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TESTING FOR ELLIPSOIDAL SYMMETRY OF A MULTIVARIATE DENSITY[1]

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Let Z be a random vector whose distribution is spherically symmetric about the origin. A random vector X which is representable as the image of Z under affine transformation is said to have an ellipsoidally symmetric distribution. The model of ellipsoidal symmetry is a useful generalization of multivariate normality. This paper proposes and studies some goodness-of-fit tests which have good asymptotic power over a broad spectrum of alternatives to ellipsoidal symmetry.

1. Introduction. The statistical model most frequently assumed in multivariate analysis is the normal distribution. A notable feature of the $N(\mu, \Sigma)$ density is the property that its constant surfaces are ellipsoids centered at μ, with orientation and shape determined by the matrix Σ. This ellipsoidal symmetry plays an important role in the geometrical interpretation of normal-model multivariate analysis. Even without normality, ellipsoidal symmetry of the data distribution can provide a rationale for the use of standard multivariate procedures (cf. Dempster (1969)).

Recent interest in robust statistical methods has led to more detailed consideration of statistical models which retain some of the features of the normal model while providing more flexibility in data-fitting. For one-dimensional data, a natural generalization of the normal model is the symmetric location model, the shape of the density being unrestricted apart from symmetry and regularity assumptions. In a p-dimensional multivariate setting, a corresponding generalization of the normal model is the model of ellipsoidal symmetry: each observation has density of the form $[\det(A)]^{-1}h[A^{-1}(x - \mu)]$, where μ is a $p \times 1$ vector, A is a $p \times p$ nonsingular matrix, and h is a density on R^p which is spherically symmetric about the origin (i.e., $h(x) = h(Ox)$ for every orthogonal transformation O). Apart from spherical symmetry and possible regularity assumptions, h is left unrestricted. To make the parametrization one-to-one, the matrix A is assumed to be lower triangular with positive diagonal elements and determinant one. (For some applications of this model, see Dempster (1969), Huber (1972), Maronna (1976)).

The problem of testing for symmetry of a one-dimensional distribution has received considerable attention in the literature. The tests studied fall into two general categories: rank and permutation tests for symmetry (see Hájek and Šidák (1967), for instance) and tests based upon the empirical cdf (Smirnov (1947), Butler

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(1969), Rothman and Woodroofe (1972), Doksum (1975), Koul and Staudte (1976), Doksum et al. (1976)). Most of these references assume that the center of symmetry is known; notable exceptions are the papers by Doksum et al. The more general problem of testing for ellipsoidal symmetry of a multivariate distribution has been studied less intensively. For known μ and A, Kariya and Eaton (1977) derive tests of ellipsoidal symmetry which are UMP against certain affine transformations of the data.

One aim of this paper, then, is to develop some broadly useful tests for ellipsoidal symmetry, including the common situation where μ and A must be estimated from the data. The basis for such tests is the following consideration. Suppose Z is a p-dimensional random vector whose distribution is spherically symmetric about the origin. Let $|\cdot|$ denote the euclidean norm in R^p. Then the random variable $|Z|$ and the random unit vector $|Z|^{-1}Z$ are independent; moreover the distribution of $|Z|^{-1}Z$ is uniform over a hypersphere of dimension $p-1$ (cf. Dempster (1969)). For $p \geqslant 2$, the unit vector $|Z|^{-1}Z$ can be represented in terms of angular polar co-ordinates $\Theta = (\theta_1, \theta_2, \cdots, \theta_{p-1})$ as follows:

$$(1.1)\quad |Z|^{-1}Z = \big(\cos(\theta_1), \sin(\theta_1)\cos(\theta_2), \cdots, \sin(\theta_1)\sin(\theta_2), \cdots, \sin(\theta_{p-1})\big),$$

with $\theta_{p-1} \in [0, 2\pi)$ and $\theta_j \in [0, \pi]$ for $1 \leqslant j \leqslant p-2$. The distribution of Θ is given by

$$(1.2)\quad \begin{aligned} P[\Theta \in B] &= \textstyle\int_B 2^{-1}\pi^{-p/2}\Gamma(p/2)\big[\prod_{i=1}^{p-2} \sin^{p-1-i}(\theta_i)\big]d\theta_1 d\theta_2 \cdots d\theta_{p-1} \\ &= m(B), \qquad \text{say.} \end{aligned}$$

The uniform measure m is hyperspherical surface area measure, normalized to a probability measure.

Let $X_1, X_2 \cdots X_n$ be a random sample of vectors in R^p. The hypothesis $H_n(\mu, A)$ of ellipsoidal symmetry postulates that the $\{X_i\}$ are independent and that each X_i has density of the form $[\det A]^{-1}h[A^{-1}(x-\mu)]$, where h is spherically symmetric about the origin and A is lower triangular with positive diagonal elements and determinant one.

Let $(\hat{\mu}_n, \hat{A}_n)$ denote an estimator of (μ, A) based upon the $\{X_i\}$. To test $H_n(\mu, A)$, form the residuals $\{\hat{A}_n^{-1}(X_i - \hat{\mu}_n)\}$. Let the $\{R_i(\hat{\mu}_n, \hat{A}_n)\}$ denote the ranks, divided by $n+1$, of the distances $\{|\hat{A}_n^{-1}(X_i - \hat{\mu}_n)|\}$. Represent the direction vectors $\{|\hat{A}_n^{-1}(X_i - \hat{\mu}_n)|^{-1}\hat{A}_n^{-1}(X_i - \hat{\mu}_n)\}$ in terms of angular polar coordinates $\{\Theta_i(\hat{\mu}_n, \hat{A}_n)\}$. Let $\{a_k; k \geqslant 1\}$ be a family of functions orthonormal with respect to Lebesgue measure on [0, 1] and orthogonal to the constant function on [0, 1]. If $p \geqslant 2$, let $\{b_m; m \geqslant 1\}$ be a family of functions orthonormal with respect to the uniform measure m on $[0, \pi]^{p-2} \times [0, 2\pi)$ and orthogonal to the constant function on this domain. For testing $H_n(\mu, A)$ when $p \geqslant 2$, we propose the statistic

$$(1.3)\quad S_n(\hat{\mu}_n, \hat{A}_n) = \Sigma_{k=1}^{K_n}\Sigma_{m=1}^{M_n}\big[n^{-\frac{1}{2}}\Sigma_{i=1}^{n} a_k(R_k(\hat{\mu}_n, \hat{A}_n))b_m(\Theta_i(\hat{\mu}_n, \hat{A}_n))\big]^2.$$

Large values of $S_n(\hat{\mu}_n, \hat{A}_n)$ indicate that the data does not support $H_n(\mu, A)$. Both

K_n and M_n are allowed to increase with the sample size n, so as to expand the class of alternatives that can be detected by the test.

An intuitive rationale for this test statistic runs as follows. Let F denote the cdf of $|A^{-1}(X_i - \mu)|$ under $H_n(\mu, A)$, let $U_i = F(|A^{-1}(X_i - \mu)|)$, and let Θ_i denote the polar angles of the direction vector $|A^{-1}(X_i - \mu)|^{-1}A^{-1}(X_i - \mu)$. Set

$$(1.4) \qquad \hat{g}_n(u, \theta) = 1 + \Sigma_{k=1}^{K_n}\Sigma_{m=1}^{M_n}\left[n^{-1}\Sigma_{i=1}^{n} a_k(R_i(\hat{\mu}_n, \hat{A}_n))b_m(\Theta_i(\hat{\mu}_n, \hat{A}_n))\right]a_k(u)b_m(\theta)$$

for $u \in [0, 1]$ and $\theta \in [0, \pi]^{p-2} \times [0, 2\pi)$. Under $H_n(\mu, A)$ and for $K_n, M_n \to \infty$ at a suitable rate as $n \to \infty$, the random function $\hat{g}_n(u, \theta)$ should estimate consistently the density of (U_i, Θ_i), which is uniform. Thus, a plausible statistic for testing the validity of the hypothesis $H_n(\mu, A)$ is

$$(1.5) \qquad n\int[\hat{g}_n(u, \theta) - 1]^2 du\, dm(\theta),$$

which equals $S_n(\hat{\mu}_n, \hat{A}_n)$.

Another plausible statistic for testing $H_n(\mu, A)$ is

$$(1.6) \qquad U_n(\hat{\mu}_n, \hat{A}_n) = \Sigma_{k=1}^{\infty}\Sigma_{m=1}^{\infty}\beta_{k,m}^2\left[n^{-\frac{1}{2}}\Sigma_{i=1}^{n} a_k(R_i(\hat{\mu}_n, \hat{A}_n))b_m(\Theta_i(\hat{\mu}_n, \hat{A}_n))\right]^2,$$

where the weights $\{\beta_{k,m}\}$ do not depend upon n and $\Sigma_{k,m}\beta_{k,m}^2 < \infty$. Statistics of this form are essentially generalizations of the Cramér-von Mises goodness-of-fit test and can be studied by methods developed for that case (see Beran (1975) for more details). A practical drawback of $U_n(\hat{\mu}_n, \hat{A}_n)$ is that its asymptotic null distribution depends in a complex way upon the specific estimators $\hat{\mu}_n, \hat{A}_n$ used in calculating the residuals. However, when (μ, A) are assumed known, comparing the asymptotic power of the U_n test with that of the S_n test proves interesting and is done in Section 3 of this paper.

For one-dimensional data, there exist statistics analogous to $S_n(\hat{\mu}_n, \hat{A}_n)$ and $U_n(\hat{\mu}_n, \hat{A}_n)$ which are suitable for testing the hypothesis of symmetry. These analogues are, respectively,

$$(1.7) \qquad \Sigma_{k=1}^{K_n}\left[n^{-\frac{1}{2}}\Sigma_{i=1}^{n} a_k(R_i(\hat{\mu}_n))\operatorname{sgn}(X_i - \hat{\mu}_n)\right]^2$$

and

$$(1.8) \qquad \Sigma_{k=1}^{\infty}\beta_k^2\left[n^{-\frac{1}{2}}\Sigma_{i=1}^{n} a_k(R_i(\hat{\mu}_n))\operatorname{sgn}(X_i - \hat{\mu}_n)\right]^2,$$

where $\hat{\mu}_n$ is a center-of-symmetry estimator, the $\{R_i(\hat{\mu}_n)\}$ are the ranks, divided by $n + 1$, of the distances $\{|X_i - \hat{\mu}_n|\}$, and the $\{\beta_k\}$ are weights which do not depend upon n and which satisfy the requirement $\Sigma_k\beta_k^2 < \infty$. No explicit results are given in this paper for the statistics (1.7) and (1.8). However, it is not difficult to establish appropriate counterparts to the results that are developed for dimension $p \geq 2$.

Primarily, this paper is concerned with the asymptotic behavior of $S_n(\hat{\mu}_n, \hat{A}_n)$ as $K_n, M_n \to \infty$ with n. Asymptotic distributions for quadratic functionals of a multivariate density estimator of window type have been derived by Rosenblatt (1975), using a Poissonization technique. In view of (1.5), the theorems established in this paper confirm the possibility of proving similar results for Fourier series density estimators. In place of Poissonization, however, the principal tool used in this paper is a central limit theorem for dependent random variables which recognizes near martingale structure in the sequence $\{S_n(\hat{\mu}_n, \hat{A}_n);\ n \geqslant 1\}$ after appropriate centering. The method has wider applicability; for instance, it can be used to weaken the assumptions for Theorem 1 in Rosenblatt (1975).

Assuming (μ, A) known, Section 2 establishes the asymptotic normality under $H_n(\mu, A)$ of $S_n(\mu, A)$. Asymptotic power of the $S_n(\mu, A)$ test is studied in Section 3. The effect on the asymptotics of estimating (μ, A) is analyzed in the final section.

2. Asymptotic null distribution for known μ, A. Our study of $S_n(\hat{\mu}_n, \hat{A}_n)$ begins with a simpler problem: the asymptotic distribution theory of the random variable $S_n(\mu, A)$, which is defined like $S_n(\hat{\mu}_n, \hat{A}_n)$ with the estimators $(\hat{\mu}_n, \hat{A}_n)$ replaced by the actual parameter values (μ, A). The transition from $S_n(\mu, A)$ to $S_n(\hat{\mu}_n, \hat{A}_n)$ is examined in Section 4. Let $\|\cdot\|$ denote the essential sup norm. The result established in this section is

THEOREM 1. *Suppose the functions* $\{a_k;\ k \geqslant 1\}$, $\{b_m;\ m \geqslant 1\}$ *are uniformly bounded, the* $\{a_k\}$ *are differentiable, and*

$$(2.1)\qquad \begin{aligned} &\text{(i) } \lim_{n\to\infty} K_n = \lim_{n\to\infty} M_n = \infty;\\ &\text{(ii) } \lim_{n\to\infty} n^{-1} M_n \Sigma_{k=1}^{K_n} \|a_k'\|^2 = 0;\\ &\text{(iii) } \lim_{n\to\infty} n^{-1}(K_n M_n) = 0. \end{aligned}$$

Then, the limiting distribution of $(K_n M_n)^{-\frac{1}{2}}[S_n(\mu, A) - K_n M_n]$ *under* $H_n(\mu, A)$ *as* $n \to \infty$ *is* $N(0, 2)$.

The proof of Theorem 1 uses two preliminary approximations which are described in the following two lemmas. Let $U_i = F(|A^{-1}(X_i - \mu)|)$, where F denotes the cdf of $|A^{-1}(X_i - \mu)|$ under $H(\mu, A)$. Let $Z_i = (U_i, \Theta_i)$, where Θ_i is the same as $\Theta_i(\mu, A)$ in the notation of the introduction, and let $c_{k,m}(Z_i) = a_k(U_i)b_m(\Theta_i)$. Under $H_n(\mu, A)$, the $\{U_i\}$ and the $\{\Theta_i\}$ are independent, the $\{U_i\}$ being i.i.d. with uniform (0, 1) distribution and the $\{\Theta_i\}$ being i.i.d. with the distribution as defined in (1.2). Define

$$(2.2)\qquad W_n(\mu, A) = \Sigma_{k,m}^{K_n, M_n}\left[n^{-\frac{1}{2}}\Sigma_{i=1}^n c_{k,m}(Z_i)\right]^2,$$

and

$$(2.3)\qquad T_n(\mu, A) = (K_n M_n)^{-\frac{1}{2}} n^{-1} \Sigma_{k,m}^{K_n, M_n} \Sigma_{i \neq j}^n c_{k,m}(Z_i) c_{k,m}(Z_j).$$

LEMMA 1. *Suppose that all the assumptions of Theorem* 1, *save* (iii), *hold. Then, under* $H_n(\mu, A)$

$$(2.4)\qquad \lim_{n\to\infty} (K_n M_n)^{-\frac{1}{2}} E|S_n(\mu, A) - W_n(\mu, A)| = 0.$$

LEMMA 2. *Suppose that the* $\{a_k\}$, $\{b_m\}$ *are uniformly bounded and*

$$\lim_{n\to\infty} n^{-1}K_n M_n = 0. \tag{2.5}$$

Then, under $H_n(\mu, A)$,

$$\lim_{n\to\infty} E\left[(K_n M_n)^{-\frac{1}{2}}[W_n(\mu, A) - K_n M_n] - T_n(\mu, A)\right]^2 = 0. \tag{2.6}$$

PROOF OF LEMMA 1. Let $d_{k,m}(Z_i) = a_k(R_i)b_m(\Theta_i)$ and let $C_{k,m} = n^{-\frac{1}{2}} \Sigma_{i=1}^n c_{k,m}(Z_i)$, $D_{k,m} = n^{-\frac{1}{2}}\Sigma_{i=1}^n d_{k,m}(Z_i)$. By the Cauchy-Schwarz inequality,

$$\begin{aligned} (2.7)\quad |S_n(\mu, A) - W_n(\mu, A)| &= |\Sigma_{k,m}^{K_n, M_n}(D_{k,m}^2 - C_{k,m}^2)| \\ &\leqslant [\Sigma_{k,m}^{K_n, M_n}(D_{k,m} - C_{k,m})^2]^{\frac{1}{2}}[\Sigma_{k,m}^{K_n, M_n}(D_{k,m} + C_{k,m})^2]^{\frac{1}{2}}, \end{aligned}$$

and, therefore,

$$\begin{aligned} (2.8)\quad [E|S_n(\mu, A) - W_n(\mu, A)|]^2 &\leqslant [\Sigma_{k,m}^{K_n, M_n}E(D_{k,m} - C_{k,m})^2][\Sigma_{k,m}^{K_n, M_n}E(D_{k,m} + C_{k,m})^2]. \end{aligned}$$

Lemma 1 is an immediate consequence of (2.8) and the following calculations:

$$\begin{aligned} (2.9)\quad E(D_{k,m} - C_{k,m})^2 &= n^{-1}E[\Sigma_{i=1}^n(a_k(R_i) - a_k(U_i))b_m(\Theta_i)]^2 \\ &= E[a_k(R_i) - a_k(U_i)]^2 \\ &\leqslant \|a_k'\|^2 E[R_i - U_i]^2 \\ &= \|a_k'\|^2 0(n^{-1}) \end{aligned}$$

and

$$E(C_{k,m}^2) = Ec_{k,m}^2(Z_i) = 1. \tag{2.10}$$

PROOF OF LEMMA 2. By definition of the random variables $W_n(\mu, A)$ and $T_n(\mu, A)$,

$$\begin{aligned} (2.11)\quad (K_n M_n)^{-\frac{1}{2}}[W_n(\mu, A) - K_n M_n] - T_n(\mu, A) &= (K_n M_n)^{-\frac{1}{2}}[n^{-1}\Sigma_{k,m}^{K_n, M_n}\Sigma_{i=1}^n c_{k,m}^2(Z_i) - K_n M_n]. \end{aligned}$$

The expectation of the right side in (2.11) is zero, while the variance equals

$$\begin{aligned} (2.12)\quad (K_n M_n)^{-1}n^{-1}\operatorname{Var}[\Sigma_{k,m}^{K_n, M_n}c_{k,m}^2(Z_i)] &\leqslant (K_n M_n n)^{-1}E[\Sigma_{k,m}^{K_n, M_n}a_k^2(U_i)b_m^2(\Theta_i)]^2 \\ &= O(n^{-1}K_n M_n), \end{aligned}$$

the last step using the uniform boundedness of the $\{a_k\}$ and $\{b_m\}$. The lemma conclusion follows.

PROOF OF THEOREM 1. In view of the two lemmas just established, it is enough to show that the limiting distribution of $T_n(\mu, A)$ under $H_n(\mu, A)$ is $N(0, 2)$. This can be done simply by applying a central limit theorem for dependent random variables, such as Theorem 2.2 of Dvoretzky (1972) or Corollary 3.8 of McLeish (1974). These theorems are appropriate here because if $\mathcal{Q}_j$ denotes the σ-algebra generated by the random variables $(Z_1, Z_2, \cdots, Z_j)$, the process $\{(T_n(\mu, A), \mathcal{Q}_n); n \geqslant 1\}$ is nearly a martingale under $H_n(\mu, A)$.

For $2 \leqslant j \leqslant n$, let

$$Y_j = 2\left(K_n M_n^{-\frac{1}{2}} n^{-1} \Sigma_{k,m}^{K_n, M_n} c_{k,m}(Z_j) \Sigma_{i<j} c_{k,m}(Z_i)\right. \tag{2.13}$$

and observe that $T_n(\mu, A) = \Sigma_{j=2}^n Y_j$. Though Y_j also depends on n, μ, and A, we omit further subscripts or arguments for convenience. Evidently $E[Y_j|\mathcal{Q}_{j-1}] = 0$ and

$$E\left[Y_j^2|\mathcal{Q}_{j-1}\right] = 4\left(K_n M_n n^2\right)^{-1} \Sigma_{k,m}^{K_n, M_n}\left[\Sigma_{i<j} c_{k,m}(Z_i)\right]^2. \tag{2.14}$$

Let $V_n = \Sigma_{j=2}^n E[Y_j^2|\mathcal{Q}_{j-1}]$. The asymptotic normality of $T_n(\mu, A)$ follows from the central limit theorems cited above once it has been verified that $V_n \to_p 2$ and

$$\Sigma_{j=2}^n E\left[Y_j^2 I(|Y_j| > \varepsilon)\right] \to 0 \tag{2.15}$$

for every $\varepsilon > 0$ as $n \to \infty$.

The random variable V_n can be written as the sum of two terms:

$$\begin{aligned} V_{n,1} &= 4\left(K_n M_n n^2\right)^{-1} \Sigma_{k,m}^{K_n, M_n} \Sigma_{j=1}^{n-1}(n-j) c_{k,m}^2(Z_j) \\ V_{n,2} &= 8\left(K_n M_n n^2\right)^{-1} \Sigma_{k,m}^{K_n, M_n} \Sigma_{t<u=2}^{n-1}(n-u) c_{k,m}(Z_t) c_{k,m}(Z_u), \end{aligned} \tag{2.16}$$

the inner summation in the definition of $V_{n,2}$ being over both t and u. Evidently, $\lim_{n\to\infty} E(V_{n,1}) = 2$ and $E(V_{n,2}) = 0$. Moreover,

$$\begin{aligned} \operatorname{Var}(V_{n,1}) &= 16\left(K_n M_n n^2\right)^{-2} \Sigma_{j=1}^{n-1}(n-j)^2 \operatorname{Var}\left[\Sigma_{k,m}^{K_n, M_n} c_{k,m}^2(Z_j)\right] \\ &= O(n^{-1}), \end{aligned} \tag{2.17}$$

since the $\{c_{k,m}\}$ are uniformly bounded, and

$$\begin{aligned} \operatorname{Var}(V_{n,2}) &= 64\left(K_n M_n n^2\right)^{-2} \Sigma_{k,m}^{K_n, M_n} \Sigma_{t<u=2}^{n-1}(n-u)^2 \\ &= 64\left(K_n M_n n^4\right)^{-1} \Sigma_{u=2}^{n-1}(u-1)(n-u)^2 \\ &= O\left[(K_n M_n)^{-1}\right]. \end{aligned} \tag{2.18}$$

Thus, $V_n \to_p 2$ under $H_n(\mu, A)$ as $n \to \infty$.

By direct calculation, $E(Y_j^4) = O(n^{-4} K_n^2 M_n^2 j) + O(n^{-4} j^2)$ and therefore $\Sigma_{j=2}^n E(Y_j^4) = O(n^{-2} K_n^2 M_n^2) + O(n^{-1})$, which tends to zero under the assumptions. Hence the Lindeberg condition (2.15) is satisfied.

The proof just given benefited from a referee's advice to use Dvoretzky's theorem in place of the original characteristic function argument.

3. Asymptotic power for known μ, A. This section studies the asymptotic behavior of $S_n(\mu, A)$ under sequences of alternatives tending to an ellipsoidally symmetric distribution. To simplify notation, these alternatives will be specified in an indirect manner. For every $n \geq 1$ and $1 \leq i \leq n$, let $Z_{i,n} = (U_{i,n}, \Theta_{i,n})$ be a random variable-random $p - 1$ vector pair which has density $g_n(z)$ with respect to the product measure μ obtained from Lebesgue measure on [0, 1] and the probability measure m defined in (1.2). Suppose also that the random vectors $\{Z_{i,n}; 1 \leq i \leq n\}$ are independent. Pick a continuous cdf F on the real line and let $W_{i,n}$ be the random vector in R^p whose radial and angular polar coordinates are, respectively, $F^{-1}(U_{i,n})$ and $\Theta_{i,n}$. Finally, set $X_{i,n} = \mu + AW_{i,n}$ for given $p \times 1$ vector μ and $p \times p$ lower triangular matrix A with positive diagonal elements and determinant one.

If, in this scheme, $g_n(z) \equiv 1$ for every $n \geq 1$, then the random vectors $\{X_{i,n}; 1 \leq i \leq n\}$ are i.i.d., each having an ellipsoidally symmetric distribution determined by the choices of μ, A and F. Let $H_n(\mu, A, F)$ denote the hypothesis that the $\{X_{i,n}\}$ have this particular ellipsoidally symmetric distribution; $H_n(\mu, A, F)$ is a sub-hypothesis of $H_n(\mu, A)$. A sequence of local alternatives to $\{H_n(\mu, A, F); n \geq 1\}$ can be constructed by taking, in the scheme of the previous paragraph,

$$g_n(z) = 1 + \alpha_n r_n(z), \tag{3.1}$$

where $\{\alpha_n; n \geq 1\}$ is a sequence of constants tending to zero as n increases, r_n is integrable, $\int r_n(z)d\mu(z) = 0$ for every $n \geq 1$, and $g_n(z)$ is nonnegative for every $n \geq 1$.

Let the $\{R_{i,n}; 1 \leq i \leq n\}$ denote the ranks, divided by $n + 1$, of the distances $\{|A^{-1}(X_{i,n} - \mu)|\}$ and define $S_n(\mu, A)$ through (1.3) with (μ, A) in place of $(\hat{\mu}_n, \hat{A}_n)$ and $(R_{i,n}, \Theta_{i,n})$ instead of (R_i, Θ_i). The following extension of Theorem 1 describes the large sample behavior of $S_n(\mu, A)$ under the alternatives to $H_n(\mu, A, F)$ defined above.

THEOREM 2. *Suppose the functions $\{a_k; k \geq 1\}$, $\{b_m; m \geq 1\}$ are uniformly bounded, the $\{a_k\}$ are twice differentiable, and*

(3.2) (i) $\lim_{n\to\infty} K_n = \lim_{n\to\infty} M_n = \infty$;

(ii) $\lim_{n\to\infty} n^{-1}M_n^{\frac{3}{2}}K_n^{\frac{1}{2}}\sum_{k=1}^{K_n}\|a_k'\|^2 = \lim_{n\to\infty} n^{-1}M_n^{\frac{3}{2}}K_n^{\frac{1}{2}}\sum_{k=1}^{K_n}\|a_k''\|^2 = 0$;

(iii) $\lim_{n\to\infty} n^{-1}(K_nM_n)^{\frac{3}{2}} = 0$;

(iv) $\lim_{n\to\infty}(K_nM_n)^{-\frac{1}{2}}n\alpha_n^2 = \alpha^2 < \infty$;

(v) $\lim\sup_n \int |r_n(z)|d\mu(z) < \infty$;

(vi) $\lim_{n\to\infty}\sum_{k,m}^{K_n,M_n}[\int c_{k,m}(z)r_n(z)d\mu(z)]^2 = B < \infty$.

Then, the limiting distribution of $(K_nM_n)^{-\frac{1}{2}}[S_n(\mu, A) - K_nM_n]$ under the sequence of alternatives specified by (3.1) *above is $N(\alpha^2B, 2)$.*

PROOF. Define centered versions of the random variables $S_n(\mu, A)$, $W_n(\mu, A)$, $T_n(\mu, A)$ as follows. Recalling that $c_{k,m}(Z_{i,n}) = a_k(U_{i,n})b_m(\Theta_{i,n})$ and $d_{k,m}(Z_{i,n}) = a_n(R_{i,n})b_m(\theta_{i,n})$, put

$$\begin{aligned}\hat{c}_{k,m}(Z_{i,n}) &= c_{k,m}(Z_{i,n}) - Ec_{k,m}(Z_{i,n})\\ \hat{d}_{k,m}(Z_{i,n}) &= d_{k,m}(Z_{i,n}) - Ed_{k,m}(Z_{i,n}),\end{aligned} \tag{3.3}$$

the expectations being taken under the alternatives. Replace the $\{c_{k,m}\}$ and $\{d_{k,m}\}$ in the definitions of $S_n(\mu, A)$, $W_n(\mu, A)$, $T_n(\mu, A)$ by the $\{\hat{c}_{k,m}\}$ and $\{\hat{d}_{k,m}\}$ respectively; call the new statistics $\hat{S}_n(\mu, A)$, $\hat{W}_n(\mu, A)$ and $\hat{T}_n(\mu, A)$ respectively.

Let $\hat{C}_{k,m} = n^{-\frac{1}{2}}\sum_{i=1}^{n}\hat{c}_{k,m}(Z_{i,n})$ and $\hat{D}_{k,m} = n^{-\frac{1}{2}}\sum_{i=1}\hat{d}_{k,m}(Z_{i,n})$. Then, letting $G_{n,1}$ denote the cdf of $U_{i,n}$,

$$\begin{aligned}E(\hat{D}_{k,m} - \hat{C}_{k,m})^2 &= \operatorname{Var}\left[(a_k(R_{i,n}) - a_k(U_{i,n}))b_m(\Theta_{i,n})\right]\\ &\leq \|b_m\|^2 E\left[a_k(R_{i,n}) - a_k(U_{i,n})\right]^2\\ &\leq \|b_m\|^2\|a_k'\|^2 E\left[R_{i,n} - U_{i,n}\right]^2\\ &\leq 2\|b_m\|^2\|a_k'\|^2\left\{E\left[R_{i,n} - G_{n,1}(U_{i,n})\right]^2 + E\left[G_{n,1}(U_{i,n}) - U_{i,n}\right]^2\right\}\\ &= O\left(\alpha_n^2\|a_k'\|^2\right).\end{aligned} \tag{3.4}$$

Thus, by an argument analogous to Lemma 1,

$$(K_nM_n)^{-\frac{1}{2}}\left[\hat{S}_n(\mu, A) - \hat{W}_n(\mu, A)\right] \to_p 0 \tag{3.5}$$

under the sequence of local alternatives.

Evidently, $E[\hat{c}_{k,m}(Z_{i,n})] = 0$, $\operatorname{Var}[\hat{c}_{k,m}(Z_{i,n})] = 1 + O(\alpha_n)$, and, for $(r, s) \neq (k, m)$, $E[\hat{c}_{k,m}(Z_{i,n})\hat{c}_{r,s}(Z_{i,n})] = O(\alpha_n)$. The argument of Lemma 2 yields, in the present context,

$$(K_nM_n)^{-\frac{1}{2}}\left[\hat{W}_n(\mu, A) - K_nM_n\right] - \hat{T}_n(\mu, A) \to_p 0 \tag{3.6}$$

as $n \to \infty$. Moreover, the proof for Theorem 1 adapts easily to show that the limiting distribution of $\hat{T}_n(\mu, A)$ is $N(0, 2)$. This result combined with (3.5) and (3.6) establishes that the limiting distribution of $(K_nM_n)^{-\frac{1}{2}}[\hat{S}_n(\mu, A) - K_nM_n]$ is $N(0, 2)$ under the alternatives.

To complete the proof of Theorem 2, it suffices to show that

$$(K_nM_n)^{-\frac{1}{2}}\left[S_n(\mu, A) - \hat{S}_n(\mu, A)\right] \to_p \alpha^2 B \tag{3.7}$$

as $n \to \infty$. With $\mu_{k,m}$ denoting $Ed_{k,m}(Z_{i,n})$ for brevity, the difference on the left side of (3.7) can be expressed as the sum of two terms:

$$\begin{aligned}B_{1,n} &= 2(K_nM_n)^{-\frac{1}{2}}\sum_{k,m}^{K_n,M_n}\mu_{k,m}\sum_{i=1}^{n}d_{k,m}(Z_{i,n})\\ B_{2,n} &= -(K_nM_n)^{-\frac{1}{2}}n\sum_{k,m}^{K_n,M_n}\mu_{k,m}^2.\end{aligned} \tag{3.8}$$

By Taylor expansion, the difference $\mu_{k,m} - Ec_{k,m}(Z_{i,n})$ equals the sum of $D_{1,n}$ and $D_{2,n}$, where

$$D_{1,n} = E[(R_{i,n} - U_{i,n})a_k'(U_{i,n})b_m(\Theta_{i,n})]$$
$$D_{2,n} = 2^{-1}E[(R_{i,n} - U_{i,n})^2 a_k''(\xi_{i,n})b_m(\Theta_{i,n})] \tag{3.9}$$

and $\xi_{i,n}$ lies between $R_{i,n}$ and $U_{i,n}$. If $I(x)$ denotes the indicator of the set $[0, \infty)$, then $(n+1)R_{i,n}$, which is the rank of $U_{i,n}$, can be represented in the form

$$(n+1)R_{i,n} = 1 + \Sigma_{j\neq i}^{n} I(U_{i,n} - U_{j,n}). \tag{3.10}$$

By using (3.10) and first computing the conditional expectation given $U_{i,n}$, we find that $D_{1,n} = O(\alpha_n^2\|a_k'\|)$. On the other hand, by an argument similar to (3.4), $D_{2,n} = O(\alpha_n^2\|a_k''\|)$. Since $Ec_{k,m}(Z_{i,n}) = \alpha_n \int c_{k,m}(z)r_n(z)d\mu(z)$, it follows now under the theorem assumptions that $\lim_{n\to\infty} B_{2,n} = -\alpha^2 B$ and that $\lim_{n\to\infty} E(B_{1,n}) = 2\alpha^2 B$.

Evidently

$$\begin{aligned}\operatorname{Var}(B_{1,n}) &\leq 4n(K_nM_n)^{-1}E[\Sigma_{k,m}^{K_n,M_n}\mu_{k,m}d_{k,m}(Z_{i,n})]^2 \\ &= L_{1,n} + L_{2,n},\end{aligned} \tag{3.11}$$

where

$$L_{1,n} = 4n(K_nM_n)^{-1}\Sigma_{k,m}^{K_n,M_n}\mu_{k,m}^2 E[d_{k,m}^2(Z_{i,n})]$$
$$L_{2,n} = 4n(K_nM_n)^{-1}\Sigma_{(k,m)\neq(r,s)}^{K_n,M_n}\mu_{k,m}\mu_{r,s}E[d_{k,m}(Z_{i,n})d_{r,s}(Z_{i,n})]. \tag{3.12}$$

Since the $\{d_{k,m}(Z_{i,n})\}$ are uniformly bounded, the result of the previous paragraph for $B_{i,n}$ implies that $L_{1,n} = O[(K_nM_n)^{-\frac{1}{2}}]$. Since $E[d_{k,m}(Z_{i,n}) - c_{k,m}(Z_{i,n})]^2 = O(\alpha_n^2\|a_k'\|^2)$ by the argument for (3.4), it follows that $E[d_{k,m}(z_{i,n})d_{r,s}(Z_{i,n})] = E[c_{k,m}(Z_{i,n})c_{r,s}(Z_{i,n})] + O(\alpha_n\|a_k'\|) + O(\alpha_n\|a_r'\|)$; moreover for $(r,s)\neq(k,m)$, $E[c_{k,m}(Z_{i,n})c_{r,s}(Z_{i,n})] = O(\alpha_n)$. A calculation which applies this and the result of the previous paragraph to $L_{2,n}$ ultimately yields the fact that $L_{2,n} = O[n^{-\frac{1}{2}}(K_nM_n)^{\frac{3}{4}}] + O[n^{-\frac{1}{2}}K_n^{\frac{1}{4}}M_n^{\frac{3}{4}}(\Sigma_k^{K_n}\|a_k'\|^2)^{\frac{1}{2}}]$, which tends to zero as $n\to\infty$. Thus, $B_{1,n} \to_p 2\alpha^2 B$ and (3.7) is verified.

Remarks. The significance of these asymptotics can be clarified by comparing the asymptotic power of the test based upon $S_n(\mu, A)$ with that of the test based upon the statistic $U_n(\mu, A)$ defined in (1.6). The limiting distribution of $U_n(\mu, A)$ under $H_n(\mu, A)$ as $n\to\infty$ is a convolution of scaled chi-square distributions (see Beran (1975) for further technical details).

Both $S_n(\mu, A)$ and $U_n(\mu, A)$ generate tests which are sensitive to a wide range of alternatives to $H_n(\mu, A)$. However, neither test dominates the other uniformly in asymptotic power. To see this, consider two particular sequences of alternatives having the general form described at the beginning of this section. The first sequence is specified by the choices

$$\alpha_n = n^{-\frac{1}{2}}(K_nM_n)^{\frac{1}{4}}, \qquad r_n(z) = (K_nM_n)^{-\frac{1}{4}}c_{r,s}(z) \tag{3.13}$$

where (r, s) is fixed, while the second sequence is obtained by taking

$$\alpha_n = n^{-\frac{1}{2}}(K_n M_n)^{\frac{1}{4}}, \qquad r_n(z) = (K_n M_n)^{-\frac{1}{2}} \Sigma_{k,m}^{K_n, M_n} c_{k,m}(z). \tag{3.14}$$

Since the $\{c_{k,m}\}$ are uniformly bounded and $\lim_{n\to\infty} n^{-1}(K_n M_n)^{\frac{3}{2}} = 0$, the alternative densities $1 + \alpha_n r_n(z)$ determined by either (3.13) or (3.14) are, in fact, nonnegative for all sufficiently large values of n. Moreover, $\int r_n(z) d\mu(z) = 0$ and $\lim \sup_n \int |r_n(z)| d\mu(z) < \infty$ in both cases.

The alternative sequence (3.13) is of a type commonly considered in asymptotic power calculations: the sequence $\{n^{\frac{1}{2}}[g_n(z) - 1]\}$ converges to a nontrivial limit as $n \to \infty$. Under the alternatives (3.13), the statistic $S_n(\mu, A)$ has the same limiting distribution as it does under $H_n(\mu, A)$ (by Theorem 2); however, the limiting distribution of $U_n(\mu, A)$ becomes a convolution of scaled chi-square distributions, one of which is noncentral (cf. Beran (1975)). Thus, $U_n(\mu, A)$ gives an asymptotically more powerful test in this case.

On the other hand, under the alternatives (3.14), the statistic $U_n(\mu, A)$ can be shown to have the same limiting distribution as it does under $H_n(\mu, A)$, while the limiting distribution of $(K_n M_n)^{-\frac{1}{2}}[S_n(\mu, A) - K_n M_n]$ becomes $N(\alpha^2, 2)$ by Theorem 2. Consequently, $S_n(\mu, A)$ is the asymptotically more powerful test statistic here.

The phenomenon just noted, that neither the $S_n(\mu, A)$ nor the $U_n(\mu, A)$ test dominates the other uniformly in asymptotic power, is logically related to a result described in Section 3 of Rosenblatt (1975): a goodness-of-fit test based upon a quadratic functional of a window density estimator neither dominates nor is dominated by a goodness-of-fit test based directly upon the empirical cdf. In fact, the test statistic $U_n(\mu, A)$ discussed above is an extended analogue of the Cramér-von Mises goodness-of-fit test (cf. Beran (1975)) while $S_n(\mu, A)$ can be viewed as a quadratic functional of a certain Fourier series density estimator (see (1.5)).

4. Effect of estimating the parameters. In most cases, the parameters μ, A are not known but can be estimated from the data, under the assumption of ellipsoidal symmetry. Let $\hat{\mu}_n$, $\hat{A}_n$ denote estimators such that $(\hat{\mu}_n - \mu, \hat{A}_n - A) = O_p(n^{-\frac{1}{2}})$ under $H_n(\mu, A)$. One construction of such estimators is given by the appropriate class of M-estimators (see Maronna (1976) for details). This section examines the asymptotic behavior of the statistic $S_n(\hat{\mu}_n, \hat{A}_n)$.

To simplify the theoretical calculations, the estimators $\hat{\mu}_n$, $\hat{A}_n$ will be replaced with discretized versions μ_n^*, A_n^* which are defined as follows. Let b be an arbitrary positive constant. Since A is nonsingular, the parameter space for (μ, A) is an open subset of a euclidean space having dimension $(p^2 + 3p)/2$. Pave this parameter space with cubes of side length $n^{-\frac{1}{2}}b$. Set (μ_n^*, A_n^*) equal to the center of the cube containing the realized value of $(\hat{\mu}_n, \hat{A}_n)$ except when the second coordinate of that center is a singular matrix; in that case define (μ_n^*, A_n^*) as the cube center estimator plus $(0, n^{-\frac{1}{2}}bI)$, where I is the $p \times p$ identity matrix. Since the difference $(\hat{\mu}_n -$

$\mu_n^*, \hat{A}_n - A_n^*) = O_p(n^{-\frac{1}{2}})$ in either case, it follows that $(\mu_n^* - \mu, A_n^* - A) = O_p(n^{-\frac{1}{2}})$ under $H_n(\mu, A)$.

The notation of the previous sections is retained in the statement and proof of the following result.

THEOREM 3. *Suppose the functions $\{a_k; k \geqslant 1\}$, $\{b_m; m \geqslant 1\}$ are differentiable and*

(4.1) (i) $\lim_{n\to\infty} K_n = \lim_{n\to\infty} M_n = \infty$;

(ii) $\lim_{n\to\infty} n^{-\frac{1}{2}} K_n^{-\frac{1}{2}} M_n^{\frac{1}{2}} \sum_{k=1}^{K_n} \|a_k'\| = \lim_{n\to\infty} n^{-\frac{1}{2}} M_n^{-\frac{1}{2}} K_n^{\frac{1}{2}} \sum_{m=1}^{M_n} \|b_m'\| = 0$;

(iii) *μ_n^*, A_n^* are defined as above;*

(iv) *the density h is continuous on R^p;*

(v) *there exists a $p \times 1$ vector valued function $\dot{h}(x)$ such that for every $p \times p$ matrix B and every $p \times 1$ vector c, the function $h^{-\frac{1}{2}}(x)(Bx + c)^T\dot{h}(x)$ is square integrable over R^p and*

$$\lim_{n\to\infty} \int \Big\{ n^{\frac{1}{2}} \Big[h^{\frac{1}{2}}\Big(x + n^{-\frac{1}{2}} B_n x + n^{-\frac{1}{2}} c_n\Big) - h^{\frac{1}{2}}(x) \Big] - 2^{-1} h^{-\frac{1}{2}}(x)(Bx + c)^T \dot{h}(x) \Big\}^2 dx = 0$$

for every sequence $\{(B_n, c_n)\}$ converging to (B, c). Then, under $H_n(\mu, A)$,

$$(K_n M_n)^{-\frac{1}{2}} \big[S_n(\mu_n^*, A_n^*) - W_n(\mu, A) \big] \to_p 0 \tag{4.2}$$

as $n \to \infty$. Moreover, if the assumptions of Theorem 1 are also satisfied, the limiting distribution of $(K_n M_n)^{-\frac{1}{2}}[S_n(\mu_n^*, A_n^*) - K_n M_n]$ under $H_n(\mu, A)$ is $N(0, 2)$.

PROOF. It suffices to establish (4.2). Let $\{(\mu_n, A_n); n \geqslant 1\}$ be an arbitrary sequence of values in the parameter space such that $(\mu_n - \mu, A_n - A) = O(n^{-\frac{1}{2}})$. We show first that

$$(K_n M_n)^{-\frac{1}{2}} \big[S_n(\mu_n, A_n) - W_n(\mu, A) \big] \to_p 0 \tag{4.3}$$

as $n \to \infty$. If (4.3) were not true, there would exist a subsequence of $\{(\mu_n, A_n); n \geqslant 1\}$ which contained no further subsequence for which (4.3) would hold. Thus, in verifying (4.3), the existence of limits $\delta\mu = \lim_{n\to\infty} n^{\frac{1}{2}}(\mu_n - \mu)$ and $\delta A = \lim_{n\to\infty} n^{\frac{1}{2}}(A_n - A)$ may be assumed without loss of generality.

Under $H_n(\mu, A)$, the joint density of the residuals $\{A_n(X_i - \mu_n); 1 \leqslant i \leqslant n\}$, which are used in computing $S_n(\mu_n, A_n)$, is $\prod_{i=1}^n h_n(x_i)$, where

$$h_n(x) = \det\big[I + n^{-\frac{1}{2}} A^{-1}(\delta A)_n\big] \cdot h\big[x + n^{-\frac{1}{2}} A^{-1}(\delta A)_n x + n^{-\frac{1}{2}} A^{-1}(\delta\mu)_n\big] \tag{4.4}$$

and $(\delta\mu)_n = n^{\frac{1}{2}}(\mu_n - \mu)$, $(\delta A)_n = n^{\frac{1}{2}}(A_n - A)$. Since the determinant in (4.4) equals $1 + n^{-\frac{1}{2}}\mathrm{tr}[A^{-1}(\delta A)] + o(n^{-\frac{1}{2}})$, it follows, using the assumed properties of the density h, that

(4.5) $$2n^{\frac{1}{2}}\left[h_n^{\frac{1}{2}}(x) - h^{\frac{1}{2}}(x)\right]$$
$$= \operatorname{tr}\left[A^{-1}(\delta A)\right]h^{\frac{1}{2}}(x)$$
$$+h^{-\frac{1}{2}}(x)\left[A^{-1}(\delta A)x + A^{-1}(\delta\mu)\right]^T\dot{h}(x) + \varepsilon_n(x),$$

where $\lim_{n\to\infty}\int\varepsilon_n^2(x)dx = 0$. This implies that the two sequences of probabilities with densities $\{\Pi_{i=1}^n h_n(x_i)\}$ and $\{\Pi_{i=1}^n h(x_i)\}$ are contiguous (cf. Le Cam (1969)). Consequently, in view of Lemma 1,

(4.6) $$(K_nM_n)^{-\frac{1}{2}}\left[S_n(\mu_n, A_n) - W_n(\mu_n, A_n)\right] \to_p 0$$

under $H_n(\mu, A)$ as $n \to \infty$.

The random variable $W_n(\mu_n, A_n)$ can be written in the form

(4.7) $$W_n(\mu_n, A_n) = \Sigma_{k,m}^{K_n,M_n}\left[n^{-\frac{1}{2}}\Sigma_{i=1}^n a_k(U_i(\mu_n, A_n))b_m(\Theta_i(\mu_n, A_n))\right]^2,$$

where $U_i(\mu_n, A_n) = F(|A_n^{-1}(X_i - \mu_n)|)$ and $\Theta_i(\mu_n, A_n)$ is the polar angle vector of $A_n^{-1}(X_i - \mu_n)$; F is still the cdf of $A^{-1}(X_i - \mu)$ under $H_n(\mu, A)$. By algebraic manipulation, $W_n(\mu_n, A_n)$ can be expressed as the sum of $W(\mu, A)$ and two other terms:

(4.8) $$J_{1,n} = n^{-1}\Sigma_{k,m}^{K_n,M_n}\Sigma_{i=1}^n\left[c_{k,m}(Z_i(\mu_n, Z_n)) - c_{k,m}(Z_i)\right]^2$$
$$J_{2,n} = 2n^{-1}\Sigma_{k,m}^{K_n,M_n}\Sigma_{i=1}^n c_{k,m}(Z_i)\left[c_{k,m}(Z_i(\mu_n, A_n))c_{k,m}(Z_i)\right],$$

where $Z_i(\mu_n, A_n) = (U_i(\mu_n, A_n), \Theta_i(\mu_n, A_n))$ and $c_{k,m}$ is defined as in Section 2. Since h is continuous, the density of F is also continuous and bounded. Consequently $Z_i(\mu_n, A_n) = Z_i + O_p(n^{-\frac{1}{2}})$ and both $J_{1,n}$ and $J_{2,n}$ are $O_p(n^{-\frac{1}{2}}K_n \Sigma_{m=1}^{M_n}\|b_m'\|) + O_p(n^{-\frac{1}{2}}M_n\Sigma_{k=1}^{K_n}\|a_k'\|)$. Therefore,

(4.9) $$(K_nM_n)^{-\frac{1}{2}}\left[W_n(\mu_n, A_n) - W_n(\mu, A)\right] \to_p 0$$

as $n \to \infty$. Combining (4.9) with (4.6) establishes (4.3).

Let $B(r)$ denote the ball of radius $n^{-\frac{1}{2}}r$ and center (μ, A) in the parameter space, viewed as an open subset of euclidean space of dimension $(p^2 + 3p)/2$. The ball $B(r)$ contains only a finite number $N(r)$ of the possible values of the discretized estimator (μ_n^*, A_n^*); let $\{(\mu_{n,i}, A_{n,i}); 1 \leqslant i \leqslant N(r)\}$ denote these possible values. The convergence (4.3) implies that

(4.10) $$\max_{1\leqslant i\leqslant N(r)}(K_nM_n)^{-\frac{1}{2}}\left[S_n(\mu_{n,i}, A_{n,i}) - W_n(\mu, A)\right] \to_p 0$$

as $n \to \infty$. Since $(\mu_n^* - \mu, A_n^* - A) = O_p(n^{-\frac{1}{2}})$ and r can be chosen arbitrarily large, (4.10) in turn implies the validity of (4.2). The theorem follows immediately.

REMARKS. While $S_n(\mu, A)$ and $S_n(\mu_n^*, A_n^*)$ have the same asymptotic distributions under the conditions of Theorem 3, their exact distributions under $H_n(\mu, A)$

differ. Moreover, the exact distribution of $S_n(\mu_n^*, A_n^*)$ depends upon the specific estimator used in place of the parameters (μ, A).

In general, the asymptotic power of the $S_n(\mu_n^*, A_n^*)$ test is not the same as that of the $S_n(\mu, A)$ test. Typical estimators (μ_n^*, A_n^*) which are root-n consistent for (μ, A) under $H_n(\mu, A)$ possess only the weaker property $(\mu_n^* - \mu, A_n^* - A) = O_p(\alpha_n)$ under the local alternatives studied in Section 3. Thus, the perturbation induced in the distrubution of the statistic $S_n(\mu, A)$ when (μ, A) is replaced by (μ_n^*, A_n^*) is of the same order as that caused by the alternatives acting directly on $S_n(\mu, A)$.

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