Testing for a change in correlation at an unknown point in time using an extended functional delta method

by

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Proposed running head:

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Abstract

We propose a new test against a change in correlation at an unknown point in time based on cumulated sums of empirical correlations. The test does not require that inputs are iid under the null. We derive its limiting null distribution using a new functional delta method argument, provide a formula for its local power for particular types of structural changes, give some Monte Carlo evidence on its finite sample behavior and apply it to recent stock returns.

Keywords: Correlation, Structural break, Functional delta method
JEL numbers: C12, C14, G12

1 Introduction and Summary

There is quite a consensus in empirical finance and elsewhere that correlations among returns of all sorts cannot be assumed to remain constant over longer stretches of time (Longin and Solnik, 1995; Krishan et al., 2009, among many others). While conditional correlations are easily modeled as time-varying in various ways (see e.g. McAleer et al., 2008), including procedures to test for this, unconditional correlations are often taken as constant, which seems to be at odds with various stylized facts from various applications. In particular, correlations among stock returns seem to increase in times of crisis, as evidenced by the most recent joint downturn in stock markets worldwide.

Yet, there is a surprising lack of methods to formally test for a change in correlation. Existing procedures either require strong parametric assumptions (Dias and Embrechts, 2004), assume that potential break points are known (Pearson and Wilks, 1933; Jennrich, 1970; Goetzmann et al., 2005), or simply estimate correlations from moving windows without giving a formal decision rule (Longin and Solnik, 1995). Only recently, Galeano and Peña (2007) and Aue et al. (2009) have proposed formal tests for a change in covariance structure that do not build upon prior knowledge as to the timing of potential shifts. Both are based on cumulated sums of second order empirical cross moments (in the vain of Ploberger et al., 1989) and reject the null of constant covariance structure if these cumulated sums fluctuate too much.

While Galeano and Peña (2007) operate in a parametric setting, the Aue et al. (2009) approach is quite similar to ours. However, the null hypotheses considered here and by Aue et al. (2009) are not identical but overlapping, none is encompassing the other. It can happen that correlations remain constant, but covariance changes and vice versa: The correlation changes, while covariance remains constant. This distinction is important for instance when testing for contagion in international finance (see e.g. Forbes and Rigobon, 2002). What is termed "shift contagion" in this literature is equivalent in certain models to a change in covariance induced by a change in correlation. Our test will then detect such types of contagion, while the Aue et al. (2009) procedure might not.

Similar to Aue et al. (2009), our test statistic is a suitably standardized cumulated sum of empirical correlation coefficients. To derive its asymptotic null distribution, we use a functional delta method argument that has not been considered before, extending conventional functional central limit theorems which are the workhorse in much of the existing literature on structural changes (see e.g. Ploberger et al., 1989, Ploberger and Krämer, 1990, 1992, or Inoue, 2001, just to name a few).

2 The test statistic and its asymptotic null distribution

Let $(X_t, Y_t), t = 0, \pm 1, ...$ be a sequence of bivariate random vectors with finite $(4+\delta)$th absolute moments for some $\delta > 0$. We want to test whether the correlation between X_t and Y_t,

$$\rho_t = \frac{Cov(X_t, Y_t)}{\sqrt{Var(X_t)}\sqrt{Var(Y_t)}},$$

is constant over time in the observation period, i.e. we test

$$H_0 : \rho_t = \rho_0 \ \forall t \in \{1, \ldots, T\} \text{ vs. } H_1 : \exists t \in \{1, \ldots, T-1\} : \rho_t \neq \rho_{t+1}$$

for a constant ρ_0. Our test statistic is

$$Q_T(X, Y) = \hat{D} \max_{2 \leq j \leq T} \frac{j}{\sqrt{T}} |\hat{\rho}_j - \hat{\rho}_T|, \tag{1}$$

where

$$\hat{\rho}_k = \frac{\sum_{i=1}^k (X_i - \bar{X}_k)(Y_i - \bar{Y}_k)}{\sqrt{\sum_{i=1}^k (X_i - \bar{X}_k)^2}\sqrt{\sum_{i=1}^k (Y_i - \bar{Y}_k)^2}}$$

and $\bar{X}_k = \frac{1}{k}\sum_{i=1}^k X_i$, $\bar{Y}_k = \frac{1}{k}\sum_{i=1}^k Y_i$. The value $\hat{\rho}_k$ is the empirical correlation coefficient from the first k observations. The scalar $\hat{D}$ is needed for the asymptotic null distribution and will be specified below. The test rejects the null hypothesis of constant correlation if the empirical correlations fluctuate too much, as measured by $\max_{2\le j\le T} |\hat{\rho}_j - \hat{\rho}_T|$, with the weighting factor $\frac{j}{\sqrt{T}}$ scaling down deviations at the beginning of the sample where the $\hat{\rho}_j$ are more volatile.

Of course, other functionals of the $\hat{\rho}_j$-series are likewise possible as suitable test statistics, such as some standardized version of

$$\max_{2\le j\le T}(\hat{\rho}_j) - \min_{2\le j\le T}(\hat{\rho}_j),$$

or simply some suitable average (see Krämer and Schotman, 1992, or Ploberger and Krämer, 1992), but for ease of exposition we stick to expression (1) for the purpose of the present paper.

The following technical assumptions are required for the limiting null distribution:

(A1) For $U_t := \Big(X_t^2 - \mathbb{E}(X_t^2),\ \ Y_t^2 - \mathbb{E}(Y_t^2),\ \ X_t - \mathbb{E}(X_t),\ \ Y_t - \mathbb{E}(Y_t),\ \ X_tY_t - \mathbb{E}(X_tY_t)\Big)'$ and $S_j := \sum_{t=1}^j U_t$, we have

$$\lim_{T\to\infty} \mathbb{E}\left(\frac{1}{T}S_T S_T'\right) =: D_1 \text{ (finite and positive definite).}$$

(A2) The r-th absolute moments of the components of U_t are uniformly bounded for some $r > 2$.

(A3) The vector (X_t, Y_t) is L_2-NED (near-epoch dependent) with size $-\frac{r-1}{r-2}$, where r from (A2), and constants $(c_t), t \in \mathbb{Z}$, on a sequence $(V_t), t \in \mathbb{Z}$, which is α-mixing of size $\phi^* := -\frac{r}{r-2}$, i.e.

$$||(X_t, Y_t) - \mathbb{E}\left((X_t, Y_t)|\sigma(V_{t-m}, \dots, V_{t+m})\right)||_2 \le c_t v_m$$

with $v_m \to 0$, such that

$$c_t \le 2||U_t||_2$$

with U_t from Assumption (A2) and the L_2-norm $||\cdot||_2$.

(A4) The moments $\mathbb{E}(X_t^2)$, $\mathbb{E}(Y_t^2)$, $\mathbb{E}(X_t)$, $\mathbb{E}(Y_t)$, $\mathbb{E}(X_tY_t)$ are uniformly bounded and "almost" constant, in the sense that the deviations d_t from the respective constants satisfy

$$\lim_{T\to\infty} \frac{1}{\sqrt{T}} \sum_{t=1}^{T} |d_t| = \lim_{T\to\infty} \frac{1}{T} \sum_{t=1}^{T} d_t^2 = 0.$$

Of course, (A4) allows for weak-stationary, i.e. $d_t = 0$ for all t. Note that our assumption (A3) is more general than the dependence assumption of Aue et al. (2009), because in their case, the $(V_t), t \in \mathbb{Z}$, have to be independent.
As an alternative to (A4), our main results also hold when variances vary more widely (although not arbitrarily widely), but are proportional to each other, as often happens in financial markets:

(A5) For a bounded function g that is not identically zero and that can be approximated by step functions,

$$\begin{aligned}\mathbb{E}(X_t^2) &= a_2 + a_2 \frac{1}{\sqrt{T}} g\left(\frac{t}{T}\right) \\ \mathbb{E}(Y_t^2) &= a_3 + a_3 \frac{1}{\sqrt{T}} g\left(\frac{t}{T}\right) \\ \mathbb{E}(X_tY_t) &= a_1 + a_1 \frac{1}{\sqrt{T}} g\left(\frac{t}{T}\right),\end{aligned}$$

while $\mathbb{E}(X_t)$ and $\mathbb{E}(Y_t)$ remain constant.

As an example, the function g might be piecewise constant with jumps in z_0 from 0 to g_0, which implies that the covariance changes in $[Tz_0]$. Assumption (A4) is violated because $\mathbb{E}(X_tY_t) = m_{xy} + d_t$ with

$$\lim_{T\to\infty} \frac{1}{\sqrt{T}} \sum_{t=1}^{T} |d_t| = \lim_{T\to\infty} \frac{1}{T} \sum_{t=1}^{t} \left| g\left(\frac{t}{T}\right) \right| = \int_0^1 |g(u)| du > 0.$$

In case of (A5), the correlation is constant and equal to $a_1/\sqrt{a_2 a_3}$. Note that we operate with a triangular array in this case (see also Section 3 and the beginning of Appendix A.2).

Assumption (A3) guarantees that

$$U_t^* := \left(X_t^2, \quad Y_t^2, \quad X_t, \quad Y_t, \quad X_tY_t\right)'$$

is L_2-NED with size $\frac{1}{2}$, see Davidson (1994). It could be modified to ϕ-mixing, requiring only finite 4-th moments, but this would admit less dependence than we allow here. In particular, assumption (A3) allows for GARCH-effects (see e.g. Hansen, 1991 or Carrasco and Chen, 2002), which are observed in financial data.

Given these assumptions, we next derive the limiting null distribution of our test statistic (1). To this purpose, we first rewrite it as $\sup_{0\leq z\leq 1} |K_T(z)|$ with

$$K_T(z) = \hat{D}\frac{\tau(z)}{\sqrt{T}}(\hat{\rho}_{\tau(z)} - \hat{\rho}_T), \tag{2}$$

where $\tau(z) = [2 + z(T-2)], z \in [0,1]$.

Theorem 1. *Under H_0 and assumptions (A1) - (A4) or (A1) - (A3) and (A5), we have*

$$\sup_{0\leq z\leq 1} |K_T(z)| \to_d \sup_{0\leq z\leq 1} |B(z)|,$$

where B is a one-dimensional Brownian bridge.

The explicit form of the distribution function of $\sup_{0\leq z\leq 1} |B(z)|$ is well known, see Billingsley (1968); its quantiles provide an asymptotic test.

The situation in (A5) explicitly allows for changing variances and covariances, while correlations remain constant. This setup is not included in the Aue et al. (2009)-test, because the test is based on a different null hypothesis.

As an illustration, consider testing for a break in correlation between S&P 500 and DAX returns, using daily data from early 2003 to the end of 2009. Panel a of Figure 1 shows the evolution of the right-hand-side of (1), with a maximum value of $Q_T = 2.593$. This is well beyond the 5% critical level (in fact, the corresponding p-value is less than 0.001). Interestingly enough, it is attained just after the Lehman-Brothers breakdown on September 15, 2008. Panel b of Figure 1 plots the evolution of the successively estimated correlations themselves. It is seen that these start at a high level, then decrease for a long time and increase again after the breakdown, approaching a final value of 0.612.

- Figure 1 here -

If we apply the correlation test for the time period early 2004 to the end of 2006, then the null hypothesis of constant correlation is not rejected ($Q_T = 0.546$ with a p-value of 0.927). This fits to the fact that this time period did not contain any dramatic economy changes which would have lead to changing correlations. However, the Aue et al. (2009)-test (applied with the critical values from Kiefer, 1959) rejects the null hypothesis of constant covariance matrix with a value of the test statistic of 3.409 (p-value less than 0.003) as well as the hypothesis of constant marginal variance of the DAX (3.194 with a p-value less than 0.004, respectively). Thus, both procedures give complementary information about the co-movement of this time series; with our test we conclude that the dependence structure remains constant, what the Aue et al. (2009)-test does not show. Note that there are of course other time periods where either both or none of the tests rejects.

The proof of Theorem 1 is in the appendix. Simply applying a standard functional central limit theorem as in Aue et al. (2009) is not possible in the present case: The proof relies on an adapted functional delta method argument which is presented and proven in the appendix. One major difficulty in the proof is that we first have to show convergence on the interval $[\epsilon, 1]$ for arbitrary $\epsilon > 0$ and then show that the statistic vanishes on the interval $[0, \epsilon]$ if ϵ tends to zero.

3 Local power

Next, we consider local alternatives of the form

$$\rho_{t,T} = \rho_0 + \frac{1}{\sqrt{T}} g\left(\frac{t}{T}\right) \quad t = 1, \ldots, T,$$

where g is as in (A5). Using a piecewise constant function g would lead to multiple change points as e.g. Inoue (2001) deals with. This form of local alternatives is similar to the ones in Ploberger and Krämer (1990) who analyze local power properties of the CUSUM and CUSUM of squares test. Now, the random vectors $(X_t, Y_t), t \in \mathbb{Z}$, and $(V_t), t \in \mathbb{Z}$, from assumption (A3) form a triangular array, but we stick to the former notation for simplicity, i.e. $(X_t, Y_t) := (X_{t,T}, Y_{t,T}), (V_t) := (V_{t,T}), t \in \mathbb{Z}; T = 1, 2, \ldots$. However, (A4) is replaced by

(A6) $\mathbb{E}(X_t Y_t) = m_{xy} + \frac{1}{\sqrt{T}} g\left(\frac{t}{T}\right)$,
$\mathbb{E}(X_t^2) = m_x^2, \mathbb{E}(Y_t^2) = m_y^2, \mathbb{E}(X_t) = \mu_x, \mathbb{E}(Y_t) = \mu_y$.

This is a special case of the general local alternative, but we stick to it for ease of exposition. As was shown for (A5), Assumption (A4) is violated here as well.

Theorem 2. *Under assumptions (A1) - (A3) and (A6),*

$$\sup_{z \in [0,1]} \left| \hat{D} \frac{\tau(z)}{\sqrt{T}} (\hat{\rho}_{\tau(z)} - \hat{\rho}_T) \right| \to_d \sup_{z \in [0,1]} |B(z) + C(z)|,$$

where $C(z)$ is a deterministic function which depends on the specific form of the local alternative under consideration, characterized by g.

With this theorem and Anderson's Lemma, we can deduce that the asymptotic level is always larger than or equal to α, see Andrews (1997) or Rothe and Wied (2011). The proof is in the appendix; it relies on similar arguments as the derivation of the null distribution.

The supremum is now taken over the absolute value of a Brownian bridge plus a deterministic function $C(z)$. Its distribution is rather unwieldy, but the local power of the test is easily established for large g. To this purpose, rewrite assumption (A6) as $g(z) = Mh(z)$ for a function h and a factor M. The function h represents the structural form of the alternative, whereas M captures its amplitude.

Corollary 1. *Let $P_{H_1}(M)$ be the rejection probability for given M under the alternative. Let $\epsilon > 0$ and h be arbitrary but not constant. Then there is a M_0 such that*

$$\lim_{T\to\infty} P_{H_1}(M) > 1 - \epsilon$$

for all $M > M_0$.

This means that local rejection probabilities become arbitrarily large as structural changes are increasing.

4 Some finite sample simulations

First, we check the test's finite sample null distribution. To that purpose, we consider both friendly (i.e. iid) and unfriendly environments defined by some serial correlation under the null. Both situations are encompassed by the $AR(1)$-model

$$\begin{pmatrix} X_t \\ Y_t \end{pmatrix} = \begin{pmatrix} \phi X_{t-1} \\ \phi Y_{t-1} \end{pmatrix} + \begin{pmatrix} \epsilon_t^1 \\ \epsilon_t^2 \end{pmatrix},$$

where $(\epsilon_t^1, \epsilon_t^2)'$ has correlation ρ and $(\epsilon_t^1\ \epsilon_t^2)'$ is iid-bivariate t_5 to make our data better resemble empirical data such as stock returns. The friendly environment is given by $\phi = 0$ (i.e. iid observations), the unfriendly environment is given by some nonzero value of ϕ which is compatible with our assumption (A3). Similar to Aue et al. (2009) we use $\phi = 0.1$. We also ran additional simulations for time-series correlations as large as $\phi = 0.8$, which are likewise allowed for under H_0, given our assumptions. As autocorrelations that large are usually not observed for stock or FX-returns, we do not dwell on the resulting figures here - they are available from the corresponding author upon request.

Table 1 gives our results for the cases relevant in practice, based on 5000 replications and a nominal significance level of five percent.

-Table 1 here -

The table shows that convergence to the nominal significance level is not monotone, and that the test overrejects quite dramatically for small samples and large values of the correlation coefficient. For correlations in the range that is relevant for e.g. financial applications, the performance is satisfactory, however. Similar results were obtained with GARCH-processes of various types and again, there is overrejection in small samples. Detailed results are available from the corresponding author upon request.

Next, we check the size-adjusted power of our test for the following alternatives, in which variances remain constantly equal to 1 and correlations change, and compare it to Aue et al. (2009) (maximum-based version, size-adjusted):

(1) $\rho_i = 0.5, i \leq \frac{T}{2}$, and $\rho_i = 0.7, i > \frac{T}{2}$,

(2) $\rho_i = 0.5, i \leq \frac{T}{4}$, and $\rho_i = 0.7, i > \frac{T}{4}$,

(3) $\rho_i = 0.5, i \leq \frac{T}{2}$, and $\rho_i = -0.5, i > \frac{T}{2}$,

(4) $\rho_i = 0.5, i \leq \frac{T}{4}$, and $\rho_i = -0.5, i > \frac{T}{4}$,

(5) $\rho_i = 0.5, i \leq \frac{T}{4}$, and $\rho_i = 0.7, \frac{T}{4} < i \leq \frac{3}{4}T$, and $\rho_i = 0.5, i > \frac{3}{4}T$.

Table 2 gives the results.

-Table 2 here -

It is seen that power increases rapidly with sample size and not surprisingly, it is highest when the break in ρ is largest (i.e ρ changes from -0.5 to 0.5). In general, the powers of both tests lie closely together, while the power of our test is often slightly higher than the power of the Aue et al. (2009)-test for rather small samples and in situations when the change points appears early and is always higher in the case of several change points.

5 Discussion

In this paper, we have proposed a fluctuation test for constant correlation which works under rather general assumptions. A major shortcoming of our test, which it shares with Aue et al. (2009), is the requirement of finite 4-th moments. Although it has by now been firmly established that second or even third moments of financial returns are finite, the existence of fourth moments remains doubtful, see Krämer (2002). If the 4-th moment of one or both of the components of (X_t, Y_t) does not exist, our functional central limit theorem, from which we derive the null distribution of our test, would not apply. As the asymptotic distribution of empirical cross-correlations would then be different (see e.g. **?**), this condition of finite fourth moments is also necessary for our limit results. A multivariate extension of our approach, i.e. testing for the constancy of a whole correlation matrix, is possible by doing pairwise comparisons and rejecting the null hypothesis e.g. if the maximum of the test statistics is too large. Circumventing the resulting multiple testing problem, however, requires some more theory which goes beyond the scope of the present paper. The problem might be overcome by relying on other multivariate dependence measures as in Schmid et al. (2010). A general approach based on multivariate empirical distributions can be found in Inoue (2001).

A Appendix

A.1 The scalar $\hat{D}$ from the test statistic

The scalar $\hat{D}$ from our test statistic (1) can be written as

$$\hat{D} = (\hat{F}_1\hat{D}_{3,1} + \hat{F}_2\hat{D}_{3,2} + \hat{F}_3\hat{D}_{3,3})^{-\frac{1}{2}}$$

where

$$\begin{pmatrix}\hat{F}_1 & \hat{F}_2 & \hat{F}_3\end{pmatrix} = \begin{pmatrix}\hat{D}_{3,1}\hat{E}_{11} + \hat{D}_{3,2}\hat{E}_{21} + \hat{D}_{3,3}\hat{E}_{31} \\ \hat{D}_{3,1}\hat{E}_{12} + \hat{D}_{3,2}\hat{E}_{22} + \hat{D}_{3,3}\hat{E}_{32} \\ \hat{D}_{3,1}\hat{E}_{13} + \hat{D}_{3,2}\hat{E}_{23} + \hat{D}_{3,3}\hat{E}_{33}\end{pmatrix}',$$

$$\hat{E}_{11} = \hat{D}_{1,11} - 4\hat{\mu}_x\hat{D}_{1,13} + 4\hat{\mu}_x^2\hat{D}_{1,33},$$

$$\hat{E}_{12} = \hat{E}_{21} = \hat{D}_{1,12} - 2\hat{\mu}_x\hat{D}_{1,23} - 2\hat{\mu}_y\hat{D}_{1,14} + 4\hat{\mu}_x\hat{\mu}_y\hat{D}_{1,34},$$

$$\hat{E}_{22} = \hat{D}_{1,22} - 4\hat{\mu}_y\hat{D}_{1,24} + 4\hat{\mu}_y^2\hat{D}_{1,44},$$

$$\hat{E}_{13} = \hat{E}_{31} = -\hat{\mu}_y\hat{D}_{1,13} + 2\hat{\mu}_x\hat{\mu}_y\hat{D}_{1,33} - \hat{\mu}_x\hat{D}_{1,14} + 2\hat{\mu}_x^2\hat{D}_{1,34} + \hat{D}_{1,15} - 2\hat{\mu}_x\hat{D}_{1,35},$$

$$\hat{E}_{23} = \hat{E}_{32} = -\hat{\mu}_y\hat{D}_{1,23} + 2\hat{\mu}_x\hat{\mu}_y\hat{D}_{1,44} - \hat{\mu}_x\hat{D}_{1,24} + 2\hat{\mu}_y^2\hat{D}_{1,34} + \hat{D}_{1,25} - 2\hat{\mu}_y\hat{D}_{1,45},$$

$$\hat{E}_{33} = \hat{\mu}_y^2\hat{D}_{1,33} + 2\hat{\mu}_x\hat{\mu}_y\hat{D}_{1,34} - 2\hat{\mu}_y\hat{D}_{1,35} + \hat{\mu}_x^2\hat{D}_{1,44} + \hat{D}_{1,55} - 2\hat{\mu}_x\hat{D}_{1,45},$$

$$\hat{D}_1 = \begin{pmatrix}\hat{D}_{1,11} & \hat{D}_{1,12} & \hat{D}_{1,13} & \hat{D}_{1,14} & \hat{D}_{1,15} \\ \hat{D}_{1,21} & \hat{D}_{1,22} & \hat{D}_{1,23} & \hat{D}_{1,24} & \hat{D}_{1,25} \\ \hat{D}_{1,31} & \hat{D}_{1,32} & \hat{D}_{1,33} & \hat{D}_{1,34} & \hat{D}_{1,35} \\ \hat{D}_{1,41} & \hat{D}_{1,42} & \hat{D}_{1,43} & \hat{D}_{1,44} & \hat{D}_{1,45} \\ \hat{D}_{1,51} & \hat{D}_{1,52} & \hat{D}_{1,53} & \hat{D}_{1,54} & \hat{D}_{1,55}\end{pmatrix} = \sum_{t=1}^{T}\sum_{u=1}^{T} k\left(\frac{t-u}{\gamma_T}\right) V_t V_u',$$

$$V_t = \frac{1}{\sqrt{T}}U_t^{***}, \gamma_T = [\log T],$$

$$U_t^{***} = \begin{pmatrix}X_t^2 - \overline{(X^2)}_T & Y_t^2 - \overline{(Y^2)}_T & X_t - \bar{X}_T & Y_t - \bar{Y}_T & X_tY_t - \overline{(XY)}_T\end{pmatrix}',$$

$$k(x) = \begin{cases}1 - |x|, & |x| \leq 1 \\ 0, & otherwise\end{cases},$$

$$\hat{\mu}_x = \bar{X}_T, \hat{\mu}_y = \bar{Y}_T, \hat{D}_{3,1} = -\frac{1}{2}\frac{\hat{\sigma}_{xy}}{\hat{\sigma}_y}\hat{\sigma}_x^{-3}, \hat{D}_{3,2} = -\frac{1}{2}\frac{\hat{\sigma}_{xy}}{\hat{\sigma}_x}\hat{\sigma}_y^{-3}, \hat{D}_{3,3} = \frac{1}{\hat{\sigma}_x\hat{\sigma}_y}$$

and

$$\hat{\sigma}_x^2 = \overline{(X^2)}_T - (\bar{X}_T)^2, \hat{\sigma}_y^2 = \overline{(Y^2)}_T - (\bar{Y}_T)^2, \hat{\sigma}_{xy} = \overline{(XY)}_T - \bar{X}_T\bar{Y}_T.$$

Proof. See the discussion preceding Lemma 3. □

A.2 Proof of Theorem 1 under (A1) - (A4)

The proof of Theorem 1 requires several lemmas. In all proofs, we assume $(X_t, Y_t), t \in \mathbb{Z}$, to be a triangular array, which generalizes the assumptions in Section 2 and corresponds to the assumptions in Section 3. The array is defined on the probability space $(\Omega, \mathfrak{A}, \mathbb{P})$.

Let I be some interval, e.g. $I = [\epsilon, 1]$ for some $\epsilon \in [0, 1)$. For an integer $k \geq 1$, let $D(I, \mathbb{R}^k)$ be the set of all functions $\theta : I \to \mathbb{R}^k$ which are càdlàg in each of the k components, equipped with the multi-dimensional supremum norm

$$||\theta||_\infty := \sup_{z \in T} ||\theta(z)||,$$

where $||\cdot||$ denotes the maximum norm in $\mathbb{R}^k$. Let in addition $m_x^2, m_y^2, \mu_x, \mu_y, m_{xy}$ be the respective constants from (A4) and

$$\sigma_x := \sqrt{m_x^2 - \mu_x^2}, \sigma_y := \sqrt{m_y^2 - \mu_y^2}, \sigma_{xy} := m_{xy} - \mu_x\mu_y.$$

The first lemma is a straightforward application of the functional central limit theorem in Davidson (1994, p. 492) which relies on a univariate invariance principle from Wooldridge and White (1988).

Lemma 1. *On $D([\epsilon, 1], \mathbb{R}^5)$, for arbitrary $\epsilon > 0$, we have*

$$\frac{1}{\sqrt{T}}\sum_{t=1}^{\tau(\cdot)} U_t^{**} := \frac{1}{\sqrt{T}}\sum_{t=1}^{\tau(\cdot)} \begin{pmatrix} X_t^2 - m_x^2 \\ Y_t^2 - m_y^2 \\ X_t - \mu_x \\ Y_t - \mu_y \\ X_tY_t - m_{xy} \end{pmatrix} = \frac{\tau(\cdot)}{\sqrt{T}} \begin{pmatrix} \overline{(X^2)}_{\tau(\cdot)} & - & m_x^2 \\ \overline{(Y^2)}_{\tau(\cdot)} & - & m_y^2 \\ \bar{X}_{\tau(\cdot)} & - & \mu_x \\ \bar{Y}_{\tau(\cdot)} & - & \mu_y \\ \overline{(XY)}_{\tau(\cdot)} & - & m_{xy} \end{pmatrix} =: U(\cdot) \to_d D_1^{\frac{1}{2}}\ W_5(\cdot),$$

where $W_k(\cdot)$ is a k-dimensional Brownian Motion and

$$D_1 = D_1' = \lim_{T\to\infty} \frac{1}{T} \sum_{t=1}^{T} \sum_{u=1}^{T}$$

$$\begin{pmatrix} Cov(X_t^2, X_u^2) & Cov(X_t^2, Y_u^2) & Cov(X_t^2, X_u) & Cov(X_t^2, Y_u) & Cov(X_t^2, X_uY_u) \\ \cdot & Cov(Y_t^2, Y_u^2) & Cov(Y_t^2, X_u) & Cov(Y_t^2, Y_u) & Cov(Y_t^2, X_uY_u) \\ \cdot & \cdot & Cov(X_t, X_u) & Cov(X_t, Y_u) & Cov(X_t, X_uY_u) \\ \cdot & \cdot & \cdot & Cov(Y_t, Y_u) & Cov(Y_t, X_uY_u) \\ \cdot & \cdot & \cdot & \cdot & Cov(X_tY_t, X_uY_u) \end{pmatrix}.$$

Proof.

$$\frac{1}{\sqrt{T}} \sum_{t=1}^{\tau(z)} \begin{pmatrix} X_t^2 - m_x^2 \\ Y_t^2 - m_y^2 \\ X_t - \mu_x \\ Y_t - \mu_y \\ X_tY_t - m_{xy} \end{pmatrix} = \frac{1}{\sqrt{T}} \sum_{t=1}^{\tau(z)} \begin{pmatrix} X_t^2 - \mathbb{E}(X_t^2) \\ Y_t^2 - \mathbb{E}(Y_t^2) \\ X_t - \mathbb{E}(X_t) \\ Y_t - \mathbb{E}(Y_t) \\ X_tY_t - \mathbb{E}(X_tY_t) \end{pmatrix} + \frac{1}{\sqrt{T}} \sum_{t=1}^{\tau(z)} \begin{pmatrix} \mathbb{E}(X_t^2) - m_x^2 \\ \mathbb{E}(Y_t^2) - m_y^2 \\ \mathbb{E}(X_t) - \mu_x \\ \mathbb{E}(Y_t) - \mu_y \\ \mathbb{E}(X_tY_t) - m_{xy} \end{pmatrix}$$

$$=: A_1 + A_2.$$

Consider the first component of A_2:

$$A_{2,1}(z) = \frac{1}{\sqrt{T}} \sum_{t=1}^{\tau(z)} \left(\mathbb{E}(X_t^2) - m_x^2\right).$$

We have

$$||A_{2,1}(z)||_\infty \leq \frac{1}{\sqrt{T}} \sum_{t=1}^{T} \left|\mathbb{E}(X_t^2) - m_x^2\right|.$$

With assumption (A4), the righthand side tends to zero. In the same way, all other components of A_2 converge to zero, hence also A_2.
The sum in A_1 can be separated into one term from $i = 1$ to $[Tz]$ called A_3 and one term from $[Tz + 1]$ to $[Tz + (1 - z)2]$, called A_4. We show that A_4 converges in probability to the zero function in the supremum norm. To this end, we first show that for fixed z, A_4 converges to zero in probability. If

$$[Tz + (1 - z)2] < [Tz + 1],$$

A_4 is equal to zero. For

$$[Tz + (1-z)2] \geq [Tz + 1]$$

our argument builds on the Markov inequality. A_4 consists of two terms at most so that we have for the first component A_{41}

$$\sup_{z\in[\epsilon,1]} |A_{41}(z)| \leq \frac{2}{\sqrt{T}} \max_{1\leq t\leq T} |X_t^2 - \mathbb{E}(X_t^2)|.$$

Now, for arbitrary $\delta > 0$

$$\mathbb{P}(\sup_{z\in[\epsilon,1]} |A_{41}(z)| > \delta) \leq T \max_{1\leq t\leq T} \mathbb{P}\left(|X_t^2 - \mathbb{E}(X_t^2)| > \frac{\sqrt{T}}{2\delta}\right).$$

By Assumption (A2) $\mathbb{E}|X_t^2 - \mathbb{E}(X_t^2)|^r$ is uniformly bounded for $r > 2$. With Markov's inequality the righthand side converges to 0.

The same argument applies for the other components of A_4. Therefore, all finite-dimensional distributions converge to zero in probability and thus in distribution. We show the tightness of the process similar to the method on page 138 in Billingsley (1968). At first, we show the tightness of every single component (without loss of generality for the first); the tightness of the whole vector follows.

$$B := \mathbb{E}\left(|A_{41}(z) - A_{41}(z_1)|^{1+\frac{\alpha}{2}} \cdot |A_{41}(z_2) - A_{41}(z)|^{1+\frac{\alpha}{2}}\right) \leq \frac{1}{T^{1+\frac{\alpha}{2}}} C$$

for $\epsilon \leq z_1 \leq z \leq z_2 \leq 1$ and a constant C because of uniform boundedness. If $[Tz_2] - [Tz_1] = 0$, then $B = 0$. If $[Tz_2] - [Tz_1] \geq 1$, we get

$$\frac{1}{T^{1+\frac{\alpha}{2}}} C \leq ([Tz_2] - [Tz_1])^{1+\frac{\alpha}{2}} \frac{1}{T^{1+\frac{\alpha}{2}}} C = C\left(\frac{[Tz_2] - [Tz_1]}{T}\right)^{1+\frac{\alpha}{2}}$$

and the condition of Theorem 15.6 in Billingsley (1968) holds. Thus, A_4 as a process converges in distribution (and also in probability) to the zero function. We apply to A_3 the multivariate invariance principle from Davidson (1994, p. 492) which relies on a univariate invariance principle from Wooldridge and White (1988). The value $c_{T,i}^{\lambda}$ in this theorem is given by $(\lambda' D_1^{-1} T^{-1} \lambda)^{\frac{1}{2}}$ in our case. The theorem is actually given for a function space equipped with Skorohod metric, but it also holds in our uniform topology, because the limit process is continuous almost surely, see Gill (1989), p. 106, for details.

With the Continuous Mapping Theorem, CMT, see van der Vaart (1998, p. 259), the lemma follows. □

Lemma 2. *On $D([\epsilon, 1], \mathbb{R})$, for arbitrary $\epsilon > 0$,*

$$\frac{\tau(\cdot)}{\sqrt{T}}(\hat{\rho}_{\tau(\cdot)} - \rho_0^*) \to_d D_3 D_2 D_1^{\frac{1}{2}} \; W_5(\cdot),$$

where

$$\rho_0^* = \frac{\sigma_{xy}}{\sigma_x \sigma_y},$$

$$D_2 = \begin{pmatrix} 1 & 0 & -2\mu_x & 0 & 0 \\ 0 & 1 & 0 & -2\mu_y & 0 \\ 0 & 0 & -\mu_y & -\mu_x & 1 \end{pmatrix} \text{ and}$$

$$D_3 = \begin{pmatrix} -\frac{1}{2}\frac{\sigma_{xy}}{\sigma_y}\sigma_x^{-3} & -\frac{1}{2}\frac{\sigma_{xy}}{\sigma_x}\sigma_y^{-3} & \frac{1}{\sigma_x \sigma_y} \end{pmatrix}.$$

The proof of Lemma 2 relies on a functional delta method argument which we present in a form slightly more general than is actually needed.

Theorem 3 (Delta method)**.** *Consider a sequence $(\theta_T)_T$ of functions in $D(I, \mathbb{R}^k)$ converging uniformly to a function $\theta \in D(I, \mathbb{R}^k)$. Furthermore, let $(s_T)_T$ be a sequence of functions $s_T : I \to \mathbb{R}\backslash\{0\}$ such that $||s_T^{-1}||_\infty \to 0$, and let M_T be stochastic processes on I with values in $\mathbb{R}^k$ and bounded sample paths such that*

$$||Z_T||_\infty = O_p(1) \text{ with } Z_T := s_T(M_T - \theta_T).$$

Furthermore, let $f : \mathbb{R}^k \to \mathbb{R}^l$ be a mapping which is continuously differentiable on an open set $\Omega \subset \mathbb{R}^k$. Suppose that

$$\overline{\theta(I)} \text{ is a compact subset of } \Omega,$$

where $\overline{\theta(I)}$ stands for the closure of the set $\{\theta(t) : t \in I\}$ in $\mathbb{R}^k$. Then it holds

1. $s_T(\cdot)\left(f(M_T(\cdot)) - f(\theta_T(\cdot))\right) = Df(\theta(\cdot))Z_T(\cdot) + R_T$
   with a stochastic process such that

   $$||R_T||_\infty = o_p(1).$$

2. *If Z_T also converges in distribution (in $D(I, \mathbb{R}^k)$) to a stochastic process Z, then*

   $$s_T(\cdot)\left(f(M_T(\cdot)) - f(\theta_T(\cdot))\right) \to_d Df(\theta(\cdot))Z(\cdot).$$

Proof. Assertion 2 immediately follows from Assertion 1 with the usual continuous mapping theorem.

To prove the expansion from Assertion 1, note that for any $z \in I$,

$$\begin{aligned} R_T(z) &:= s_T(z)\left(f(M_T(z)) - f(\theta_T(z))\right) - Df(\theta(z))Z_T(z) \\ &= s_T(z)\left(f\left(\theta_T(z) + s_T^{-1}(z)Z_T(z)\right) - f(\theta_T(z))\right) - Df(\theta(z))Z_T(z) \\ &= \int_0^1 Df\left(\theta_T(z) + us_T^{-1}(z)Z_T(z)\right) Z_T(z)du - Df(\theta(z))Z_T(z) \\ &= \int_0^1 \left(Df\left(\theta_T(z) + us_T^{-1}(z)Z_T(z)\right) - Df(\theta(z))\right) du \cdot Z_T(z), \end{aligned} \tag{3}$$

provided that

$$r_T := ||\theta_T - \theta||_\infty + ||s_T^{-1}||_\infty ||Z_T||_\infty = o_p(1)$$

is smaller than

$$\rho := \inf_{x \in \overline{\theta(I)}, y \in \mathbb{R}^k \setminus \Omega} ||x - y|| > 0.$$

The latter condition is needed for (3) to be well defined.

Hence

$$||R_T||_\infty \leq \sup\left\{||Df(y) - Df(x)|| : x \in \overline{\theta(I)}, y \in \mathbb{R}^k, ||y - x|| \leq r_T\right\} \cdot ||Z_T||_\infty. \tag{4}$$

Here $||Df(y) - Df(x)||$ is the usual operator norm of the matrix $Df(y) - Df(x)$ in case of $y \in \Omega$. (In case of $y \notin \Omega$ define $||Df(y) - Df(x)|| = \infty$.) One can easily deduce from continuity of $Df(\cdot)$ on Ω, compactness of $\overline{\theta(T)} \in \Omega$ and $r_T = o_p(1)$ that the right hand side of (4) converges to zero in probability. □

Proof of Lemma 2

We apply the generalized delta method from Theorem 3.2 twice to $U(\cdot)$ with $I = [\epsilon, 1]$ for arbitrary $\epsilon > 0$ and $s_T(\cdot) = \frac{\tau(z)}{\sqrt{T}}$. The first transformation of $U(\cdot)$ is

$$f_1 : \mathbb{R}^5 \to \mathbb{R}^3,$$

$$f_1(x_1, x_2, x_3, x_4, x_5) = \begin{pmatrix} x_1 - x_3^2 \\ x_2 - x_4^2 \\ x_5 - x_3 x_4 \end{pmatrix}$$

with, for $\theta = (\theta_1, \theta_2, \theta_3, \theta_4, \theta_5)$,

$$Df_1(\theta) = \begin{pmatrix} 1 & 0 & -2\theta_3 & 0 & 0 \\ 0 & 1 & 0 & -2\theta_4 & 0 \\ 0 & 0 & -\theta_4 & -\theta_3 & 1 \end{pmatrix}.$$

The second transformation is

$$f_2 : \mathbb{R}^3 \to \mathbb{R},$$
$$f_2(x_1, x_2, x_3) = \frac{x_3}{\sqrt{x_1 x_2}}$$

with, for $\theta = (\theta_1, \theta_2, \theta_3)$,

$$Df_2(\theta) = \begin{pmatrix} -\frac{1}{2}\frac{\theta_3}{\sqrt{\theta_2}}\theta_1^{-\frac{3}{2}} & -\frac{1}{2}\frac{\theta_3}{\sqrt{\theta_1}}\theta_2^{-\frac{3}{2}} & \frac{1}{\sqrt{\theta_1\theta_2}} \end{pmatrix}.$$

The lemma follows then from the fact that $U(\cdot)$ converges in distribution to the stochastic process $D_1^{\frac{1}{2}}\, W_5(\cdot)$. $\square$

We need the restriction on $[\epsilon, 1]$ because the function $r_T(z) = \frac{\sqrt{T}}{\tau(z)}$ would not tend to 0 in the supremum norm on $[0, 1]$.

Now, one can show that

$$(D_3 D_2 D_1 D_2' D_3')^{-\frac{1}{2}} \frac{\tau(\cdot)}{\sqrt{T}}(\hat{\rho}_{\tau(\cdot)} - \rho_0^*) \to_d W_1(\cdot)$$

on $D[\epsilon, 1]$, where W_1 is a one-dimensional Brownian Motion. The parameter $(D_3 D_2 D_1 D_2' D_3')^{-\frac{1}{2}}$ has to be estimated consistently. It is a continuous composition of moments of X_i and Y_i that appear in the matrices D_3 and

$$E = \begin{pmatrix} E_{11} & E_{12} & E_{13} \\ E_{21} & E_{22} & E_{23} \\ E_{31} & E_{32} & E_{33} \end{pmatrix} := D_2 D_1 D_2'$$

with

$$E_{11} = D_{1,11} - 4\mu_x D_{1,13} + 4{\mu_x}^2 D_{1,34},$$

$$E_{12} = E_{21} = D_{1,12} - 2\mu_x D_{1,23} - 2\mu_y D_{1,14} + 4\mu_x\mu_y D_{1,34},$$

$$E_{22} = D_{1,22} - 4\mu_y D_{1,24} + 4{\mu_y}^2 D_{1,44},$$

$$E_{13} = E_{31} = -\mu_y D_{1,13} + 2\mu_x\mu_y D_{1,33} - \mu_x D_{1,14} + 2{\mu_x}^2 D_{1,34} + D_{1,15} - 2\mu_x D_{1,35},$$

$$E_{23} = E_{32} = -\mu_y D_{1,23} + 2\mu_x\mu_y D_{1,44} - \mu_x D_{1,24} + 2{\mu_y}^2 D_{1,34} + D_{1,25} - 2\mu_y D_{1,45},$$

$$E_{33} = -{\mu_y}^2 D_{1,33} + 2\mu_x\mu_y D_{1,34} - 2\mu_y D_{1,35} + {\mu_x}^2 D_{1,44} + D_{1,55} - 2\mu_x D_{1,45}.$$

In addition,

$$D_3 D_2 D_1 D_2' = \begin{pmatrix} D_{3,1}E_{11} + D_{3,2}E_{21} + D_{3,3}E_{31} \\ D_{3,1}E_{12} + D_{3,2}E_{22} + D_{3,3}E_{32} \\ D_{3,1}E_{13} + D_{3,2}E_{23} + D_{3,3}E_{33} \end{pmatrix}'$$
$$=: \begin{pmatrix} F_1 & F_2 & F_3 \end{pmatrix}$$

and

$$(D_3 D_2 D_1 D_2' D_3')^{-\frac{1}{2}} = (F_1 D_{3,1} + F_2 D_{3,2} + F_3 D_{3,3})^{-\frac{1}{2}}.$$

Thus, $(D_3 D_2 D_1 D_2' D_3')^{-\frac{1}{2}}$ is a continuous composition of moments of X_i and Y_i from the matrices D_3 and E. The only non-trivial task is to estimate the matrix D_1 consistently because the other elements can easily be estimated by Maximum-Likelihood. We solve the problem with a kernel estimator proposed by de Jong and Davidson (2000) using the bandwidth $\gamma_T = [\log T]$ and the Bartlett-kernel $k(\cdot)$ with

$$k(x) = \begin{cases} 1 - |x|, & |x| \leq 1 \\ 0, & otherwise \end{cases}.$$

Other choices would also be possible, but we restrict to our choice for ease of exposition. Because of assumption (A4), D_1 is asymptotically equivalent to

$$\hat{\Sigma}_T = \sum_{t=1}^{T} \sum_{u=1}^{T} k\left(\frac{t-u}{\gamma_T}\right) V_t {V_u}'$$

with $V_t = \frac{1}{\sqrt{T}} U_t^{**}$ and U_t^{**} from Lemma 1. The vector U_t^{**} depends on $\theta_0 = \begin{pmatrix} m_x^2 & m_y^2 & \mu_x & \mu_y & m_{xy} \end{pmatrix} \in (\mathbb{R} \cap (0,\infty))^2 \times \mathbb{R}^3$; a consistent estimator for this is the sequence

$$\theta_T = \begin{pmatrix} \overline{(X^2)}_T & \overline{(Y^2)}_T & \bar{X}_T & \bar{Y}_T & \overline{(XY)}_T \end{pmatrix}.$$

Thus, from de Jong and Davidson (2000), we get a consistent estimator given in Appendix A.1.

Now, we extend the convergence result to the interval $[0,1]$.

Lemma 3. *On* $D([0,1],\mathbb{R})$,

$$W_T(\cdot) := \hat{D} \frac{\tau(\cdot)}{\sqrt{T}} (\hat{\rho}_{\tau(\cdot)} - \rho_0^*) \to_d W_1(\cdot).$$

Proof. Consider the following functions:

$$W_T^\epsilon(z) = \begin{cases} W_T(z), & z \geq \epsilon \\ 0 & z < \epsilon \end{cases},$$

$$W^\epsilon(z) = \begin{cases} W_1(z), & z \geq \epsilon \\ 0 & z < \epsilon \end{cases}.$$

The previous lemmas then imply that

$$W_T^\epsilon(\cdot) \to_d W^\epsilon(\cdot)$$

for $T \to \infty$ on $D[0,1]$ and also

$$W^\epsilon(\cdot) \to_d W_1(\cdot)$$

for rational $\epsilon \to 0$ on $D[0,1]$. The convergence of $W_T(\cdot)$ on $D([0,1],\mathbb{R})$ then follows from Theorem 4.2 in Billingsley (1968) if we can show that

$$\lim_{\epsilon\to 0}\limsup_{T\to\infty}\mathbb{P}(\sup_{z\in[0,1]}|W_T^\epsilon(z) - W_T(z)| \geq \eta) = \lim_{\epsilon\to 0}\limsup_{T\to\infty}\mathbb{P}(\sup_{z\in[0,\epsilon]}|W_T(z)| \geq \eta) = 0$$

for all $\eta > 0$. Note that the separability condition of this theorem is not necessary in our case, because $\sup_{z\in I}|A(z)|$ is always a random variable when $A(\cdot)$ is a right-continuous random function. Now,

$$\begin{aligned}
&\sup_{z\in[0,\epsilon]}|W_T(z)| \\
&= \sup_{z\in[0,\epsilon]}\left|\frac{\frac{1}{\sqrt{T}}\sum_{t=1}^{\tau(z)}(X_t - \bar{X}_{\tau(z)})(Y_t - \bar{Y}_{\tau(z)}) - \frac{\rho_0}{\sqrt{T}}\sqrt{\frac{1}{\tau(z)}\sum_{t=1}^{\tau(z)}(X_t - \bar{X}_{\tau(z)})^2\frac{1}{\tau(z)}\sum_{t=1}^{\tau(z)}(Y_t - \bar{Y}_{\tau(z)})^2}}{\sqrt{\frac{1}{\tau(z)}\sum_{t=1}^{\tau(z)}(X_t - \bar{X}_{\tau(z)})^2\frac{1}{\tau(z)}\sum_{t=1}^{\tau(z)}(Y_t - \bar{Y}_{\tau(z)})^2}}\right| \\
&=: \sup_{z\in[0,\epsilon]}\left|\frac{D_1(z)}{D_2(z)}\right|.
\end{aligned}$$

By the strong law of large numbers and the CMT, D_2 goes to $\sigma_x\sigma_y$ almost surely for fixed $z > 0$. The same holds for $\bar{X}_T$ and $\bar{Y}_T$ with the limit μ_x and μ_y. Now let $\delta > 0$ be arbitrary. By Egoroff's Theorem, see Davidson (1994, theorem 18.4), there is a set $\Omega_\delta \subset \Omega$ with $\mathbb{P}(\Omega_\delta) \geq 1 - \delta$ and a number $M(\delta) > 0$ so that $|D_1(z) - \sigma_x\sigma_y| < \delta$,

$|\bar{X}_{\tau(z)} - \mu_x| < \delta$ and $|\bar{Y}_{\tau(z)} - \mu_y| < \delta$ on Ω_δ for $\tau(z) \geq M(\delta)$. Hence, for $z \geq \frac{M(\delta)}{T}$, for large enough T,

$$\sup_{z\in[\frac{M(\delta)}{T},\epsilon]} \left|\frac{1}{D_2(z)}\right| \leq \frac{1}{\sigma_x\sigma_y - \delta} < \infty.$$

Straightforward calculation yields

$$\sup_{z\in[\frac{M(\delta)}{T},\epsilon]} |D_1(z)| \leq C_1(\delta) \sup_{z\in[\frac{M(\delta)}{T},\epsilon]} D_3(z)$$

for some constant $C_1(\delta)$, where $D_3(z)$ is the sum of finitely many functions $D_3^i(z)$ with

$\sup_{z\in[\frac{M(\delta)}{T},\epsilon]} |D_3^i(z)| \to_d \sup_{z\in[0,\epsilon]} |W_1(z)|$.

We have

$$\sup_{z\in[0,\frac{M(\delta)}{T}]} |W_T(z)| \leq \frac{C_2(\delta)}{\sqrt{T}} \to 0$$

for a constant $C_2(\delta)$.

Since $W(0) = 0$ $\mathbb{P}$-almost everywhere, we have

$$\lim_{\epsilon\to 0} \limsup_{T\to\infty} \mathbb{P}(\sup_{z\in[0,\epsilon]} |W_T(z)| \geq \eta) = 0$$

on Ω_δ. Since $\delta > 0$ was arbitrary, the lemma follows. □

Lemma 4. *On* $D([0,1],\mathbb{R})$,

$$B_T(\cdot) := \hat{D}\frac{\tau(\cdot)}{\sqrt{T}}(\hat{\rho}_{\tau(\cdot)} - \rho_T) \to_d B(\cdot),$$

where $B(\cdot)$ *is a one-dimensional Brownian bridge.*

Proof. Define

$$W_T(\cdot) := \hat{D}\frac{\tau(\cdot)}{\sqrt{T}}(\hat{\rho}_{\tau(\cdot)} - \rho_0^*)$$

and

$$B_T(z) = W_T(z) - \frac{\tau(z)}{T}W_T(1) =: h\left(W_T(z), \frac{\tau(z)}{T}\right).$$

Since $\frac{\tau(z)}{T}$ converges to z, the lemma follows with the CMT and the definition of the Brownian bridge. □

Applying the CMT another time proves Theorem 1.

A.3 Proofs of the local power

Proof of Theorem 2

Transferring the proof of Lemma 1, we obtain that $U(\cdot)$ converges to $D_1^{\frac{1}{2}}\ W_5(\cdot) + A$ with $A = \begin{pmatrix} 0 & 0 & 0 & 0 & \int_0^z g(u)du \end{pmatrix}'$ because

$$A_2 = \frac{1}{\sqrt{T}} \sum_{i=1}^{\tau(z)} \begin{pmatrix} 0 & 0 & 0 & 0 & \frac{1}{\sqrt{T}} g(\frac{i}{T}) \end{pmatrix}'.$$

The fifth component converges as a process to the deterministic function $\int_0^{\cdot} g(u)du$. Also all other proofs can be transferred and it holds

$$\begin{aligned} \frac{\tau(\cdot)}{\sqrt{T}}(\hat{\rho}_{\tau(\cdot)} - \rho_0^*) &\to_d D_3 D_2 D_1^{\frac{1}{2}}\ W_5(\cdot) + D_3 D_2 A \\ &\stackrel{L}{=} (D_3 D_2 D_1 D_2' D_3')^{\frac{1}{2}} W_1(\cdot) + D_3 D_2 A \end{aligned}$$

and

$$\begin{aligned} (D_3 D_2 D_1 D_2' D_3')^{-\frac{1}{2}} \frac{\tau(\cdot)}{\sqrt{T}}(\hat{\rho}_{\tau(\cdot)} - \rho_0^*) &\to_d W_1(\cdot) + (D_3 D_2 D_1 D_2' D_3')^{-\frac{1}{2}} D_3 D_2 A \\ &\stackrel{L}{=} W_1(\cdot) + (D_3 D_2 D_1 D_2' D_3')^{-\frac{1}{2}} \cdot \frac{\int_0^{\cdot} g(u)du}{\sigma_x \sigma_y}. \end{aligned}$$

The estimator $\hat{D}$ converges in probability to $(D_3 D_2 D_1 D_2' D_3')^{-\frac{1}{2}}$. Thus,

$$\hat{D} \frac{\tau(\cdot)}{\sqrt{T}}(\hat{\rho}_{\tau(\cdot)} - \rho_T) \to_d B(\cdot) + C(\cdot),$$

where

$$C(z) = \frac{(D_3 D_2 D_1 D_2' D_3')^{-\frac{1}{2}}}{\sigma_x \sigma_y} \left(\int_0^z g(u)du - z \int_0^1 g(u)du \right),$$

a deterministic function depending on z.

Proof of Corollary 1

Similar to the proof of Theorem 2, we have

$$\begin{aligned} &\sup_{z\in[0,1]} \left| \hat{D} \frac{\tau(z)}{\sqrt{T}}(\hat{\rho}_{\tau(z)} - \hat{\rho}_T) \right| \to_d \sup_{z\in[0,1]} |B(z) + MC(z)| \\ &= M \sup_{z\in[0,1]} \left| \frac{B(z)}{M} + C(z) \right|, \end{aligned}$$

where $C(z) \neq 0$ for at least one z. Hence,

$$M \sup_{z\in[0,1]} \left| \frac{B(z)}{M} + C(z) \right| \geq MC_2$$

for a constant C_2. Thus, the test statistic becomes arbitrarily large, in particular, larger than every quantile of the distribution under H_0.

It is necessary that h is not constant because the test statistic would be equal to $\sup_{z\in[0,1]} |B(z)|$ otherwise. Since we integrate Mh from 0 to 1, even late structural changes are detected asymptotically if M is sufficiently large.

A.4 Proof of Theorem 1 under (A1) - (A3) and (A5)

Analogously to A.3, we transfer the proof of Lemma 1 with

$$A_2 = \frac{1}{\sqrt{T}} \sum_{i=1}^{\tau(z)} \Big(a_2 \tfrac{1}{\sqrt{T}} g(\tfrac{i}{T}) \quad a_3 \tfrac{1}{\sqrt{T}} g(\tfrac{i}{T}) \quad 0 \quad 0 \quad a_1 \tfrac{1}{\sqrt{T}} g(\tfrac{i}{T}) \Big)'.$$

Straightforward calculation yields that $C(z)$ then equals to 0.

References

Andrews, D., 1997. A conditional Kolmogorov test. Econometrica 65 (5), 1097–1128.

Aue, A., Hörmann, S., Horvath, L., Reimherr, M., 2009. Break detection in the covariance structure of multivariate time series models. Annals of Statistics 37(6B), 4046–4087.

Billingsley, P., 1968. Convergence of probability measures. Wiley, New York.

Carrasco, M., Chen, X., 2002. Mixing and moment properties of various GARCH and stochastic volatility models. Econometric Theory 18, 17–39.

Davidson, J., 1994. Stochastic limit theory. Oxford University Press.

de Jong, R., Davidson, J., 2000. Consistency of kernel estimators of heteroscedastic and autocorrelated covariance matrices. Econometrica 68(2), 407–424.

Dias, A., Embrechts, P., 2004. Change point analysis for dependence structures in finance and insurance. In: Szegö, G. (Ed.), Risk measures of the 21th century. Wiley, New York.

Forbes, K., Rigobon, R., 2002. No contagion, only interdependence: Measuring stock market comovements. Journal of Finance 57, 2223–2261.

Gill, R., 1989. Non- and semi-parametric maximum likelihood estimators and the von Mises method - Part 1. Scandinavian Journal of Statistics 16, 97–128.

Goetzmann, W., Li, L., Rouwenhorst, K., 2005. Long-term global market correlations. Journal of Business 78(1), 1–38.

Hansen, B., 1991. GARCH(1,1) processes are near-epoch dependent. Economics Letters 36, 181–186.

Inoue, A., 2001. Testing for distributional change in time series. Econometric Theory 17, 156–187.

Jennrich, R., 1970. An asymptotic chi-square test for the equality of two correlation matrices. Journal of the American Statistical Association 65, 904–912.

Kiefer, J., 1959. K-sample analogues of the kolmogorov-smirnov and cramer-v. mises tests. Annals of Mathematical Statistics 30(2), 420–447.

Krishan, C., Petkova, R., Ritchken, P., 2009. Correlation risk. Journal of Empirical Finance 16, 353–367.

Krämer, W., 2002. Statistische Besonderheiten von Finanzzeitreihen. Jahrbücher für Nationalökonomie und Statistik 222, 210–229.

Krämer, W., Schotman, P., 1992. Range vs. maximum in the OLS-based version of the CUSUM test. Economics Letters 40, 379–381.

Longin, F., Solnik, B., 1995. Is the correlation in international equity returns constant: 1960-1990? International Money and Finance 14(1), 3–26.

McAleer, M., Chan, F., Hoti, S., Lieberman, O., 2008. Generalized autoregressive conditional correlation. Econometric Theory 24, 1554–1583.

Mikosch, T., Resnick, S., Samarodnitsky, G., 2000. The maximum of the periodogram for a heavy-tailed sequence. Annals of Probability 28, 885–908.

P. Galeano and D. Peña, 2007. Covariance changes detection in multivariate time series. Journal of Statistical Planning and Inference 137(1), 194–211.

Pearson, E., Wilks, S., 1933. Methods of statistical analysis appropriate for k samples of two variables. Biometrika 25, 353–378.

Ploberger, W., Krämer, W., 1990. The local power of the cusum and cusum of squares tests. Econometric Theory 6, 335–347.

Ploberger, W., Krämer, W., 1992. The CUSUM-test with OLS residuals. Econometrica 60(2), 271–285.

Ploberger, W., Krämer, W., Kontrus, K., 1989. A new test for structural stability in the linear regression model. Journal of Econometrics 40, 307–318.

Rothe, C., Wied, D., 2011. Misspecification testing in a class of conditional distributional models. SFB 823 Discussion paper, TU Dortmund 03/11.

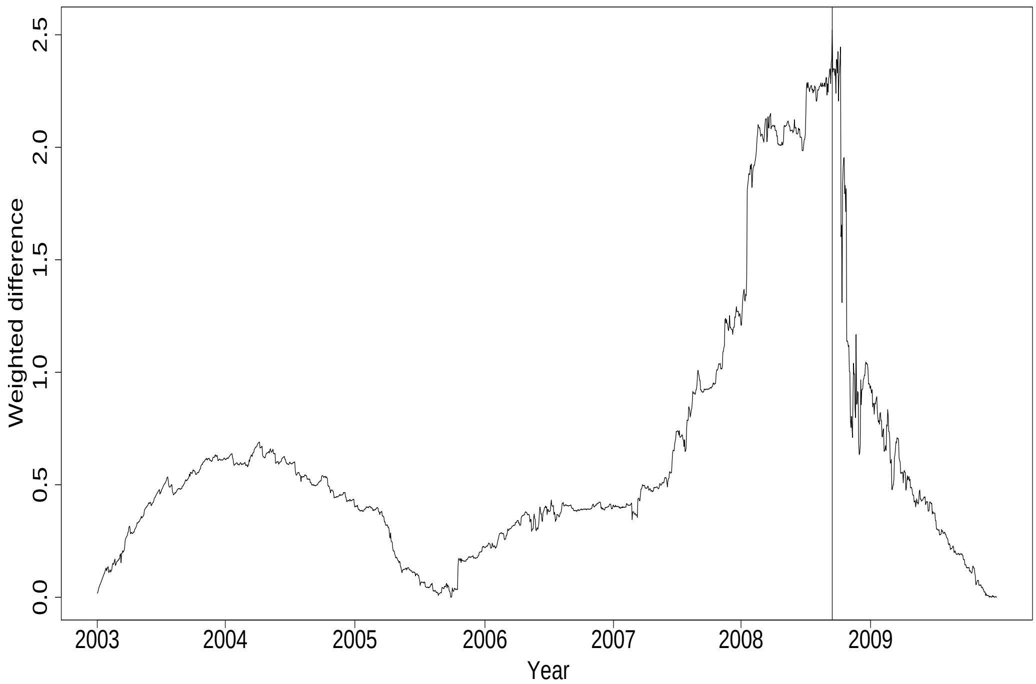
Schmid, F., Schmidt, R., Blumentritt, T., Gaißler, S., Ruppert, M., 2010. Copula-based measures of multivariate association. In: Durante, F., Härdle, W., Jaworski, P., Rychlik, T. (Eds.), Workshop on Copula Theory and its Applications. Springer, Berlin/Heidelberg.

van der Vaart, A., 1998. Asymptotic statistics. Cambridge University Press, Cambridge.

Wooldridge, J., White, H., 1988. Some invariance principles and central limit theorems for dependent heterogeneous processes. Econometric Theory 4, 210–230.

Figure 1: Evolution of cumulated deviations and successive empirical correlations

(a) Evolution of cumulated deviations



(b) Evolution of successive empirical correlations

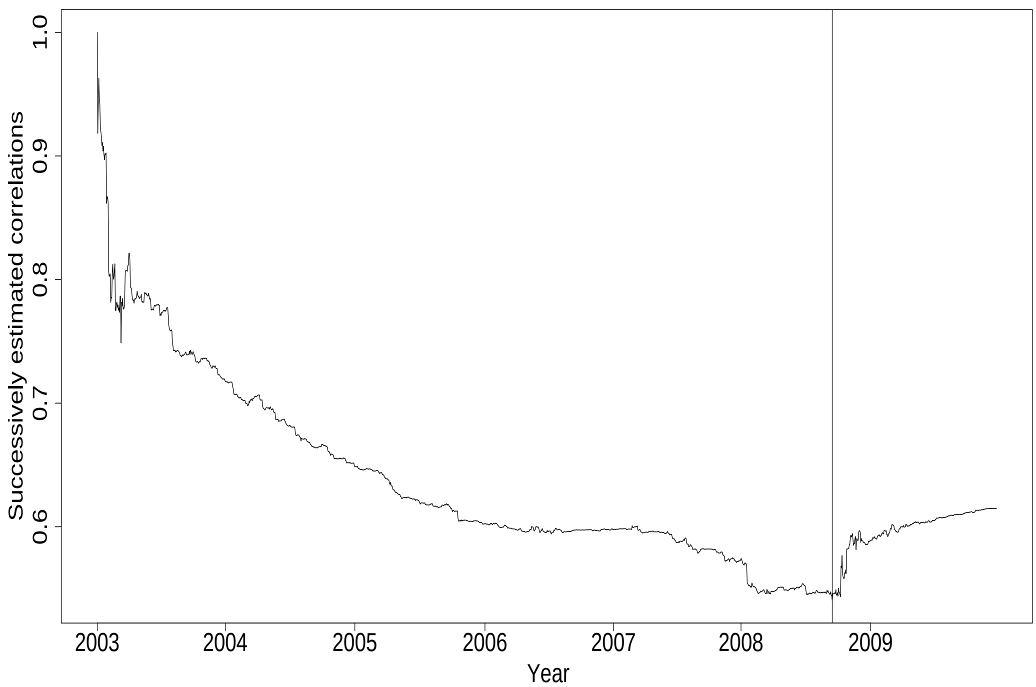


Table 1: Empirical rejection frequencies under the null hypothesis

T	ρ				
	-0.9	-0.5	0	0.5	0.9
			a) iid-case		
200	0.144	0.054	0.039	0.053	0.142
500	0.064	0.040	0.035	0.041	0.064
1000	0.048	0.038	0.034	0.039	0.049
2000	0.043	0.038	0.036	0.038	0.043
			b) serial dependence		
200	0.150	0.052	0.038	0.053	0.144
500	0.070	0.049	0.037	0.044	0.064
1000	0.055	0.046	0.039	0.044	0.053
2000	0.046	0.043	0.043	0.045	0.051

Table 2: Empirical size-adjusted rejection frequencies when correlations change

T	Alternative				
	1	2	3	4	5
			a) Our test		
200	0.309	0.255	0.953	0.880	0.101
500	0.587	0.488	0.996	0.989	0.207
1000	0.830	0.733	0.998	0.998	0.422
2000	0.967	0.928	1	0.999	0.750
			b) Aue et al. (2009)		
200	0.241	0.159	0.977	0.842	0.083
500	0.586	0.400	1	0.998	0.163
1000	0.858	0.690	1	1	0.284
2000	0.980	0.929	1	1	0.609