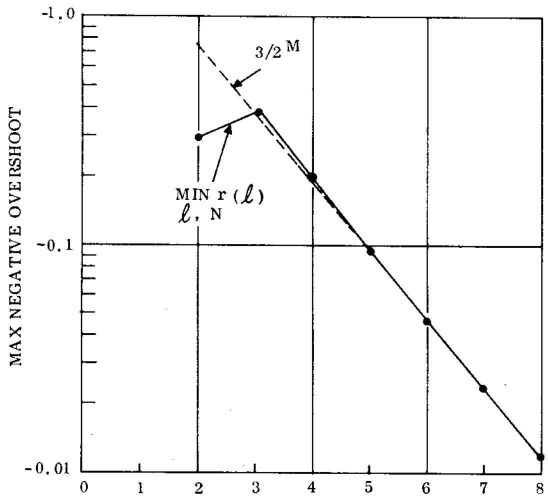

Fig. 2. Maximum negative overshoot of M-bit binary word autocorrelation function.

III. Results

Equation (14) is plotted in Fig. 1 for several values of the word length M. Each set of curves approaches a limiting curve for large N in accordance with (12) for $\alpha = 2$. In addition, successive sets of curves for progressively larger word lengths can be seen to approach the limiting exponential 2^{-l} in accordance with (13).

It can be shown that, except for the single case $N = M = 2$, the maximum negative overshoot of the word autocorrelation function for any word length M is given by the value $r(M)$ for $N = M$, as can be seen in Fig. 1. For $\alpha = 2$, this value is given by

$$r_y(M) = -\frac{3(2^M - 1)}{2^M(2^M + 1) - 3(2^M - 1)}, \qquad N = M. \tag{16}$$

For large word length M, the limiting form of (16) is

$$r_y(M) \xrightarrow[N=M\to\infty]{} -3/2^M. \tag{17}$$

Equations (16) and (17) are plotted in Fig. 2 as a function M. As can be seen, the largest overshoot occurs for $M = 3$ and for $M > 4$; the asymptotic behavior of (16) closely follows the limiting expression (17).

IV. Conclusions

In this correspondence, a closed-form expression has been obtained for the autocorrelation function of successive M-bit digital words taken from a single PN sequence of length $L = 2^N - 1$. This function $R_y(l)$ specifies the correlation between words separated in time by l clock cycles. For any given values of l and M, this expression can be used to find the value of N which will minimize the correlation between words.

Alternatively, a word length M can be specified such that successive words separated by $l \geq M$ clock cycles will be correlated to less than a specified amount.

References

[1] S. W. Golomb, *Shift Register Sequences.* San Francisco: Holden-Day, 1967.

[2] F. J. MacWilliams and N. J. A. Sloane, "Pseudo-random sequences and arrays," *Proc. IEEE,* vol. 64, pp. 1715–1729, Dec. 1976.

[3] A. C. Davies, "Properties of waveforms obtained by nonrecursive digital filtering of pseudo-random binary sequences," *IEEE Trans. Comput.,* vol. C-20, pp. 270–281, Mar. 1971.

Testing for Single Intermittent Failures in Combinational Circuits by Maximizing the Probability of Fault Detection

JACOB SAVIR

***Abstract*—Intermittent faults in combinational circuits may appear and disappear randomly; hence, their detection requires many repeated applications of test vectors. Since testing reduces the time available for computation, it is necessary to efficiently minimize the time required for a test, while still achieving a high degree of fault detection.**

This paper presents an optimal random test procedure to detect intermittent failures. The algorithm maximizes the probability of fault detection by optimally choosing the input vector probabilities.

***Index Terms*—Error latency, intermittent fault detection, irredundant circuit, maximum likelihood estimator, random testing.**

I. Introduction

Many systems in the field today perform critical functions which require extremely high availability (for example, computers in airplanes, hospitals, etc.). One method of achieving this goal is by periodic testing of each subsystem, and whenever a fault is detected, that faulty subsystem is repaired or replaced.

Intermittent failures (IF's) can be defined as failures that change the normal function of a circuit during randomly occurring intervals of time.

IF's may have a variety of causes: external effects such as temperature, humidity, thermal vibration, mechanical vibration, power fluctuation, pollution, or pressure may cause loose connections. Other IF's may appear as hazards, races, etc.

Studies [1], [5] indicate that IF's have comprised over 30 percent of predelivery failures and almost 90 percent of field failures in several computer systems; therefore, an efficient way to detect IF's is required.

Numerous papers have been published in the literature regarding models and testing procedures for detecting IF's. Continuous-time models have been used by Breuer [2], Spillman [22], and Su [11], while discrete-time models have been used by Kamal [6], [7], Koren [10], and Savir [15]–[20]. The testing strategies can be categorized as either deterministic or probabilistic in nature. Kamal [6], [7] and Koren [10] describe properties of deterministic test procedures, while Savir [15]–[20] describes random test procedures for detecting IF's.

When dealing with random testing, it is useful to distinguish between *time-variant random testing* and *time-invariant random testing* [16]. Time-invariant random testing refers to testing for which input vectors are applied with a constant probability distribution during the entire test procedure, while time-variant random testing refers to a strategy which allows the probability distribution to change as the test proceeds. Only time-invariant random testing will be considered in this paper.

This paper presents an optimal algorithm to detect IF's in combinational circuits. It is optimal in the sense that the algorithm suggested maximizes the probability of fault detection. Optimal detection is achieved by applying input vectors according to an optimal probability distribution. It is shown that the set of input vectors used for optimal detection of the IF's is not necessarily a minimal solid fault detection experiment for the corresponding permanent faults.

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Section II presents the basic definitions and assumptions related to optimal testing. In Section III, the optimal intermittent fault detection experiment is developed. Section IV presents the algorithm and provides a few examples and Section V concludes with a brief summary.

II. Basic Definitions and Assumptions

In this paper, we consider only *stationary* intermittent failures, meaning that the same fault will physically change the circuit in the same way each time it appears. When a certain malfunction occurs in the circuit, we say that a certain intermittent fault is *present* in it (we assume that the circuit is fault free with respect to permanent faults). When an IF is present in the circuit, it can be active at some times, and inactive at other times. We say that an IF is *active* at a certain time if it changes the function of the circuit at that time (justified by Assumption 3 to follow).

Definition 1: An IF f_i, when active, partitions the set of all possible input vectors into two disjoint subsets. The elements of the *affected subset* AF_i of input vectors yield an incorrect output under the influence of the fault. Elements of the *unaffected subset* UAF_i yield the correct output.

Definition 2: The *error probability* [21] a_i of a fault f_i is the conditional probability of an incorrect output under the application of a randomly chosen input vector, given that fault f_i is active.

$$a_i = \Pr\{\text{input vector} \in AF_i\}. \tag{1}$$

Definition 3: The *error latency* [21] EL_i of an IF f_i is the number of input vectors applied to a digital circuit while f_i is present until the first incorrect output due to f_i is observed.

In practice, the testing of a combinational circuit will be done by a fast clocked machine where one test will be applied per each clock cycle. As suggested in Definition 3, we can measure the error latency by the number of clock cycles elapsed until the first error is observed rather than by the time elapsed. Hence, the error latency is a discrete random variable taking on values in the set of positive integers.

The following are assumed to hold for the circuit under test.

Assumption 1: The faults are *well behaved,* namely, during an application of an input vector, the circuit behaves as if it is fault free or else a certain intermittent fault is active [2].

Assumption 2: The application of input vectors is random and independent of the activity times of the IF's [15].

Assumption 3: The circuit is *irredundant,* namely, if the IF's were solid, every single fault would be detectable.

Assumption 4: Only one out of m possible IF's may be present in the circuit. The set of possible faults $F = \{f_1, f_2, \cdots, f_m\}$ is assumed to be known to the test designer.

Assumption 5: The fault parameters are known or estimated (see the Appendix), namely,

a) $\Pr\{f_i \text{ is present} \mid \text{circuit is faulty}\}$

$$= w_i > 0, \qquad i = 1, 2, \cdots, m \tag{2}$$

where

$$\sum_{i=1}^{m} w_i = 1. \tag{3}$$

w_i is called the *probability of presence*

b) $\Pr\{f_i \text{ is active in any clock cycle} \mid f_i \text{ is present}\}$

$$= p_i > 0, \qquad i = 1, 2, \cdots, m. \tag{4}$$

p_i is called the *probability of activity* [16].

Note that the probabilities of activity do not form a probability distribution. In fact, the following inequality holds:

$$0 < \sum_{i=1}^{m} p_i \leq m$$

where the equality sign holds for the case of solid faults.

III. The Optimal Intermittent Fault Detection Experiment

Definition 4: An *optimal assignment of input probabilities* is an input probability distribution which maximizes the probability of fault detection, with the constraint that no more than n input vectors are applied.

Clearly, the optimal assignment of input probabilities is not unique because different tests from the same affected subset might be used to detect a certain fault. For example, in the case of only one fault with $AF = \{t_1, t_2\}$, all the distributions of the form $\Pr\{t_1\} = u$, $\Pr\{t_2\} = 1 - u$, $0 \leq u \leq 1$ are valid optimal assignments. In these cases, we agree to use either $u = 0$ or $u = 1$ for the optimal distribution. As implied in Definition 4, the optimal assignment will, in general, depend on n.

Definition 5: A set $S \subseteq \cup_{i=1}^{m} AF_i$ is called a *generating set* if it contains at least one element from each affected subset.

Definition 6: An *IF detection experiment* for F is a finite string of n tests which are all members of a generating set S. The length n of this string is called the *length of the experiment.*

Generating sets and IF detection experiments are usually not unique. In general, different experiments yield different detection probabilities. Note that the shortest string formed using all members of any generating set is a permanent-fault detection experiment of F. Such a string formed from the smallest possible generating set (smallest cardinality) is the minimal permanent-fault detection experiment for F.

In principle, any generating set can be used to generate an IF detection experiment because every such set covers all faults under consideration. However, there is a substantial difference in the testing quality achieved for various generating sets. The following definition describes those generating sets which will be used in designing the optimal experiment.

Definition 7: An *optimal generating set* S^* is a generating set for which an optimal assignment of probabilities exists which assigns zero probability to every test not in S^*.

Because every IF detection experiment must have finite length, there is no guarantee that at the end of any experiment, a definite conclusion regarding the state of the circuit (faulty or fault free) could be drawn. When enough consecutive correct outputs are observed, it is necessary to stop and decide with a certain confidence that the circuit is fault free. Such a stopping decision rule should be based on the confidence that the circuit possesses no IF's, which is a monotonically increasing function of the number of consecutive correct outputs that are observed. The testing quality and hence the decision rule for stopping the experiment, is based on the escape probability defined below.

Definition 8: The *escape probability* P_s of a fault set F is the conditional probability that the fault (which is a member of F) will go undetected during an application of n randomly selected input vectors, given that the circuit is faulty [16].

In order to carry out the optimization procedure, we need an expression for the cumulative distribution function (cdf) of the error latency F_{EL} for the case of detection of single faults from a fault set F.

$$F_{EL}(n) = \Pr\{EL \leq n \mid \text{circuit is faulty}\}. \tag{5}$$

In [16], the cdf of the error latency of a specific fault f_j is derived to be

$$F_{EL_j}(n) = \Pr\{EL_j \leq n \mid f_j \text{ is present}\} = 1 - (1 - b_j)^n \tag{6}$$

where

$$b_j = a_j \cdot p_j. \tag{7}$$

Hence,

$$F_{EL}(n) = \sum_{j=1}^{m} \Pr\{EL_j \leq n \mid f_j \text{ is present}\} \cdot \Pr\{f_j \text{ is present} \mid \text{circuit is faulty}\} \tag{8}$$

$$F_{EL}(n) = \sum_{j=1}^{m} w_j \cdot F_{EL_j}(n). \tag{9}$$

Using (6) and (9), we obtain

$$F_{EL}(n) = 1 - \sum_{j=1}^{m} w_j \cdot (1 - b_j)^n = 1 - P_s. \tag{10}$$

The problem can now be specified in more exact terms: we are looking for an optimal assignment that maximizes $F_{EL}(n)$ (or equivalently minimizes P_s) for a given n.

Lemma 1: In a combinational circuit with $F = \{f_1, f_2\}$ and conjoint affected subsets AF_1 and AF_2, an optimal generating set is the one consisting of a single test $t \in AF_1 \cap AF_2$.

Proof: See [17].

Lemma 1 indicates what would be an optimal generating set in the most general case. The members of the optimal generating set should be selected out of the *minimal intersections* of the affected subsets generated by Procedure 1.

Procedure 1:

Step 1: Create all intersections between AF_i and AF_j for all $i \neq j$, and add them to the collection of the original sets $\{AF_1, AF_2, \cdots, AF_m\}$. If all intersections are empty, stop—those sets are the minimal intersections. Otherwise, mark all original sets which have nonempty intersections with at least one other set.

Step 2: Delete all the marked sets. If two or more intersections are identical, delete all but one copy of that intersection.

Step 3: Rename the remaining sets $\{AF_1, AF_2, \cdots\}$ and go to Step 1.

Fig. 1 shows an example of generating the minimal intersections.

Let q be the number of minimal intersections generated by Procedure 1, and denote them by $\{T_1, T_2, \cdots, T_q\}$. An optimal generating set will be

$$S^* = \{t_1, t_2, \cdots, t_q\}; \qquad t_i \in T_i, i = 1, 2, \cdots, q. \tag{11}$$

Note that at most there exists one minimal intersection for each two affected subsets; hence, the following inequality holds:

$$|S^*| = q \leq \binom{m}{2} = \frac{m(m-1)}{2}. \tag{12}$$

The testing information embedded in the optimal generating set can be summed up in the *testing matrix E*. The testing matrix is defined to have m rows corresponding to the m possible faults and q columns corresponding to the members of the optimal generating set. The entries e_{ij} are obtained by

$$e_{ij} = \begin{cases} 1, & \text{if } t_j \text{ is a test for fault } f_i \\ 0, & \text{otherwise.} \end{cases}$$

Let x_j be the probability assignment to the test t_j, namely,

$$x_j = \Pr\{\text{input vector} = t_j\}.$$

Since all tests not in S^* are given zero probability, we have

$$\sum_{j=1}^{q} x_j = 1.$$

The error probabilities can be calculated from E:

$$a_i = \Pr\{\text{input vector} \in AF_i\} = \sum_{j=1}^{q} e_{ij} \cdot x_j = \mathbf{e}_i \cdot \mathbf{x}^{tr} \tag{13}$$

where

$$\mathbf{e}_i = (e_{i1}, e_{i2}, \cdots, e_{iq}), \tag{14}$$

$$\mathbf{x} = (x_1, x_2, \cdots, x_q), \tag{15}$$

and $\mathbf{x}^{tr}$ denotes $\mathbf{x}$—transposed.

Notice that, in general, $\sum_{i=1}^{m} a_i \geq 1$, with the equality holding only when the affected subsets are disjoint. The optimal assignment of input probabilities can be obtained by minimizing the following function:

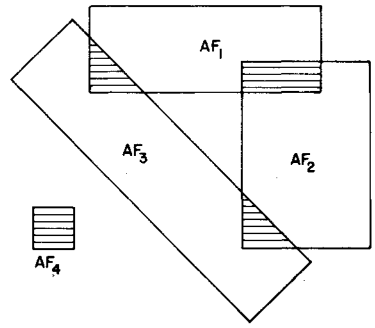

Fig. 1. Set of minimal intersections: $\{AF_1 \cap AF_2, AF_1 \cap AF_3, AF_2 \cap AF_3, AF_4\}$.

$$P_s = \sum_{j=1}^{m} w_j (1 - p_j \mathbf{e}_j \mathbf{x}^{tr})^n \tag{16}$$

under the constraints

$$x_j \geq 0 \qquad \text{for all } j = 1, 2, \cdots, q$$

and

$$\sum_{j=1}^{q} x_j = 1. \tag{17}$$

Lemma 2: The escape probability is a convex function of the probability assignment to the input vectors.

Proof: Let $n > 1$ (for $n = 1$, the optimal assignment is always on the boundary, and therefore is not an interesting case). We have to show that

$$P_s[\beta \mathbf{x}_1 + (1-\beta)\mathbf{x}_2] \leq \beta P_s(\mathbf{x}_1) + (1-\beta) P_s(\mathbf{x}_2) \qquad \text{for all}$$

$$\mathbf{x}_1, \mathbf{x}_2 \in \left\{ \mathbf{x} \middle| x_i \geq 0, \sum_{i=1}^{q} x_i = 1 \right\}$$

$$\text{such that } \mathbf{x}_1 \neq \mathbf{x}_2, 0 \leq \beta \leq 1. \tag{18}$$

Statement (18) can be proved by simply showing that each term in (16) is a convex function of $\mathbf{x}$. For this purpose, we use the inequality

$$\left(\sum_{i=1}^{k} \alpha_i z_i^t\right)^{1/t} < \left(\sum_{i=1}^{k} \alpha_i z_i^s\right)^{1/s}, \qquad 0 < t < s \tag{19}$$

where

$$z_i \geq 0, \alpha_i \geq 0, \sum_{i=1}^{k} \alpha_i = 1.$$

Let

$$Q_j = w_j (1 - p_j \mathbf{e}_j \mathbf{x}^{tr})^n.$$

Thus,

$$\frac{1}{w_j} [\beta Q_j(\mathbf{x}_1) + (1-\beta) Q_j(\mathbf{x}_2)]^{1/n} > \beta(1 - p_j \mathbf{e}_j \mathbf{x}_1^{tr})$$

$$+ (1-\beta)(1 - p_j \mathbf{e}_j \mathbf{x}_2^{tr}) = \frac{1}{w_j} \{Q_j[\beta \mathbf{x}_1 + (1-\beta)\mathbf{x}_2]\}^{1/n}. \quad \text{Q.E.D.}$$

The problem of minimizing (16) under the constraints described in (17) is equivalent to that of minimizing the function

$$z^* = \sum_{j=1}^{m} w_j (1 - p_j \mathbf{e}_j \mathbf{x}^{tr})^n + \mu_0 \left(\sum_{j=1}^{q} x_j - 1\right) + \sum_{j=1}^{q} \mu_j (y_j^2 - x_j) \tag{20}$$

under the constraint

$$\sum_{j=1}^{q} x_j = 1 \text{ and } y_j^2 - x_j = 0 \qquad \text{for all } j = 1\ 2, \cdots, q.$$

Note that since the function z^* is a sum of P_s and linear terms in $\mathbf{x}$, the stationary solution of z^* is also the global minimum.

The optimization procedure can be carried out by defining two sets of indices S_0 and S_1 where the members of S_0 are those indices k of tests $t_k \in S^*$ for which $x_k = 0$, and the members of S_1 are those for which $x_k > 0$. The following procedure describes the algorithm of deriving the optimal assignment of input probabilities.

Procedure 2:

Step 1: Let $S_0 = \phi$ and let $S_1 = \{1, 2, \cdots, q\}$.

Step 2: Calculate $x_j, j \in S_1$ by solving

$$-\sum_{j=1}^{m} nw_jp_je_{ji}(1 - p_j\,\mathbf{e}_j\,\mathbf{x}^{tr})^{n-1} + \mu_0 = 0, \qquad i \in S_1, n > 1$$

$$\sum_{j \in S_1} x_j = 1. \tag{21}$$

Step 3: If $x_j \le 0$, transfer the index j from S_1 to S_0. Go to Step 2.

Step 4: If $x_j > 0$ for all $j \in S_1$, then the current probability assignment is the optimal one.

The solution to (21) requires proper numerical methods [3].

A. Detection of Intermittent Faults which have Disjoint Affected Subsets

This is the case where the testing matrix E is square and diagonal. For this simple case, it is possible to analyze the behavior of the optimal solution. In this case, the solution to (21) is given by

$$x_j = \frac{1}{p_j} \cdot \left[1 - \frac{1}{(w_jp_j)^{1/(n-1)}} \cdot \frac{\sum_{k \in S_1} (1/p_k) - 1}{\sum_{k \in S_1} 1/p_k(w_kp_k)^{1/(n-1)}}\right], \qquad j \in S_1. \tag{22}$$

In most practical circuits, the length of the experiment will be quite large. Note that 1 s of testing time with a 100 ns/instruction machine is equivalent to $n = 10^7$. Since testing for permanent faults in a real size circuit takes at least few seconds of computer time, it is evident that testing for intermittent faults will take at least as much (in order to achieve a reasonable escape probability). Thus, for most practical applications, the actual optimal assignment of input vector probabilities will be very close to the asymptotic one derived below.

Definition 9: The *effective probability of activity* P_{ef} is defined to be

$$P_{ef} = \frac{1}{\sum_{j=1}^{m} 1/p_j}. \tag{23}$$

Lemma 3: The effective probability of activity lies in the region $(0, 1/m]$ and satisfies

$$P_{ef} \le \min_k (p_k).$$

Proof: See [17].

Lemma 4: If the affected subsets are disjoint, then as the length of the experiment becomes large, the optimal assignment of input vector probabilities approaches a limit distribution which is independent of w_j for all $j = 1, 2, \cdots, m$ and is given by

$$\lim_{n \to \infty} x_j = \frac{P_{ef}}{p_j}, \qquad j = 1, 2, \cdots, m. \tag{24}$$

Proof: The proof follows directly from Procedure 2 and (22).

Q.E.D.

Letting n be arbitrarily large, an asymptotic expression for the cdf of the error latency is obtained:

$$F_{EL}^{\mathrm{OPT}}(n) \cong 1 - \frac{1}{P_{ef}} \cdot (1 - P_{ef})^n. \tag{25}$$

The escape probability achieved under optimal probability assignment for the affected subsets asymptotically approaches

$$\Pr\{\text{escape} | \text{circuit is faulty}\} = 1 - F_{EL}^{\mathrm{OPT}}(N) \cong \frac{1}{P_{ef}} \cdot (1 - P_{ef})^N. \tag{26}$$

Hence, the least upper bound (lub) to the experiment length needed to gain an escape probability of P_s is given by

$$N_s \cong \frac{\ln (P_{ef} \cdot P_s)}{\ln (1 - P_{ef})}. \tag{27}$$

An important conclusion of the above treatment is that the optimal generating set is not necessarily the generating set used to form a minimal permanent fault detection experiment for F (see Example 1). In general, if a test set designed to detect permanent faults is used to detect intermittent faults, it will result in a poor testing quality unless this test set happens to also be an optimal generating set.

IV. The Algorithm

Assuming that the test designer has the following data at hand: F, $\{w_i: i = 1, 2, \cdots, m\}$, $\{p_i: i = 1, 2, \cdots, m\}$, and P_s, he can design the optimal IF detection experiment according to the following steps.

Procedure 3:

Step 1: Find the optimal generating set S^*, the optimal assignment $\mathbf{x}^{\mathrm{OPT}}$, and the lub of the experiment length N_s by jointly solving (16) with Procedure 2.

Step 2: Set counter to $N = 1$.

Step 3: Apply input vectors from S^* generated by the optimal assignment $\mathbf{x}^{\mathrm{OPT}}$. If an output error is observed, go to Step 6; otherwise, continue.

Step 4: If $N = N_s$, go to Step 7; otherwise, continue.

Step 5: Increase counter to $N + 1$. Go to Step 3.

Step 6: Conclude circuit is faulty and stop.

Step 7: Conclude circuit is fault free and stop.

Example 1: The following testing matrix has been derived for a combinational circuit having three possible faults:

$$E = \begin{array}{c} \\ f_1 \\ f_2 \\ f_3 \end{array} \begin{array}{c} \begin{array}{ccc} t_1 & t_2 & t_3 \end{array} \\ \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} \end{array}.$$

The fault parameters are given:

$$w_i = 1/3; \qquad i = 1, 2, 3$$
$$p_i = p; \qquad i = 1, 2, 3.$$

The optimal generating set, and a generating set which constitutes a minimal permanent-fault detection experiment for F, are, respectively,

$$S^* = \{t_1, t_2, t_3\}$$

and

$$S_m = \{t_1, t_2\}.$$

The optimal assignment of input probabilities is (1/3, 1/3, 1/3), which gives

$$F_{EL,S^*}(n) = 1 - \left(1 - \frac{2p}{3}\right)^n.$$

On the other hand, $F_{EL,S_m}(n)$ is maximized by the distribution (1/2, 1/2, 0), giving

$$F_{EL,S_m}(n) = 1 - \frac{1}{3}\left[2\left(1 - \frac{p}{2}\right)^n + (1-p)^n\right].$$

TABLE I
GAIN IN TEST QUALITY VERSUS EXPERIMENT LENGTH FOR EXAMPLE 1

n	10	20	50	100
Gain	2	7	240	87×10^3

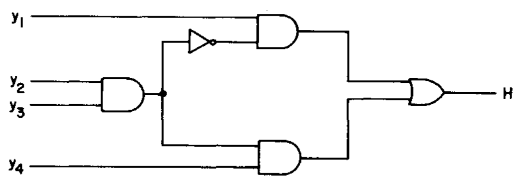

Fig. 2. Example 2.

TABLE II
THE OPTIMAL ASSIGNMENT FOR INPUT VECTOR 7 AS A FUNCTION OF THE EXPERIMENT LENGTH IN EXAMPLE 2

n	6	11	21	51	101	501
x	0.487	0.529	0.550	0.563	0.567	0.570

The gain in test quality achieved by using S^* over S_m (the ratio of the two escape probabilities which result from the two generating sets) for $p = 1/2$ and various values of n is shown in Table I.

The gain increases very fast, asymptotically approaching $2/3 \cdot (9/8)^n$.

This example emphasizes the importance of using an optimal generating set rather than some other arbitrary generating set in the detection procedure.

Example 2: Consider the combinational circuit of Fig. 2 with the intermittent stuck-at-faults (we denote by $f/0$ fault f stuck-at 0 and by $f/1$ fault f stuck-at 1).

$$F = \{y_2/0; y_4/0; y_4/1\}.$$

Denote

$$f_1 = y_2/0; f_2 = y_4/0; f_3 = y_4/1.$$

The following fault parameters are given:

$$w_i = \frac{1}{3}; \qquad i = 1, 2, 3$$

$$p_1 = \frac{1}{2}; p_2 = \frac{1}{4}; p_3 = \frac{1}{3}.$$

The affected subsets of F can be obtained by using the Boolean difference technique [26]:

$$y_2 \cdot \frac{\partial H}{\partial y_2} = 1 \text{ which implies } AF_1 = \{7, 14\}.$$

In the same way, we obtain

$$y_4 \cdot \frac{\partial H}{\partial y_4} = 1 \Rightarrow AF_2 = \{7, 15\}$$

$$\bar{y}_4 \cdot \frac{\partial H}{\partial y_4} = 1 \Rightarrow AF_3 = \{6, 14\}.$$

Here, the tests are represented in decimal notation. For instance, 7 $\Leftrightarrow y_1 = 0, y_2 = y_3 = y_4 = 1$. Applying Procedure 1 yields the unique optimal generating set

$$S^* = \{7, 14\}.$$

The corresponding testing matrix is

$$E = \begin{array}{c c} & \begin{array}{cc} 7 & 14 \end{array} \\ \begin{array}{c} f_1 \\ f_2 \\ f_3 \end{array} & \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \end{array}$$

Let the optimal assignment be

$$\Pr\{\text{input vector} = 7\} = x$$

$$\Pr\{\text{input vector} = 14\} = 1 - x.$$

Procedure 2 yields the following optimal assignment:

$$x = \frac{(3/4)^{1/n-1} - 2/3}{1/3 + 1/4 \cdot (3/4)^{1/n-1}}; \qquad n > 1.$$

The asymptotic optimal assignment is $(4/7; 3/7)$. Table II shows the optimal assignment as a function of the experiment length.

Notice that the optimal assignment for $n = 501$ is only 0.1 percent different from the asymptotic value. From (16), we find that the lub of the experiment length needed to achieve an escape probability of $P_s = 2.6 \times 10^{-4}$ is $N_s = 50$.

V. SUMMARY AND CONCLUSIONS

The assumption of single faults is a first-order approximation to the situation for real circuits. This assumption is often used by authors because it makes the mathematics tractable and leads to satisfactory results.

The optimal solution is achieved in two steps. In the first step, the optimal generating set is found. This optimal generating set depends on the circuit structure and fault set. In the second step, the optimal distribution for the members of the optimal generating set is calculated. The optimal distribution depends on the optimal generating set and fault parameters.

The proposed algorithm for the optimal detection procedure is relatively easy to implement and provides the shortest experiment for achieving a given test quality. No calculation is necessary during the running time of the test procedure, therefore reducing the cost of the testing process.

APPENDIX
ESTIMATION OF THE FAULT PARAMETERS

The estimation of the fault parameters is based on experience with circuits of the same kind. Formally speaking, we assume that a record of previous faults was kept containing the following information.

1) Out of n^* failures of the circuit in a given period of time, fault f_i was present n_i times, $i = 1, 2, \cdots, m$ and

$$\sum_{i=1}^{m} n_i = n^*. \tag{28}$$

2) When the fault f_i was present, $i = 1, 2, \cdots, m$ out of a time period of T_i, it was active a total period of τ_i.

These two statistics can be obtained by testing a standby circuit for a long period of time and then using the data obtained for all other circuits. Certainly, we implicitly assume that the fault mechanism is ergodic from a statistical point of view.

When trying to estimate any sort of parameters, the estimation procedure should be the "best" in some sense; thus, the "maximum likelihood estimator" procedure is used to estimate the fault parameters [14].

Maximum Likelihood Estimation of the Probabilities of Presence

Let N_i be the random variable describing the number of times fault f_i was present out of a total of n^* observations. The observed value of N_i will be denoted by n_i. The set of random variables $\{N_1, \cdots, N_m\}$ have a joint multinomial distribution. The likelihood function is

$$L(w_1, \cdots, w_m) = \Pr\{N_1 = n_1, \cdots, N_m = n_m\} = \binom{n^*}{n_1, n_2, \cdots, n_m}^{\dagger} \cdot w_1^{n1} \cdot w_2^{n2} \cdot \cdots \cdot w_m^{nm}. \quad (29)$$

In order to find the maximum likelihood estimator (MLE), we have to maximize L, which requires maximizing the function

$$L^* = \binom{n^*}{n_1, n_2, \cdots, n_m} \cdot w_1^{n1} \cdot w_2^{n2} \cdot \cdots \cdot w_m^{nm} + M\left(\sum_{i=1}^{m} w_i - 1\right) \quad (30)$$

where M is the Lagrange multiplier.

$$\frac{\partial L^*}{\partial w_i} = \binom{n^*}{n_1, n_2, \cdots, n_m} \cdot n_i \cdot w_1^{n1} \cdot w_2^{n2} \cdot \cdots \cdot w_{i-1}^{n_{i-1}} \cdot w_i^{n_i - 1} \cdot w_{i+1}^{n_{i+1}} \cdot \cdots \cdot w_m^{nm} + M = 0; \qquad i = 1, 2, \cdots, m \quad (31)$$

$$\frac{\partial L^*}{\partial M} = \sum_{i=1}^{m} w_i - 1 = 0. \quad (32)$$

The MLE of w_i, $\hat{w}_i$ is found by solving the following equations:

$$\frac{n_i}{\hat{w}_i} \cdot L + M = 0; \qquad i = 1, 2, \cdots, m \quad (33)$$

$$\sum_{i=1}^{m} \hat{w}_i = 1. \quad (34)$$

The solution to (33) and (34) is given by

$$\hat{w}_i = \frac{n_i}{n^*}; \qquad i = 1, 2, \cdots, m. \quad (35)$$

Notice that this estimator is also unbiased. The random variable N_i is of the binomial type with probability of success w_i. Hence,

$$E\left\{\frac{N_i}{n^*}\right\} = \frac{1}{n^*} E\{N_i\} = w_i. \quad (36)$$

Maximum Likelihood Estimation of the Probabilities of Activity

Let the clock period be Δ. Assuming that Δ is small, the periods T_i and τ_i are approximately integer multiplies of Δ. We therefore denote

$$T_i = A_i \cdot \Delta; \qquad i = 1, 2, \cdots, m \quad (37)$$

$$\tau_i = B_i \cdot \Delta; \qquad i = 1, 2, \cdots, m \quad (38)$$

where B_i is an integer random variable having a binomial distribution with probability of success p_i. Let the observed value of B_i be β_i. The likelihood function will be, therefore,

$$L_i = \Pr\{B_i = \beta_i\} = \binom{A_i}{\beta_i} p_i^{\beta_i} \cdot (1 - p_i)^{A_i - \beta_i}; \qquad i = 1, 2, \cdots, m. \quad (39)$$

† $\binom{n^*}{n_1, n_2, \cdots, n_m} = \frac{n^*!}{n_1!\, n_2! \cdots n_m!}$.

The MLE of p_i, $\hat{p}_i$ is achieved by equating the derivative of L_i with respect to p_i to zero for all i, $i = 1, 2, \cdots, m$. Hence,

$$\frac{dL_i}{dp_i} = \binom{A_i}{\beta_i} [\beta_i \cdot p_i^{\beta_i - 1} \cdot (1 - p_i)^{A_i - \beta_i} - (A_i - \beta_i) \cdot (1 - p_i)^{A_i - \beta_i - 1} \cdot p_i^{\beta_i}] = 0; \qquad i = 1, 2, \cdots, m. \quad (40)$$

The solution to (40) is

$$\hat{p}_i = \frac{\beta_i}{A_i}; \qquad i = 1, 2, \cdots, m. \quad (41)$$

Writing (41) in terms of T_i and τ_i yields

$$\hat{p}_i = \frac{\tau_i}{T_i}; \qquad i = 1, 2, \cdots, m. \quad (42)$$

Here, also, the estimator $\hat{p}_i$ is unbiased:

$$E\left\{\frac{B_i}{A_i}\right\} = \frac{1}{A_i} E\{B_i\} = p_i. \quad (43)$$

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REFERENCES

[1] M. Ball and F. Hardie, "Effects and detection of intermittent failures in digital systems," in *1969 Fall Joint Comput. Conf., AFIPS Conf. Proc.*, vol. 35. Montvale, NJ: AFIPS Press, 1969, pp. 329–335.

[2] M. A. Breuer, "Testing for intermittent faults in digital circuits," *IEEE Trans. Comput.*, vol. C-22, pp. 241–246, Mar. 1973.

[3] G. Dahlquist and A. Björck, *Numerical Methods.* Englewood Cliffs, NJ: Prentice-Hall, 1974.

[4] L. H. Goldstein, "A probabilistic analysis of multiple faults in LSI circuits," IEEE Comput. Rep. R77-304.

[5] F. H. Hardie and R. J. Suhocki, "Design and use of fault simulation for Saturn computer design," *IEEE Trans. Electron. Comput.*, vol. EC-16, pp. 412–428, Aug. 1967.

[6] S. Kamal and C. V. Page, "Intermittent faults: A model and detection procedure," *IEEE Trans. Comput.*, vol. C-23, pp. 713–719, July 1974.

[7] S. Kamal, "An approach to the diagnosis of intermittent faults," *IEEE Trans. Comput.*, vol. C-24, pp. 502–505, May 1975.

[8] S. Karlin and H. M. Taylor, *A First Course in Stochastic Processes*, 2nd ed. New York: Academic, 1975.

[9] W. Komorowoski, "The influence of intermittent failures on the output signal in combinational logic networks," *Archiwum Automatyki i Telemekhanik*, vol. 18, pp. 335–340, 1973.

[10] I. Koren and Z. Kohavi, "Diagnosis of intermittent faults in combinational networks," *IEEE Trans. Comput.*, vol. C-26, pp. 1154–1158, Nov. 1977.

[11] S. Y. H. Su *et al.*, "A continuous-parameter Markov model and detection procedures for intermittent faults," *IEEE Trans. Comput. (Special Issue on Fault-Tolerant Computing)*, pp. 567–570, June 1978.

[12] S. Mallela and G. M. Masson, "Intermittent fault diagnosability of digital systems," in *Proc. 12th Annu. Conf. Inform. Sci. Syst.*, John Hopkins Univ., Baltimore, MD, 1978, pp. 286–291.

[13] ——, "Diagnosable systems for intermittent fault," *IEEE Trans. Comput. (Special Issue on Fault-Tolerant Computing)*, pp. 560–566, June 1978.

[14] A. M. Mood, F. A. Graybill, and D. C. Boes, *Introduction to the Theory of Statistics*, 3rd ed. New York: McGraw-Hill, 1974.

[15] J. Savir, "Optimal random testing of single intermittent failures in combinational circuits," in *Dig. 1977 Int. Symp. Fault-Tolerant Comput.*, Los Angeles, CA, June 1977, pp. 180–185.

[16] ——, "Model and random testing properties of intermittent faults in combinational circuits," *J. Design Automation and Fault-Tolerant Comput.*, pp. 215–230, July 1978.

[17] ——, "Testing for single intermittent failures in combinational circuits by maximizing the probability of fault detection," Digital Syst. Lab., Stanford Univ., Stanford, CA, Tech. Rep. 145, Sept. 1977.
[18] ——, "Testing for multiple intermittent failures in combinational circuits by maximizing the probability of fault detection," in *Dig. 1978 Int. Symp. Fault-Tolerant Comput.*, Toulouse, France, June 1978, p. 212.
[19] ——, "Detection of intermittent faults in sequential circuits," Digital Syst. Lab., Stanford Univ., Stanford, CA, Tech. Rep. 120, Mar. 1978.
[20] ——, "Testing for intermittent failures in combinational circuits by minimizing the mean testing time for a given test quality," in *Proc. 3rd U.S.A.-Japan Conf.*, San Francisco, CA, Oct. 1978, pp. 155-161.
[21] J. J. Shedletsky and E. J. McCluskey, "The error latency of a fault in combinational digital circuits," in *Dig. 1975 Int. Symp. Fault-Tolerant Comput.*, Paris, France, June 1975, pp. 210-214.
[22] R. J. Spillman, "A Markov model of intermittent faults in digital systems," in *Dig. 1977 Int. Symp. Fault-Tolerant Comput.*, Los Angeles, CA, June 1977, pp. 157-161.
[23] O. Tasar and V. Tasar, "A study of intermittent faults in digital computers," presented at the Nat. Comput. Conf., 1977.
[24] P. K. Varshney, "Intermittent faults in digital systems—analytical models," IEEE Comput. Rep. R78-6.
[25] ——, "Statistical characterization of intermittent faults in digital systems," in *Proc. 15th Allerton Conf.*, 1977.
[26] S. S. Yau and Y.-S. Tang, "An efficient algorithm for generating complete test sets for combinational logic circuits," *IEEE Trans. Comput.*, vol. C-20, pp. 1245-1251, Nov. 1971.