.4640 \end{pmatrix} .$$

Equation (21) requires us to determine $\overline{F}$, the smallest eigenvalue of $\underline{V}_s\underline{V}_d^{-1}$. It is found that $\overline{F} = 1.45017$. With $df_1 = df_2 = 648$ this value is significant at a level $\alpha < .01$. We conclude that the trait captured by vocabulary tests depends on speed conditions.

The significance level can also be determined by means of a t table if we use a result obtained by Cacoullos (1965) and later independently by Kristof (in press). This result states that $t = \sqrt{\nu}\ (\sqrt{F} - 1/\sqrt{F})/2$ follows a t distribution with $df = \nu$ when F follows an F distribution with $df_1 = df_2 = \nu$. Above $\bar{F}$ corresponds to $\bar{t} = 4.76$ with $df = 648$. Considering that a one-sided t test is required we determine that $\alpha < .0005$.

Case III: We find from the initial data that

$$\underline{V}_{11} = \begin{pmatrix} 86.3979 & 56.8651 \\ 56.8651 & 97.2850 \end{pmatrix}, \qquad \underline{V}_{22} = \begin{pmatrix} 86.2632 & 59.6683 \\ 59.6683 & 97.8192 \end{pmatrix},$$

and, after symmetrization,

$$\underline{V}_{12} = \begin{pmatrix} 57.7751 & 65.7976 \\ 65.7976 & 73.8201 \end{pmatrix}.$$

Quartic (26) was found to have two real and two complex roots. The minimal correlation $\bar{r}$ as required by (21) became .1839. We convert this into $\bar{t} = \bar{r}\sqrt{N-2} / \sqrt{1-\bar{r}^2}$ and obtain $\bar{t} = 4.76$. This value is to be interpreted as a one-sided t with $df = 647$. We see that it is identical to the value of $\bar{t}$ in case II. The difference in degrees of freedom is immaterial. Hence the two methods gave the same result as expected for the present example.

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