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TESTING AND ESTIMATION FOR A CIRCULAR STATIONARY MODEL[1]

BY I. OLKIN AND S. J. PRESS

Stanford University and University of Chicago

1. Introduction. Tests of hypotheses for the means of a p-variate normal distribution, given that the covariance matrix $\Sigma = (\sigma_{ij})$ has some special structure, or that the covariance matrix has a special structure, have been considered in the literature. Some of the covariance structures that have received attention are:

$$H_1: \Sigma = \Sigma_0, \quad \text{a known matrix}, \qquad H_2: \Sigma = \sigma^2 \Sigma_0,$$

$$H_3: \sigma_{ii} = \sigma^2, \quad \sigma_{ij} = \sigma^2 \rho, \quad (i \neq j), \qquad H_4: \Sigma \text{ unrestricted}.$$

The test of H_2 versus H_4, the sphericity test, was considered by Mauchly (1940), Girshick (1941), and more recently by Gleser (1966). The test of H_3 versus H_4, and the test for the equality of means when H_3 holds, form the basis of the hypotheses of Wilks (1946). An analog to Hotelling's test that the means are zero when H_3 holds has been considered by Geisser (1963).

Alternative models which have been considered are those for which the covariance kernel is weakly stationary, i.e., $\sigma_{ij} = \sigma_{|i-j|}$, or which arise from stochastic difference equations, e.g., $\sigma_{ij} = \sigma^2 \rho^{|i-j|}$. The model considered here combines a circular symmetry condition with weak stationarity. Although the genesis of the model stems from a physical situation (described below), there are other applications, e.g., in time series, for which the model is suitable.

Consider a point source located at the geocenter of a regular polygon of p sides, from which a signal is transmitted. Identical signal receivers (with identical noise characteristics) are positioned at the p vertices, denoted sequentially by $V_1, \cdots, V_p$. The signal received at vertex V_i is denoted by x_i. The main assumption is that the signal strength is the same in all directions, and that covariances depend only on the number of vertices separating the two receivers, so that

$$\text{(1.1)} \quad \operatorname{Var}(x_i) = \sigma_0^2, \qquad \operatorname{Cov}(x_i, x_{i+k}) = \sigma_0^2 \rho_k, \qquad \rho_k = \rho_{p-k},$$
$$i = 1, \cdots, p; \qquad k = 1, \cdots, p-1; \qquad 2 \leqq i + k \leqq p.$$

If the points $V_1, \cdots, V_p$ were on a line, (1.1) would be equivalent to stationarity, but because V_p is adjacent to V_1, a circularity is introduced.

Specifically, the assumptions are that $x = (x_1, \cdots, x_p)$ is a random vector having a p-variate normal distribution with mean vector $\mu = (\mu_1, \cdots, \mu_p)$ and covariances given by (1.1). The covariance matrices differ in the middle terms

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for p even or odd. For example, for $p = 4$ and $p = 5$, we have the structure

$$\sigma_0^2 \begin{bmatrix} 1 & \rho_1 & \rho_2 & \rho_1 \\ \rho_1 & 1 & \rho_1 & \rho_2 \\ \rho_2 & \rho_1 & 1 & \rho_1 \\ \rho_1 & \rho_2 & \rho_1 & 1 \end{bmatrix}, \quad \sigma_0^2 \begin{bmatrix} 1 & \rho_1 & \rho_2 & \rho_2 & \rho_1 \\ \rho_1 & 1 & \rho_1 & \rho_2 & \rho_2 \\ \rho_2 & \rho_1 & 1 & \rho_1 & \rho_2 \\ \rho_2 & \rho_2 & \rho_1 & 1 & \rho_1 \\ \rho_1 & \rho_2 & \rho_2 & \rho_1 & 1 \end{bmatrix}.$$

Such covariance matrices have the property of being symmetric and cyclic, and for brevity we refer to them as circular.

The circularity condition may be introduced from temporal rather than spatial considerations. This leads to a stochastic process in which the covariance kernel is Cov $(x_s, x_{s+L}) = \sigma^2\rho_L$, where $\rho_{N+L} = \rho_{N-L} = \rho_L$. The stationary process which yields a Laurent covariance kernel Cov $(x_i, x_j) =$ Cov (x_{i+L}, x_{j+L}) has been studied in detail by Whittle (1951). Following his development, Wise (1955) considered the modification to a circular process for which Cov $(x_i, x_{i+L}) = \sigma^2\rho_L$, where $\rho_{N+L} = \rho_{N-L} = \rho_L$.

The concern in this paper is with: (i) tests for symmetries in the covariance matrix, and with (ii) tests of hypotheses for the means when the covariance matrix is circular. The particular symmetries of interest are $\rho_j = \rho_{p-j}$, the circular symmetry model; $\rho_1 = \cdots = \rho_{p-1} = \rho$, which is the intraclass correlation model; and $\rho_1 = \cdots = \rho_{p-1} = 0$, which is the spherical model. For each hypothesis we obtain the likelihood ratio test and the asymptotic distribution of the likelihood ratio statistic (LRS) under the hypothesis and alternative.

2. Canonical forms and estimation. We first review the notation used. Row vectors are generally denoted by lower case letters, matrices by capital letters. The dimensionality of a matrix is indicated by the symbol $A : r \times s$. By $A > 0$ we mean that the (symmetric) matrix A is positive definite. The special vector $(1, \cdots, 1)$ is denoted by e.

To emphasize that a matrix A is circular and therefore has the form

$$A = \begin{bmatrix} a_1 & a_2 & a_3 & \cdots & a_p \\ a_p & a_1 & a_2 & \cdots & a_{p-1} \\ & & & \cdots & \\ a_2 & a_3 & a_4 & \cdots & a_1 \end{bmatrix},$$

we sometimes write $A_C(a_1, \cdots, a_p)$ or simply A_C. Since circular matrices are basic to the ensuing analysis, several properties of such matrices are now reviewed. Whittle (1951) provides a general discussion of Laurent matrices, and specialization to the case when the circular matrix is positive definite (symmetric) is discussed by Wise (1955) and Press (1964).

Let $r_k = \exp 2\pi i(k-1)/p$ for $k = 1, \cdots, p$ denote the p roots of unity. Then

$$\tau_k = a_1 r_k^0 + \cdots + a_p r_k^{p-1}$$

are the eigenvalues of $A_C(a_1, \cdots, a_p)$. If A_C is symmetric, then $a_{p-j+2} = a_j$, $(j = 2, \cdots, p)$. In this case the eigenvalues are real and are given by

$$(2.1) \qquad \tau_k = \sum_{j=1}^p a_j \cos 2\pi p^{-1}(k-1)(p-j+1), \qquad k = 1, \cdots, p.$$

If

$$(2.2) \qquad \gamma_{jk} = p^{-\frac{1}{2}}[\cos 2\pi p^{-1}(j-1)(k-1) + \sin 2\pi p^{-1}(j-1)(k-1)],$$

then $\Gamma = (\gamma_{jk})$ is orthogonal and transforms A_C to diagonal form, namely, $A_C = \Gamma D_\tau \Gamma'$; where $D_\tau = \operatorname{diag}(\tau_1, \cdots, \tau_p)$. A critical point is the fact that this diagonalization is achieved by an orthogonal matrix whose elements are constants, independent of the elements of A_C. (Note that the first row (and column) of Γ is $p^{-\frac{1}{2}}e$ and that Γ is symmetric.)

The condition of positive definiteness, namely, $\tau_j > 0$, is equivalent to linear constraints in the elements a_j, as given by (2.1). From (2.1), with $a_j = a_{p-j+2}$, $j = 2, \cdots, p$, and from trigonometric identities, it follows that $\tau_j = \tau_{p-j+2}$, $j = 2, \cdots, p$.

2.1. *Canonical form for circular model.* Given a sample $(x_{1\alpha}, \cdots, x_{p\alpha})$, $\alpha = 1, \cdots, N$, of size N from a p-variate $N(\mu, \Sigma)$ distribution, we may (by sufficiency) consider the mean vector $\bar{x} = (\bar{x}_1, \cdots, \bar{x}_p)$ and covariance matrix S/n, $S = (s_{ij})$, $s_{ij} = \sum_{\alpha=1}^N (x_{i\alpha} - \bar{x}_i)(x_{j\alpha} - \bar{x}_j)$, $n = N - 1$, as our starting point. Denoting by $\mathcal{L}(Z)$ the law of the random matrix (or vector) Z, we note that $\bar{x}$ and S are independently distributed, with

$$\mathcal{L}(N^{\frac{1}{2}}\bar{x}) = N(N^{\frac{1}{2}}\mu, \Sigma), \qquad \mathcal{L}(S) = W(\Sigma, p; n),$$

i.e., S has density function

$$p(S) = c(p, n)|\Sigma|^{-n/2}|S|^{(n-p-1)/2}e^{-\frac{1}{2}\operatorname{tr}\Sigma^{-1}S}, \qquad S > 0, \Sigma > 0,$$
$$c(p, n) = 2^{-pn/2}\pi^{-p(p-1)/4}\left[\prod_1^p \Gamma(\tfrac{1}{2}(n-i+1))\right]^{-1}.$$

Making the transformations $y = N^{\frac{1}{2}}\bar{x}\Gamma$, $V = \Gamma' S \Gamma$, where Γ is defined by (2.2), we have that y and $V \equiv (v_{ij})$ are independently distributed with

$$(2.3) \qquad \mathcal{L}(y) = N(\eta, \bar{\Sigma}), \qquad \mathcal{L}(V) = W(\bar{\Sigma}, p; n),$$
$$\eta = N^{\frac{1}{2}}\mu\Gamma, \qquad \bar{\Sigma} = \Gamma'\Sigma\Gamma.$$

When Σ is circular, $\bar{\Sigma} = D_\tau \equiv \operatorname{diag}(\tau_1, \cdots, \tau_p)$, $\tau_j = \tau_{p-j+2}$, $j = 2, \cdots, p$, and

$$(2.4) \qquad \mathcal{L}(y) = N(\eta, D_\tau), \qquad \mathcal{L}(V) = W(D_\tau, p; n).$$

Because $\tau_j = \tau_{p-j+2}$, $j = 2, \cdots, p$, we can achieve a further reduction by noting that the minimal sufficient statistic for $(\eta, \tau_1, \cdots, \tau_p)$ is given by

$$(2.5\text{a}) \qquad (y, v_1, v_2, \cdots, v_m, v_{m+1})$$
$$\equiv (y, v_{11}, v_{22} + v_{pp}, \cdots, v_{mm} + v_{m+2,m+2}, v_{m+1,m+1}),$$

$$(2.5b) \qquad \equiv (y, v_{11}, v_{22} + v_{pp}, \cdots, v_{m+1,m+1} + v_{m+2,m+2}),$$

when $p = 2m$ and $p = 2m + 1$, respectively. For later use define $v_j = v_{p-j+2}$, $j = 2, \cdots, p$.

Since the y's and v's are independently distributed when Σ is circular, we have as a canonical model:

$$(2.6) \qquad \mathcal{L}(y_1) = N(\eta_1, \tau_1), \ \mathcal{L}(y_j) = N(\eta_j, \tau_j), \ \tau_j = \tau_{p-j+2}, \ j = 2, \cdots, p;$$

$$(2.7a) \quad \mathcal{L}(v_1/\tau_1) = \chi_n^2, \quad \mathcal{L}(v_k/\tau_k) = \chi_{2n}^2, \quad k = 2, \cdots, m, \quad \mathcal{L}(v_{m+1}/\tau_{m+1}) = \chi_n^2,$$

when $p = 2m$,

$$(2.7b) \quad \mathcal{L}(v_1/\tau_1) = \chi_n^2, \qquad \mathcal{L}(v_k/\tau_k) = \chi_{2n}^2, \qquad k = 2, \cdots, m + 1,$$

when $p = 2m + 1$.

For each of the hypotheses considered the starting point is either (2.3) or (2.6), and (2.7), from which the LRT is obtained. In all the problems considered, the LRS is distributed as a product of independent beta variates when the hypothesis is true. Consequently, except for some special cases, it is not feasible to obtain exact distributions in closed form, and we obtain an asymptotic approximation accurate to $O(N^{-3})$. Under the alternative hypothesis, we use the delta method to obtain the asymptotic distribution of a suitably normalized function of the LRS.

2.2. *Estimation of parameters for the circular model.* For the model in which $(x_{1\alpha}, \cdots, x_{p\alpha})$, $\alpha = 1, \cdots, N$ is a sample of size N from $N(\mu, \Sigma)$, where Σ is circular with elements given by (1.1), we obtain MLE and confidence intervals for the mean μ, the common variance σ_0^2, and for the circular correlation coefficients $(\rho_1, \cdots, \rho_{p-1})$ where $\rho_j = \rho_{p-j}, j = 2, \cdots, p$.

From (2.6) and (2.7), the statistics $T_j \equiv (y_j - \eta_j)/v_j$ are distributed as $t_n/n^{\frac{1}{2}}$ or $t_{2n}/(2n)^{\frac{1}{2}}$ depending on whether $\mathcal{L}(v_j/\tau_j) = \chi_n^2$ or χ_{2n}^2. The statistics T_i and T_{p-i+2} are dependent and have a bivariate t-distribution (e.g., see Dunnett and Sobel (1954)) but are independent of all other statistics. Because $P\{T_i \, \varepsilon \, \mathcal{A}, T_{p-i+2} \, \varepsilon \, \mathcal{B}\} \geqq P\{T_i \, \varepsilon \, \mathcal{A}\}P\{T_{p-i+2} \, \varepsilon \, \mathcal{B}\}$, we obtain a conservative confidence region for $\eta_1, \cdots, \eta_p$ by choosing $P\{|y_j - \eta_j| \leqq v_j c_j\} = 1 - (1 - \alpha)^{1/p} \equiv 1 - \gamma, j = 1, \cdots, p$, where $c_j = t_n(\gamma)/n^{\frac{1}{2}}$ or $t_{2n}(\gamma)/(2n)^{\frac{1}{2}}$. A confidence region for μ is then easily found from $\mu = \eta\Gamma'/N^{\frac{1}{2}}$, where Γ is given by (2.2).

The MLE of σ_0^2 and ρ_j are obtained indirectly. From (2.7) it is clear that $\hat{\tau}_j = v_j/n$ or $v_j/2n$, depending upon whether $(v_j/\tau_j) = \chi_n^2$ or χ_{2n}^2. Using (2.1), let

$$(\tau_1, \cdots, \tau_{m+1}) = \sigma_0^2(1, \rho_1, \cdots, \rho_m)B,$$

where $B = (b_{kj})$ is the $(m + 1) \times (m + 1)$ matrix with

$$b_{kj} = \alpha_j \cos[(2\pi/p)(k - 1)(p - j + 1)],$$

$\alpha_1 = 1, \alpha_2 = \cdots = \alpha_m = 2$, and $\alpha_{m+1} = 1$ if $p = 2m$, $\alpha_{m+1} = 2$ if $p = 2m + 1$. Consequently, the MLE of $(\sigma_0^2\rho_j)$ are

$$(\hat{\sigma}_0^2, \hat{\sigma}_0^2\hat{\rho}_1, \cdots, \hat{\sigma}_0^2\hat{\rho}_m) = (\hat{\tau}_1, \cdots, \hat{\tau}_{m+1})B^{-1},$$

that is,

$$\hat{\sigma}_0^2 = \sum_{i=1}^{m+1} \hat{\tau}_i b^{i1},$$

and

$$\hat{\rho}_j = (\sum_{i=1}^{m+1} \hat{\tau}_i b^{i,j+1})/(\sum_{i=1}^{m+1} \hat{\tau}_i b^{i1}), \qquad j = 1, \cdots, m,$$

where $B^{-1} \equiv (b^{ij})$.

The assumption of circularity implies that each $z_j \equiv \hat{\sigma}_0^2\hat{\rho}_j$, $j = 0, 1, \cdots, m$, $\rho_0 \equiv 1$, is distributed as a linear combination of central chi-square variates (with coefficients of negative and positive sign). Hence, interval estimates of σ_0^2 and ρ_j are difficult to obtain when N is small. However, in the large sample situation, the limiting normal distribution provides a useful approximation. Accordingly, we may use the asymptotic result

$$\lim_{N\to\infty} \mathcal{L}\{N^{\frac{1}{2}}[(z_0, \cdots, z_m) - (\sigma_0^2, \sigma_0^2\rho_1, \cdots, \sigma_0^2\rho_m)]\} = N(0, \Omega),$$

where $\Omega = (\omega_{ij})$, $i, j = 0, 1, \cdots, m$; $\Omega = 2B^{-1}D_\tau B'^{-1}$, $D_\tau = \text{diag}(\tau_1, \cdots, \tau_{m+1})$. In particular,

$$\lim_{N\to\infty} \mathcal{L}[N^{\frac{1}{2}}(\hat{\sigma}_0^2 - \sigma_0^2)] = N(0, \omega_{00}),$$
$$\lim_{N\to\infty} \mathcal{L}[N^{\frac{1}{2}}(\hat{\rho}_j - \rho_j)] = N(0, \delta_j^2),$$

where

$$\delta_j^2 = (\rho_j^2\omega_{00} - 2\omega_{0j} + \omega_{jj})/\delta_0^4, \qquad \text{for all } j = 1, \cdots, m.$$

The above representation for z requires the determination of B^{-1}. The following alternative avoids this calculation, and may at times be useful. It also has the virtue of being more intuitive. Write

$$z_j = \text{tr}\,\Delta_j\hat{\Sigma} \equiv \text{tr}\,\Gamma'\Delta_j\Gamma\,\bar{D}_{\hat{\tau}} \equiv \text{tr}\,G_j\bar{D}_{\hat{\tau}},\ \bar{D}_{\hat{\tau}} = \text{diag}(\hat{\tau}_1, \cdots, \hat{\tau}_p),$$

where the elements of Δ_j consist of zeros except in those positions corresponding to where ρ_j occurs in Σ. At these positions there is a normalizing constant so that the sum of all the elements is unity. For example, for $p = 4$,

$$\Delta_0 = \frac{1}{4}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \Delta_1 = \frac{1}{8}\begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}, \quad \Delta_2 = \frac{1}{4}\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

Consequently, $z_j = \sum_{\alpha=1}^{p} g_{\alpha\alpha}^{(j)}\hat{\tau}_\alpha \equiv \sum_{\beta=1}^{m+1} h_{\beta j}\,\hat{\tau}_\beta$, $j = 0, 1, \cdots, m+1$, $G_\alpha = (g_{ij}^{(\alpha)})$, $h_{1j} = g_{11}^{(j)}$, $h_{2j} = g_{22}^{(j)} + g_{pp}^{(j)}$, etc. Writing $z = (\hat{\tau}_1, \cdots, \hat{\tau}_{m+1})H$, where $H = (h_{ij})$, the covariance matrix of the asymptotic distribution of $N^{\frac{1}{2}}z$ is $2HD_\tau H'$.

2.3. *Distribution preliminaries.* In this section we present some distributional results which permit a representation of the LRS as a product of independent beta variables. In order to obtain asymptotic approximations, we modify the procedure of Box (1949) to suit the needs of the statistics considered.

If $z_1, \cdots, z_m$ are independently distributed with $\mathcal{L}(z_i) = N(\xi_i, 1)$, then $\mathcal{L}(\Sigma z_i^2) = \chi_m^2(\delta)$ denotes the noncentral chi-square distribution with m degrees of freedom and noncentrality parameter $\delta = \sum_1^m \xi_i^2$. (We write $\chi_m^2 = \chi_m^2(0)$.)

If u and v are independently distributed, $\mathcal{L}(u) = \chi_{2a}^2$, $\mathcal{L}(v) = \chi_{2a}^2(\delta)$, then the random variable $z = u/(u + v)$ has the noncentral beta distribution with a and b degrees of freedom and noncentrality parameter δ, denoted by $\beta(a, b; \delta)$, i.e., with density

$$p(z) = \sum_{j=0}^{\infty} [e^{-\delta/2}(\delta/2)^j z^{a-1}(1 - z)^{b+j-1}]/[j!\, B(a, b + j)], \qquad 0 < z < 1.$$

(The central beta distribution has $\delta = 0$ and is denoted by $\beta(a, b)$.)

The following lemmas permit us to represent certain statistics as products of independent beta variates and are used repeatedly in the analysis. The proofs of the first two lemmas are based on equating moments—which, because the variables are bounded, determine the distribution uniquely. In so doing, it is helpful to use the Dirichlet integral

$$\int_{0<x_i} \prod_1^r x_i^{c_i-1} f(\Sigma x_i)\, dx = \{\Gamma(c)\}^{-1} \prod_1^r \Gamma(c_i) \int_0^\infty z^{c-1} f(z)\, dz,$$

where $c = \sum_1^r c_i$, and the duplication formula

$$\Gamma(rk + a) = r^{\frac{1}{2}-rk-a}(2\pi)^{-(r-1)/2} \prod_1^r \Gamma(k + (a + j - 1)/r),$$

where r is an integer.

LEMMA 1. *Let $U_1, \cdots, U_{m_1}, V_1, \cdots, V_{m_2}$ be independently distributed, $\mathcal{L}(U_j) = \chi_n^2$, $\mathcal{L}(V_j) = \chi_{2n}^2$. If*

$$L = [M^M \prod_1^{m_1} U_i \prod_1^{m_2} V_j^2]/[2^{2m_2}(\Sigma U_i + \Sigma V_j)^M], \qquad M = m_1 + 2m_2,$$

then $\mathcal{L}(L) = \mathcal{L}(\prod_1^{M-1} X_j)$, where $X_1, \cdots, X_{M-1}$ are independently distributed,

$$\mathcal{L}(X_j) = \beta(\tfrac{1}{2}n, j/M), \qquad j = 1, \cdots, M - m_2 - 1,$$

$$\mathcal{L}(X_j) = \beta(\tfrac{1}{2}(n + 1), j/M - \tfrac{1}{2}), \qquad j = M - m_2, \cdots, M - 1.$$

LEMMA 2. *Let $W_0, W_1, \cdots, W_r$ be independently distributed, $\mathcal{L}(W_j) = \chi_{2a_j}^2$, $j = 0, 1, \cdots, r$. If*

$$L = r^r(\prod_1^r W_j)(W_0 + \sum_1^r W_j)^{-r},$$

then $\mathcal{L}(L) = \mathcal{L}(\prod_1^r X_j)$, where $X_1, \cdots, X_r$ are independently distributed, $\mathcal{L}(X_j) = \beta(a_j, (a + j - 1)r^{-1} - a_j)$, $a = \sum_0^r a_j$.

LEMMA 3. *Let $Z_1, \cdots, Z_r$ be independently and identically distributed, each as $\beta(q, 1)$. If*

$$L = \prod_1^r Z_i^M, \qquad \text{then} \quad P\{-2 \log L \leqq t\} = P\{\chi_{2r}^2 \leqq qt/M\}.$$

PROOF. $EL^{-t} = (EZ^{-tM})^r = (1 - tM/q)^{-r}$, so that $\mathcal{L}(-2qM^{-1} \log L) = \chi_{2r}^2$. □

If $\mathcal{L}(Z) = \beta(a, b)$, then $EZ^t = B(a + t, b)/B(a, b)$. Suppose W, $0 \leqq W \leqq 1$, is a random variable whose distribution is that of a product of independent beta variables, so that

$$(2.8)\quad EW^h = k \prod_1^a \Gamma[x(1+h)+\xi_j]/\Gamma[x(1+h)+\eta_j], \qquad h = 0, 1, \cdots,$$

where k is determined from $EW^0 = 1$, and (x, ξ_j, η_j) are known quantities. The procedure of Box (1949) for finding the asymptotic distribution of W is based on expanding the characteristic function to yield a linear combination of chi-square variables:

$$(2.9)\quad P\{-2\rho \log W \leqq t\} = (1+\omega)P\{\chi_f^2 \leqq t\} - \omega P\{\chi_{f+4}^2 \leqq t\} + O(N^{-3}),$$

where $f = 2\Sigma(\eta_j - \xi_j)$, $\rho = (x - b)/x$, and $b = f^{-1}\Sigma(\xi^2 - \eta^2) + \frac{1}{2}$. The factor ω is obtained from

$$\omega = [6(x-b)^2]^{-1} \sum_1^a [B_3(b+\xi_j) - B_3(b+\eta_j)],$$

where $B_3(z) = (2z^3 - 3z^2 + z)/2$ is the Bernoulli polynomial of degree 3.

Because the computations become tedious, the following representation may prove useful:

$$(2.10)\quad \begin{aligned} B_3(z+1) - B_3(z) &= 3z^2, \\ B_3(z+\tfrac{1}{2}) - B_3(z) &= 3z(2z-1)/4, \\ \sum_{j=1}^r [B_3(z+j/c) - B_3(z)] &= r(r+1)(4c)^{-1}[r(r+1)c^{-2} \\ &\quad + (2r+1)(2z-1)c^{-1} + 6z^2 - 6z + 1]. \end{aligned}$$

To obtain the noncentral asymptotic distribution of $c_n g(V)$, where $\mathcal{L}(V) = W(\Sigma, p; n)$ and c_n is a normalizing constant, we use the standard delta method (Cramér (1946), pp. 354, 366). First let $Z = \Sigma^{-\frac{1}{2}}V\Sigma^{-\frac{1}{2}}/n$, so that $\mathcal{L}(Z) = W(nI, p; n)$ and $g(V) = g(n\Sigma^{\frac{1}{2}}Z\Sigma^{\frac{1}{2}}) \equiv h(nZ)$. If h is scale invariant, i.e., $h(cZ) = h(Z)$, and satisfies mild regularity conditions (which is the case for the problems considered) then

$$(2.11)\quad \mathcal{L}\{n^{\frac{1}{2}}[h(Z) - h(I)]\} \to N(0, 2 \operatorname{tr} H^2),$$

where $H = (h_{ij})$, $h_{ij} = \frac{1}{2}\partial h/\partial z_{ij}|_{Z=I}$, for $i \neq j$, and $h_{ii} = \partial h/\partial z_{ii}|_{Z=I}$. The asymptotic variance is given by

$$v_\infty = \sum_{i,j} \sum_{k,\ell} h_{ij} h_{k\ell}\, n \operatorname{Cov}(z_{ij}, z_{k\ell}).$$

Since $\operatorname{Cov}(z_{ij}, z_{k\ell}) = n^{-1}[\delta_{ik}\delta_{j\ell} + \delta_{i\ell}\delta_{jk}]$, where δ_{ij} is the Kronecker delta, we obtain

$$(2.12)\quad v_\infty = 2 \sum_{i,j} h_{ij}^2 = 2 \operatorname{tr} H^2.$$

3. Tests for symmetries in the covariance structure. The hypotheses

$$H_1 : \Sigma = \sigma^2 I, \qquad H_2 : \sigma_{ii} = \sigma^2, \qquad \sigma_{ij} = \sigma^2\rho\,(i \neq j),$$

$$H_3 : \Sigma = \Sigma_c, \qquad H_4 : \Sigma > 0,$$

represent various degrees of specialization in the covariance matrix. The tests for H_1 versus H_4 (sphericity), H_2 versus H_4 (Wilks' hypotheses) are known. The test for sphericity given homogeneity of variances and covariances (H_1 versus H_2) is the test that the intraclass correlation is zero and is a standard F-test. Our concern is mainly with tests involving circularity: sphericity given circularity (H_1 versus H_3), homogeneity of covariances given circularity (H_2 versus H_3), and test for circularity (H_3 versus H_4).

In this section we assume the means are unknown. A slight modification in the development yields analogous results when the means are known. From the canonical model (2.6), (2.7), the hypotheses on Σ are translated to hypotheses on $\tau = (\tau_1, \cdots, \tau_p)$ as represented by the regions

$$\omega_1 = \{\tau : \tau_1 = \cdots = \tau_{m+1}, \quad 0 < \tau_1 < \infty\},$$

$$\omega_2 = \{\tau : 0 < \tau_1 < \infty, \quad \tau_2 = \cdots = \tau_{m+1}, \quad 0 < \tau_2 < \infty\},$$

$$\omega_3 = \{\tau : 0 < \tau_j < \infty, \quad j = 1, 2, \cdots, m+1\},$$

$$\omega_4 = \{\Sigma : \Sigma > 0\}.$$

It is straightforward to obtain the maxima of the likelihood functions $L(V, y)$.

Spherical model. From (2.6) and (2.7),

$$\sup_{\omega_1} L(V, y) = c(V) e^{-\frac{1}{2}pN} (pN)^{pN/2} \left(\textstyle\sum_1^p V_{ii}\right)^{-\frac{1}{2}pN} \tag{3.1}$$

where $c(V) = (2\pi)^{-p/2} c(p, n) |V|^{(n-p-1)/2}$.

Intraclass correlation model. From (2.6) and (2.7),

$$\sup_{\omega_2} L(V, y) = c(V) e^{-\frac{1}{2}pN} [N^p (p-1)^{p-1}]^{\frac{1}{2}N} [v_1 (\textstyle\sum_2^p v_{ii})^{p-1}]^{-\frac{1}{2}N}. \tag{3.2}$$

Circular model. From (2.6) and (2.7a) or (2.7b)

$$\sup_{\omega_3} L(V, y) = c(V) e^{-\frac{1}{2}pN} N^{\frac{1}{2}pN} 2^{N(p-m-1)} \left(\textstyle\prod_1^p v_j\right)^{-\frac{1}{2}N}. \tag{3.3}$$

General Σ. In this case we start with (2.3),

$$\sup_{\omega_4} L(V, y) = c(V) e^{-\frac{1}{2}pN} e^{-\frac{1}{2}pN} N^{\frac{1}{2}pN}. \tag{3.4}$$

Note that (3.1)–(3.4) have been given in a form whereby the factor $c(V)$ is common.

The various LRS in this section are denoted by

$$\lambda_{ij} = \sup_{\omega_i} L(V, y) / \sup_{\omega_j} L(V, y).$$

3.1. *Test for sphericity versus circularity.* From (3.1) and (3.3), the LRS is given by

$$\lambda_{13}^{2/N} = p^p 2^{-2(p-m-1)} \textstyle\prod_1^p v_j \left(\sum_1^{m+1} v_j\right)^{-p}.$$

When the hypothesis is true, from Lemma 1, we have the representation $\mathcal{L}(\lambda_{13}^{2/N}) = \mathcal{L}(\prod_1^{p-1} T_j)$, where the T_j are independent,

$$\mathcal{L}(T_j) = \beta(\tfrac{1}{2}n, j/p), \qquad j = 1, \cdots, m,$$

$$\mathcal{L}(T_j) = \beta(\tfrac{1}{2}(n+1), j/p - \tfrac{1}{2}), \qquad j = m+1, \cdots, p-1.$$

In its canonical form, this problem is that of testing for homogeneity of variances, for which Bartlett (1937) suggested replacing N by n. This problem has been considered by many authors (see e.g., Box (1949), Rao (1952)). For completeness, we merely note the result, obtained as a specialization of the multivariate problem, as given by Anderson (1958, p. 255).

If $V_{13} = \lambda_{13}^{n/N}$, then

$$P\{-2\rho \log V_{13} \leqq z\} = (1+\omega)P\{\chi_f^2 \leqq z\} - \omega P\{\chi_{f+4}^2 \leqq z\} + O(n^{-3}),$$

where

$$p = 2m, \quad f = p/2,$$
$$\rho = 1 - (p^2+6p-4)/(6p^2 n), \qquad \omega = [p(1-\rho)^2/(8\rho^2),$$
$$p = 2m+1, \quad f = (p-1)/2,$$
$$\rho = 1-(p+4)/(6pn), \qquad \omega = [(p-1)(1-\rho)^2]/(8\rho^2).$$

To obtain the noncentral asymptotic distribution let (when $p = 2m+1$) $z_{jj} = v_{jj}/\tau_j$. Define

$$h(z) \equiv 2N^{-1}\log \lambda_{13} = \log\left[z_1 \prod_2^{m+1}\left(\frac{z_j + z_{p-j+2}}{2}\right)^2 \prod_1^p \tau_i \Big/ \left(\sum_1^p \tau_j z_j/p\right)^p\right],$$

where $z_j \equiv z_{jj}$. Then from (2.14), $h(I) = \Sigma \log \tau_i - p \log \bar{\tau}$, where $\bar{\tau} = \Sigma \tau_i/p$, so that $h_{ii} = 1 - \tau_i/\bar{\tau}$, $h_{ij} = 0$ $(i \neq j)$. Consequently,

$$\mathcal{L}\{n^{\frac{1}{2}}[h(Z) - h(I)]\} \to N(0, 2\textstyle\sum_1^p (\tau_i - \bar{\tau})^2/\bar{\tau}^2).$$

The case $p = 2m$ yields the same result.

3.2. *Test for intraclass structures versus circularity.* From (3.2) and (3.3), the LRS is given by

$$\lambda_{23}^{2/N} = (p-1)^{p-1}2^{-2(p-m-1)} \textstyle\prod_2^p v_j (\sum_2^{m+1} v_j)^{-(p-1)}.$$

When the hypothesis is true, from Lemma 1, we have the representation $\mathcal{L}(\lambda_{23}^{2/N}) = \mathcal{L}(\prod_1^{p-2} T_j)$, where the T_j are independent and

$$\mathcal{L}(T_j) = \beta(\tfrac{1}{2}n, j/(p-1)), \qquad j = 1, \cdots, p-m-1,$$
$$\mathcal{L}(T_j) = \beta(\tfrac{1}{2}(n+1), j/(p-1) - \tfrac{1}{2}), \qquad j = p-m, \cdots, p-2.$$

We note that this problem also reduces to a test for homogeneity of variances. This is because τ_1 is unrestricted in both ω_2 and ω_3, so that the hypothesis is, in effect, that $\tau_2 = \cdots = \tau_{m+1}$. As in Section 3.1 (Anderson (1958), p. 255) we obtain the asymptotic distribution when the hypothesis is true to be

$$P\{-2\rho \log V_{23} \leqq z\} = (1+\omega)P\{\chi_f^2 \leqq z\} - \omega P\{\chi_{f+4}^2 \leqq z\} + O(n^{-3}),$$

where $V_{23} = \lambda_{23}^{n/N}$,

$$p = 2m, \quad f = (p-2)/2,$$
$$\rho = 1 - (p+3)/[6(p-1)n], \qquad \omega = [(p-2)(1-\rho)^2]/(8\rho^2),$$
$$p = 2m+1, \quad f = (p-3)/2,$$
$$\rho = 1-(p+1)/[6n(p-1)], \qquad \omega = [(p-3)(1-\rho)^2]/(8\rho^2).$$

The noncentral asymptotic distribution is obtained as in Section 3.1, namely,

$$\mathfrak{L}\{n^{\frac{1}{2}}[h(Z) - h(I)]\} \to N(0, 2 \textstyle\sum_2^p (\tau_i - \bar{\bar{\tau}})^2/\bar{\bar{\tau}}^2),$$

where $h(Z) = (2/N) \log \lambda_{23}$, $h(I) = \sum_2^p \log \tau_i - (p - 1) \log \bar{\bar{\tau}}$, where $\bar{\bar{\tau}} = \sum_2^p \tau_i/(p - 1)$.

3.3. *Circular versus general structure.* From (3.3) and (3.4), the LRS is given by

$$\lambda_{34}^{2/N} = 2^{2(p-m-1)} |V|/(\textstyle\prod_1^p v_j) = 2^{2(p-m-1)} |R| \prod_1^p v_{jj}/(\prod_1^p v_j), \tag{3.5}$$

where R is the correlation matrix. In order to obtain a representation as a product of independent beta random variables, let $V = TT'$, where T is lower triangular. Then $|V| = \prod_1^p t_{ii}^2$, $v_j = \sum_{\alpha=1}^j t_{j\alpha}^2$, $j = 2, \cdots, p$. If we let

$$\bar{G}_{m+1} = t_{m+1}/(t_{m+1} + q_{m+1}),$$

$$G_j = 2^2 t_j t_{p-j+2}/(t_j + t_{p-j+2} + q_j + q_{p-j+2}), \qquad j = 2, \cdots, m + 1,$$

where $t_j = t_{jj}^2$, $q_j = \sum_{\alpha=1}^{j-1} t_{j\alpha}^2$, $j = 2, \cdots, p$, then

$$\lambda_{34}^{2/N} = (\textstyle\prod_2^m G_j)\bar{G}_{m+1}, \qquad p = 2m,$$

$$= (\textstyle\prod_2^m G_j)G_{m+1}, \qquad p = 2m + 1.$$

When H_3 is true (the covariance matrix is circular), it may be directly verified that all the t_{ij} are independently distributed, so that the G_j are independent. Also, because $\tau_j = \tau_{p-j+2}, j = 2, \cdots, p$, the elements of the jth and $(p - j + 2)$th rows of T have the same scale, namely,

$$\mathfrak{L}(t_{j\alpha}^2/\tau_j) = \mathfrak{L}(\chi_1^2), \qquad \alpha = 1, \cdots, j - 1, \alpha \neq j$$

$$\mathfrak{L}(t_{jj}^2/\tau_j) = \mathfrak{L}(\chi_{n-j+1}^2), \qquad j = 2, \cdots, p.$$

Because G_j is invariant under a scale change $t_j \to ct_j$, we may (under H_3) assume $\tau_j = 1$. From Lemma 2, we then obtain (for both even and odd p), that $\mathfrak{L}(\lambda_{34}^{2/N}) = \mathfrak{L}(\prod_1^{p-1} T_j)$, where the T_j are independent,

$$\mathfrak{L}(T_j) = \beta((n - j)/2, j/2), \qquad j = 1, \cdots, m, \tag{3.6}$$

$$\mathfrak{L}(T_j) = \beta((n - j)/2, (j + 1)/2), \qquad j = m + 1, \cdots, p - 1.$$

When H_3 is true, we obtain from the representation (3.5)

$$E\lambda_{34}^h = c \textstyle\prod_1^{p-1} \Gamma(A - \frac{1}{2}(j + 1))/\{[\Gamma(A - \frac{1}{2})]^m[\Gamma(A)]^{p-m-1}\}, \qquad A = \frac{1}{2}N(1 + h),$$

where c is a normalizing constant so that $E\lambda_{34}^0 = 1$. From (2.8) we make the association

$$x = N/2, \qquad \rho = 1 - 2b/N, \qquad \xi_j = -(j + 1)/2, \ j = 1, \cdots, p - 1,$$

$$\eta_1 = \cdots = \eta_m = -\tfrac{1}{2}, \qquad \eta_{m+1} = \cdots = \eta_{p-1} = 0.$$

Hence (when H_3 is true)

$$P\{-2\rho \log \lambda_{34} \leq z\} = (1 + \omega)P\{\chi_f^2 \leq z\} - \omega P\{\chi_{f+4}^2 \leq z\} + O(N^{-3}),$$

where f and b are determined from (2.10), and (2.11):

$$f = [p(p+1) - 2(m+1)]/2, \qquad b = [p(p+1)(2p+7) - 18(m+1)]/24f.$$

From (2.9) and (2.10),

$$6(x-b)^2\omega = \sum_1^p [B_3(b-j/2) - B_3(b)] + (m+1)[B_3(b) - B_3(b-\tfrac{1}{2})]$$
$$\equiv a_0 + a_1 b + a_2 b^2,$$

where $a_2 = -3f/2$, $a_1 = 3bf$, $a_0 = [24(m+1) - p(p+1)^2(p+8)]/32$, so that

$$6(x-b)^2\omega = \tfrac{3}{2}b^2 f + [24(m+1) - p(p+1)^2(p+8)]/32.$$

Therefore, when

$$p = 2m, \qquad f = (p^2-2)/2, \qquad b = [2p^3 + 9p^2 - 2p - 18]/[12(p^2-2)];$$

when $p = 2m+1$, $f = (p^2-1)/2$, $b = (2p+9)/12$. In either case, ω simplifies to $\omega = -(p^2-1)(2p^2-9)/8(6n-2p-3)^2$.

To obtain the noncentral asymptotic distribution, let $Z = \Sigma^{-\frac{1}{2}}V\Sigma^{-\frac{1}{2}}$. Denote the jth row of $\Sigma^{\frac{1}{2}}$ by s_j, so that $v_{jj} = s_j A s_j' \equiv \operatorname{tr}(ZA_j)$, where $A_j = s_j' s_j$. When $p = 2m+1$, we have

$$h(Z) = 2N^{-1}\log\lambda_{34} = \log\{2^{p-1}|\Sigma|\,|Z|/((\operatorname{tr} ZA_1)\textstyle\prod_2^p [\operatorname{tr} Z(A_j + A_{p-j+2})])\},$$

and

$$\mathcal{L}\{N^{\frac{1}{2}}[2N^{-1}\log\lambda_{34} - h(I)]\} \to N(0, v_\infty), \tag{3.7}$$

where

$$h(I) = \log\{|P|\textstyle\prod_2^p [(\sigma_j\sigma_{p-j+2})/\tfrac{1}{2}(\sigma_j^2 + \sigma_{p-j+2}^2)]\},$$

$\sigma_j^2 \equiv \sigma_{jj}$ and $P = (\rho_{ij})$ is the correlation matrix. To obtain $v_\infty = 2\operatorname{tr} H^2$, note that

$$H = I - \textstyle\sum_{j=1}^p [(A_j + A_{p-j+2})/\operatorname{tr}(A_j + A_{p-j+2})] \equiv I - B,$$

where $A_{p+1} \equiv 0$. Since $\operatorname{tr} B = p$, $\operatorname{tr} H^2 = \operatorname{tr} B^2 - p$. From

$$\operatorname{tr} A_i A_j = \operatorname{tr}(s_i' s_i)(s_j' s_j) = \sigma_{ij}^2, \qquad \operatorname{tr} A_i = \sigma_{ii},$$

we immediately obtain

$$\operatorname{tr} B^2 = \textstyle\sum_{i,j}\{(\operatorname{tr} A_i A_j)/[\operatorname{tr}(A_i + A_{p-i+2})\operatorname{tr}(A_j + A_{p-j+2})]\}$$
$$= \textstyle\sum_{i,j}\{(\sigma_{ij})/[(\sigma_i^2 + \sigma_{p-i+2}^2)(\sigma_j^2 + \sigma_{p-j+2}^2)]\},$$

where $\sigma_{p+1} \equiv 0$. Consequently, $v_\infty = 2(\operatorname{tr} B^2 - p)$.

The result (3.7) is valid for $p = 2m$, if we replace $2\sigma_{m+1}^2$ by σ_{m+1}^2.

4. Tests for means given that the covariance matrix is circular. In this section we are concerned with testing that the means are zero and that the means are equal, given that the covariance matrix is circular. From (2.3), the test that

$\mu_1 = \cdots = \mu_p = \mu^*$ is equivalent to $\eta = N^{\frac{1}{2}}\mu\Gamma = N^{\frac{1}{2}}\mu^* e\Gamma$, where Γ is given by (2.2). But Γ is orthogonal with first row $p^{-\frac{1}{2}}e$, so that $e\Gamma = p^{\frac{1}{2}}(1, 0, \cdots, 0)$. Thus we have as our starting point (2.6) and (2.7), and regions

$$\omega_1^* = \{\eta, \tau : \eta = 0, 0 < \tau_j\},$$
$$\omega_2^* = \{\eta, \tau : \eta = (\eta_1, 0, \cdots, 0), 0 < \tau_j\},$$
$$\omega_3^* = \{\eta, \tau : -\infty < \eta_j < \infty, 0 < \tau_j\}.$$

The LRS for testing $(\eta, \tau) \varepsilon \omega_i^*$ versus ω_j^* is denoted by ℓ_{ij}.

4.1. *Test that means are zero.* From (2.6) and (2.7) the LRS is given by

$$\ell_{13}^{2/N} = \textstyle\prod_1^p [(v_j/(v_j + w_j))],$$

where for $p = 2m + 1$, $w_1 = y_1^2$, $w_j = y_j^2 + y_{p-j+2}^2$, $j = 2, \cdots, m + 1$; and for $p = 2m$, $w_1 = y_1^2$, $w_j = y_j^2 + y_{p-j+2}^2$, $j = 2, \cdots, m$, $w_{m+1} = y_{m+1}^2$. From Section 2.1 and the fact that if $\mathcal{L}(X) = \beta(a, 1)$, then $\mathcal{L}(X^2) = \beta(a/2, 1)$, we obtain the representation

$$\mathcal{L}(\ell_{13}^{2/N}) = \mathcal{L}((\textstyle\prod_1^m T_j)\bar{T}_{m+1}), \qquad p = 2m,$$
$$= \mathcal{L}(\textstyle\prod_1^{m+1} T_j), \qquad p = 2m + 1,$$

where the T_j are independently distributed as

$$(4.1) \quad \mathcal{L}(T_1) = \beta(n/2, \tfrac{1}{2}), \qquad \mathcal{L}(T_j) = \beta(n/2, 1),$$
$$j = 2, \cdots, m, \qquad \mathcal{L}(\bar{T}_{m+1}) = \beta(n/2, \tfrac{1}{2}).$$

When the hypothesis of zero means is true, we obtain from the representation (4.2),

$$(4.2) \quad E\ell_{13}^h = [\Gamma(A - \tfrac{1}{2})]^{m+1}/\{[\Gamma(A)]^{2m-p+2}[\Gamma(A + \tfrac{1}{2})]^{p-m-1}\},$$
$$A = \tfrac{1}{2}N(1 + h).$$

From (2.8) make the association: $x = \frac{1}{2}N$, $\rho = 1 - 2b/N$, $\xi_1 = \cdots = \xi_{m+1} = -\frac{1}{2}$, $\eta_1 = \cdots = \eta_{p-m-1} = \frac{1}{2}$, $\eta_{p-m} = \cdots = \eta_{m+1} = 0$. Hence $P\{-2\rho \log \ell_{13} \leqq z\} = (1 + \omega)P\{\chi_f^2 \leqq z\} - \omega P\{\chi_{f+4}^2 \leqq z\} + O(N^{-3})$, where f and b are determined from (2.9), (2.10): $f = p$, $b = (2m + 2 + p)/4p$.

$$6(x - b)^2\omega$$
$$= (m + 1)B_3(b - \tfrac{1}{2}) - (2m + 2 - p)B_3(b) - (p - m - 1)B_3(b + \tfrac{1}{2})$$
$$= -(p - m - 1)[B_3(b + \tfrac{1}{2}) - B_3(b)] - (m + 1)[B_3(b) - B_3(b - \tfrac{1}{2})]$$
$$= a_0 + a_1 b + a_2 b^2,$$

where $a_2 = -3p/2$, $a_1 = +3pb$, $a_0 = -3(m + 1)/4$. Hence

$$\omega = \tfrac{1}{4}p\{(2m + 2 - p)/[2Np - (2m + 2 + p)]\}^2.$$

To obtain the noncentral asymptotic distribution, note that under the alternative $\ell_{13}^{2/N}$ is the product of independent random variables. Using the delta method on each term, we obtain
$\mathcal{L}\{N^{1/2}[\log [v_j/(v_j + y_j^2 + y_{p-j+2}^2)] - \log [1/(1 + \delta_j)]]\} \to N(0, 1 - (1 + \delta_j)^{-2})$, where $\delta_1 = \eta_1^2/\tau_1$, $\delta_j = (\eta_j^2 + \eta_{p-j+2}^2)/\tau_j$, for $j = 2, \cdots, m + 1$, and $\delta_{m+1} = \eta_{m+1}^2/\tau_{m+1}$. A convolution then yields

$$(4.3)\quad \mathcal{L}\{(\tfrac{1}{2}N)^{\frac{1}{2}}[\log \ell_{13}^{2/N} - \log \textstyle\prod_1^p 1/(1 + \delta_j)]\} \to N(0, p - \sum_1^p 1/(1 + \delta_j)^2).$$

4.2. *Tests that the means are equal.* From (2.6) and (2.7) the LRS is given by

$$(4.4)\qquad \ell_{23}^{2/N} = \textstyle\prod_2^p [(v_j/(v_j + w_j))].$$

By Lemma 2 and Section 4.1,

$$(4.5)\qquad \begin{aligned} \mathcal{L}(\ell_{23}^{2/N}) &= \mathcal{L}((\textstyle\prod_1^m T_j)\bar{T}_{m+1}), && p = 2m, \\ &= \mathcal{L}(\textstyle\prod_1^{m+1} T_j), && p = 2m + 1, \end{aligned}$$

where the T_j are independently distributed, and are defined by (4.1).

When $p = 2m + 1$, the exact result

$$P\{-2 \log \ell_{23} \leqq z\} = P\{\chi_{p-1}^2 \leqq (N - 1)/Nz\}$$

is obtained from Lemma 3 with the association $(r, q, M) = (m, n/2, N)$.

When $p = 2m$, we have from (4.2) that

$$E\ell_{23}^h = \left[\frac{\Gamma\left(A - \frac{1}{2}\right)}{\Gamma\left(A + \frac{1}{2}\right)}\right]^m \qquad A = \tfrac{1}{2}N(1 + h).$$

From (2.8) make the association

$$x = N/2, \quad \rho = 1 - 2b/N, \quad \xi_1 = \cdots = \xi_m = -\tfrac{1}{2}, \quad \eta_1 = \cdots = \eta_m = \tfrac{1}{2}.$$

Hence $f = p - 1$, $b = (2p - 1)/4(p - 1)$, and

$$\begin{aligned} 6(x - b)^2\omega &= -m[B_3(b + \tfrac{1}{2}) - B_3(b - \tfrac{1}{2})] + [B_3(b + \tfrac{1}{2}) - B_3(b)] \\ &\equiv a_0 + a_1 b + a_2 b^2, \end{aligned}$$

where $a_2 = -3f/2$, $a_1 = 3fb$, $a_0 = -3p/8$; thus

$$\omega = (p - 1)/\{4[2n(p - 1) - 1]^2\}.$$

The noncentral asymptotic distribution is obtained as in Section 4.1 to yield

$$(4.6)\quad \mathcal{L}\{N^{\frac{1}{2}}[\log \ell_{23}^{2/N} - \log \textstyle\prod_2^p 1/(1 + \delta_j)]\} \to N(0, p - 1 - \sum_1^{p-1} 1/(1 + \delta_j)^2).$$

5. Tests for means and covariances. There are various combinations of tests which can be performed. We only consider two tests, (i) that the means are zero and the covariance matrix is circular, (ii) that the means are equal and the covariance matrix is circular, both against general alternatives. If we let

$$\Omega_1 = \{\mu, \Sigma : \mu = 0, \Sigma = \Sigma_c\},$$
$$\Omega_2 = \{\mu, \Sigma : \mu = \mu^* e, \Sigma = \Sigma_c\},$$
$$\Omega = \{\mu, \Sigma : -\infty < \mu_j < \infty, \Sigma > 0\},$$

then we wish to test that $(\mu, \Sigma) \varepsilon \Omega_1$ or Ω_2 versus Ω. Denote the LRS by L_1 and L_2, respectively. Because of a nesting of the various regions in Section 3 and Section 4, we obtain (Anderson (1958), Lemma 10.3.1) that

$$(5.1) \qquad L_1 = \ell_{13}\lambda_{34}, \qquad L_2 = \ell_{23}\lambda_{34}.$$

5.1. *Simultaneous test that means are zero and covariance matrix is circular.* Relating (5.1) to (3.9) and (4.1), we obtain

$$L_1^{2/N} = 2^{2(p-m-1)}|R| \prod_1^p ((v_{jj})/(v_j + w_j)),$$

where R is the sample correlation matrix corresponding to V. Using the development of Section 3.3, we obtain the representation

$$\mathcal{L}(L_1^{2/N}) = \mathcal{L}(\prod_1^p U_j),$$

where the U_j's are independently distributed,

$$(5.2) \quad \mathcal{L}(U_j) = \begin{cases} \beta\left(\frac{n-j+1}{2}, \frac{j}{2}\right), & j = 1, \cdots, m+1, \\ \beta\left(\frac{n-j+1}{2}, \frac{j+1}{2}\right), & j = m+2, \cdots, p. \end{cases}$$

When the hypothesis is true, we obtain from (5.2),

$$EL_1^h = c \prod_1^p \Gamma(A - j/2)/\{[\Gamma(A)]^{m+1}[\Gamma(A + \tfrac{1}{2})]^{p-m-1}\}, \qquad A = \tfrac{1}{2}N(1+h),$$

where c is a normalizing constant so that $EL_1^0 = 1$.

From (2.8) we make the association

$$x = N/2, \quad \rho = 1 - 2b/N, \quad \xi_j = -j/2, \quad j = 1, \cdots, p,$$
$$\eta_1 = \cdots = \eta_{m+1} = 0, \quad \eta_{m+2} = \cdots = \eta_{p-m-1} = \tfrac{1}{2}.$$

Hence

$$P\{-2\rho \log L_1 \leqq z\} = (1+\omega)P\{\chi_f^2 \leqq z\} - \omega P\{\chi_{f+4}^2 \leqq z\} + O(N^{-3}),$$

where f, b and ω are determined from (2.9):

$$f = p - m - 1 + \tfrac{1}{2}p(p+1),$$
$$b = (1/24f)[p(p+1)(2p+1)(2p+7) + 6(p-m-1)],$$
$$6(x-b)^2\omega = \sum_1^p [B_3(b - j/2) - B_3(b)] - (p-m-1)(B_3(b+\tfrac{1}{2}) - B_3(b))$$
$$\equiv a_0 + a_1 b + a_2 b^2,$$

where $a_2 = -3f/2$, $a_1 = 3bf$, $a_0 = -p(p+1)(p^2+p+4)/32$. Hence $6(x-b)^2\omega = -p(p+1)(p^2+p+4)/32 + 3bf/2$.

REMARK. The statistics ℓ_{13} and λ_{34} are not independent (which is often the case with nested hypotheses) so that the subhypotheses should not be tested individually.

Because plim log $|R| = 0$ under the hypothesis that Σ is circular, the asymptotic distribution of $L_1^{2/N} 2^{-2(p-m-1)}$ is the same as the asymptotic distribution of $\ell_{13}^{2/N}$, given by (4.3).

5.2. *Simultaneous test that means are equal and covariance matrix is circular.* Relating (5.1) to Sections (3.3) and (4.2), we obtain

$$L_2^{2/N} = 2^{2(p-m-1)}|R| \prod_2^p (v_j/(v_j + w_j)),$$

where R is the correlation matrix. Following the development of Section 3.3, we obtain the representation $\mathfrak{L}(L_2^{2/N}) = \mathfrak{L}(\prod_2^p U_j)$, where the U_j's are defined by (5.2).

When the hypothesis is true, we obtain from (5.2)

$$EL_2^h = c \prod_2^p \Gamma(A - j/2)/\{[\Gamma(A)]^m[\Gamma(A+\tfrac{1}{2})]^{p-m-1}\}, \qquad A = \tfrac{1}{2}N(1+h),$$

and c is a normalizing constant so that $EL_2^0 = 1$.

From (2.8) make the association

$$x = N/2, \qquad \rho = 1 - 2b/N, \qquad \xi_j = -(j+1)/2, \qquad j = 2, \cdots, p,$$

$$\eta_1 = \cdots = \eta_m = 0, \qquad \eta_{m+1} = \cdots = \eta_{p-m-1} = \tfrac{1}{2}.$$

Hence

$$P\{-2\rho \log L_2 \leqq z\} = (1+\omega)P\{X_f^2 \leqq z\} - \omega P\{\chi_{f+4}^2 \leqq z\} + O(N^{-3}),$$

where f, b, and ω are determined from (2.9):

$$f = \tfrac{1}{2}[p(p+1) + 2(p-m-2)],$$

$$b = (1/24f)[p(p+1)(2p+7) + 6(p-m) - 24],$$

$$6(x-b)^2\omega = \sum_1^p [B_3(b - j/2) - B_3(b)] - (p-m)[B_3(b+\tfrac{1}{2}) - B_3(b)] + [B_3(b+\tfrac{1}{2}) - B_3(b-\tfrac{1}{2})]$$

$$\equiv a_0 + a_1 b + a_2 b^2,$$

where $a_2 = -3f/2$, $a_1 = 3bf$, $a_0 = -p(p+1)^2(p+8)/32 + \tfrac{3}{4}$. Hence

$$6(x-b)^2\omega = [48fb^2 + 24 - p(p+1)^2(p+8)]/32.$$

The asymptotic distribution of $L_2^{2/N} 2^{-2(p-m-1)}$ is the same as the asymptotic distribution of $\ell_{23}^{2/N}$ given by (4.6).

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