TESTING ONE SIMPLE HYPOTHESIS AGAINST ANOTHER

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1. Summary and introduction. For the problem of testing one simple hypothesis against another, of all tests whose probabilities of incorrectly accepting the first hypothesis and of incorrectly accepting the second hypothesis are bounded from above by given bounds, the familiar Wald sequential probability ratio test gives the smallest expectation of sample size under either hypothesis. In this paper, a "generalized sequential probability ratio test" is introduced which differs from the Wald test only in that the same limits (A, B in the usual notation) are not necessarily used at each stage of the sampling, but at the ith stage A_i and B_i are used, where these numbers are predetermined constants. It is shown that for any given test T, there is a generalized sequential probability ratio test G whose probabilities of incorrectly accepting either hypothesis are no larger than the corresponding probabilities for T, and such that the cumulative distribution function of the number of observations required to come to a decision when using G is never below the corresponding distribution function when using T, under either hypothesis. We may then say that "G is uniformly better than T."

2. Assumptions and notation. In this paper we deal with the problem of testing one simple hypothesis H_1 against another simple hypothesis H_2. We assume that under H_i the chance variable X has a distribution with density function $f_i(x)$. Both $f_1(x)$ and $f_2(x)$ are everywhere bounded and have at most a finite number of discontinuities. We make the test by means of a sequence of independent chance variables ($X_1, X_2, \cdots$), each having the same distribution (the density function of each is $f_i(x)$ under H_i). We assume that for any n and any finite nonzero c,

$$\iint \cdots \iint \prod_{j=1}^{n} f_i(x_j)\, dx_j \to 0 \qquad \text{as} \quad \Delta c \to 0, \quad \text{for} \quad i = 1, 2,$$

the region of integration being

$$c \leqq \frac{\prod^{n} f_2(x_j)}{\prod^{n} f_1(x_j)} \leqq c + \Delta c.$$

The only tests we shall consider are those not involving randomization, and such that in the space of the first n chance variables the regions where H_1 is accepted and H_2 is accepted are Borel sets, for any n.

We define a "generalized sequential probability ratio test" as follows. There

Received 7/3/52.

are two sequences of predetermined nonnegative constants $(\bar{A}_1, \bar{A}_2, \cdots)$ and $(\bar{B}_1, \bar{B}_2, \cdots)$ such that $\bar{A}_i \geqq \bar{B}_i$ for all i. The value ∞ is not excluded. As long as

$$\bar{B}_n < \prod_{j=1}^{n} f_2(x_j) \Big/ \prod_{j=1}^{n} f_1(x_j) < \bar{A}_n$$

we continue sampling. The first time that this does not happen, we accept H_2 if the upper bound was violated, accept H_1 if the lower bound was violated. If $\bar{A}_m = \bar{B}_m$, while for all $i < m$ we have $\bar{A}_i > \bar{B}_i$, the test is truncated at the mth step. In general, any test is said to be truncated at the mth step if the probability of continuing sampling beyond the mth observation is zero under either hypothesis when using the test.

We use the following notation.

$T:D_i(n)$ is the probability that the sample size required to come to a decision is less than or equal to n, when the test T is used and H_i is true.

$T:A_i(n)$ is the region in the space of $(X_1, X_2, \cdots, X_n)$ where we accept H_i when the test T is used. To be in this region, we must have taken an nth observation.

$T:A_i$ is the region in the ∞-dimensional space where we accept H_i when using the test T.

$T:C(n)$ is the region in the space of $(X_1, X_2, \cdots, X_n)$ where we continue sampling when using the test T.

$P_i(R)$ is the probability of falling in any region R when H_i is true.

$P_i(R \mid S)$ is the conditional probability of falling in R, given that we are in S, when H_i is true.

$Q(X, n)$ is $\prod_{j=1}^{n} f_2(x_j)/\prod_{j=1}^{n} f_1(x_j)$. In specifying that we are using a certain test T, we shall not keep repeating the symbol "T:" throughout an expression, but shall use it once at the beginning of the expression and understand that it modifies everything coming after it, until we reach a symbol denoting another test. Thus if T and T' are two tests, the inequality $T:P_1(A_1(m)) + P_1(A_2(m)) > T':P_1(A_1)$ means that the probability of coming to a decision at the mth step when using T and H_1 is true exceeds the probability of accepting H_1 when it is true and T' is used.

3. Existence of a sequence of generalized sequential probability ratio tests uniformly better than a given test in the limit.

THEOREM 1. *If T is any test of H_1 against H_2 such that*

$$\lim_{n \to \infty} T:D_i(n) = 1 \qquad \textit{for } i = 1, 2,$$

there is a sequence $(G_1, G_2, \cdots)$ of generalized sequential probability ratio tests such that

$$G_j:D_i(n) \geqq T:D_i(n) \qquad \textit{for all } n, \textit{ all } j, \textit{ and } i = 1, 2; \textit{ and}$$

$$\lim_{j \to \infty} G_j:P_i(A_i) \geqq T:P_i(A_i) \qquad \textit{for } i = 1, 2.$$

PROOF. (At certain points in the proof, our statements should really be modified for certain sets of probability zero under both H_1 and H_2. The fact that we have neglected to do this in no way affects the proof.) To prove the theorem, we form a sequence of tests $(T_1, T_2 \cdots)$, as follows. T_j coincides with T until the jth observation. If a jth observation is taken, T_j says accept H_2 if $Q(X, j) \geqq 1$, else accept H_1. Then we have

$$T_j:D_i(n) \geqq T:D_i(n) \qquad \text{for all } n, \text{ all } j, \text{ and } i = 1, 2; \text{ and}$$

$$\lim_{j \to \infty} T_j:P_i(A_i) = T:P_i(A_i) \qquad \text{for } i = 1, 2.$$

Now for each j, we will replace T_j by a generalized sequential probability ratio test G_j, such that

$$G_j:D_i(n) \geqq T_j:D_i(n) \qquad \text{for all } n, \text{ all } j, \text{ and } i = 1, 2; \text{ and} \tag{4.1}$$

$$G_j:P_i(A_i) \geqq T_j:P_i(A_i) \qquad \text{for all } j \text{ and for } i = 1, 2. \tag{4.2}$$

This, with an obvious application of the Bolzano-Weierstrass theorem, will complete the proof of our theorem. Whenever two tests T and T' stand in the same relation to each other as do G_j and T_j in (4.1) and (4.2) we shall write T^*D^*T'. Thus we can state (4.1) and (4.2) more concisely as $G_j^*D^*T_j$ for all j.

Take any integer j and hold it fixed. Let us assume that for some integer m above 1 but not exceeding j we know that for any given test T_j truncated at the jth step there is a test $T_j'(m)$, also truncated at the jth step and coinciding with T_j before the mth observation, such that $T_j'(m)^*D^*T_j$, and also $T_j'(m)$ has the property $W(m)$ defined as follows:

$$T_j'(m):A_1(n) \text{ is given by } Q(X, n) \leqq B_n';$$

and

$$T_j'(m):A_2(n) \text{ is given by } Q(X, n) \geqq A_n',$$

for all n between m and j inclusive. We shall then show that all this is true for $m - 1$. Since it is certainly true for $m = j$ (with $A_j' = B_j'$, since a decision must be reached by the jth step), by working back to $m = 1$ we will obtain G_j and thus complete the proof of the theorem.

If in the space of the first $m - 1$ chance variables we consider only those points for which we stop sampling at the $(m - 1)^{st}$ observation, we can always transfer points so that in $T_j:A_1(m - 1)$ we have $Q(X, m - 1) \leqq c$, while in $T_j:A_2(m - 1)$ we have $Q(X, m - 1) > c$, for some nonnegative c, without making the distribution of the sample size or the probability of accepting a true hypothesis less favorable in any respect. This simply requires the application of the Neyman-Pearson lemma to the set of points under consideration. We shall assume that this is done. Suppose that we then find that there is a number r such that the subset S_I of $T_j:C(m - 1)$ where $Q(X, m - 1) > r$ and the subset S_{II} of $T_j:A_2(m - 1)$ where $Q(X, m - 1) \leqq r$ are both nonempty. We assume that $P_1(S_I) > 0$, else we would incorporate S_I into $T_j:A_2(m - 1)$, which could not make the situation less favorable. Similarly, we may assume $P_2(S_{II}) > 0$, and hence $P_1(S_{II}) > 0$, else we incorporate S_{II} into $T_j:A_1(m - 1)$. With these assumptions, $c < r < \infty$. Then we can find a number R, with $c < R < \infty$, such that if s_I is the subset of $T_j:C(m - 1)$ where $Q(X, m - 1) > R$, and s_{II} is the subset of $T_j:A_2(m - 1)$ where $Q(X, m - 1) \leqq R$, we have $P_1(s_I) = P_1(s_{II}) > 0$. It is clear that s_I and s_{II} are Borel sets. From now on, when we write X we shall understand the generic point $(x_1, x_2, \cdots, x_{m-1})$ of $m - 1$ dimensional space. To each point X_{II} of s_{II} we assign a nonnegative number $r(X_{II})$ as follows: $r(X_{II})$ is the greatest lower bound of the set of numbers v such that $P_1[X \text{ in } s_I \text{ and } Q(X, m - 1) \leqq v] \geqq P_1[X \text{ in } s_{II} \text{ and } Q(X, m - 1) \leqq Q(X_{II}, m - 1)]$.

Now let us assume that we are using a test $T'_j(m)$ defined above, where the acceptance and continuation regions from the first to the $(m - 1)^{st}$ observation are given by T_j. We modify this test into $T''_j(m)$ as follows. Transfer s_I into $T''_j(m):A_2(m - 1)$, s_{II} into $T''_j(m):C(m - 1)$, and for any X in s_{II}, we act in the future as though $Q(X, m - 1)$ were equal to $r(X)$. In all other respects, $T''_j(m)$ coincides with $T'_j(m)$. We shall show that $T''_j(m)*D*T'_j(m)$. Note that $T''_j(m)$ does not in general have the property $W(m)$.

We need the following lemma. For any given u,

$$P_1[X \text{ in } s_I \text{ and } Q(X, m - 1) \leqq u] = P_1[X \text{ in } s_{II} \text{ and } r(X) \leqq u].$$

It clearly suffices to prove the lemma for u between g.l.b. $Q(X, m - 1)$ for X in s_I and l.u.b. $Q(X, m - 1)$ for X in s_I. The proof of the lemma is given in five short sections.

(1) Given any point X' in s_{II}, we have from the definition of $r(X')$:

$$P_1[X \text{ in } s_I \text{ and } Q(X, m - 1) \leqq r(X')] = P_1[X \text{ in } s_{II} \text{ and } Q(X, m - 1) \leqq Q(X', m - 1)].$$

(2) $P_1[X \text{ in } s_{II} \text{ and } r(X) = u] = 0$. For suppose $r(X') = r(X'') = u$, and $Q(X', m - 1) < Q(X'', m - 1)$. Then we must have $P_1[X \text{ in } s_{II} \text{ and } Q(X', m - 1) \leqq Q(X, m - 1) \leqq Q(X'', m - 1)] = 0$, else we could not have $r(X') = r(X'')$, by (1). We define

$$Q_1 = \text{g.l.b. } Q(X, m - 1) \text{ for } X \text{ in } s_{II} \text{ and } r(X) = u,$$

$$Q_2 = \text{l.u.b. } Q(X, m - 1) \text{ for } X \text{ in } s_{II} \text{ and } r(X) = u.$$

Then $P_1[X \text{ in } s_{II} \text{ and } Q_1 \leqq Q(X, m - 1) \leqq Q_2] \geqq P_1[X \text{ in } s_{II} \text{ and } r(X) = u]$. Since $P_1[Q(X, m - 1) \leqq c]$ is a continuous function of c for c in the open interval $(0, \infty)$, and we can clearly assume that we accept H_2 as soon as $Q(X, n) = \infty$ and accept H_1 as soon as $Q(X, n) = 0$, we have $P_1[X \text{ in } s_{II} \text{ and } Q_1 \leqq Q(X, m - 1) \leqq Q_2] = 0$, which proves the first sentence of (2).

(3) $r(X)$ is a nondecreasing function of $Q(X, m - 1)$, and therefore if X' is in s_{II} the set of points in s_{II} such that $Q(X, m - 1) \leqq Q(X', m - 1)$ is the

set of points in s_{II} such that $r(X) \leqq r(X')$ (ignoring sets of probability zero under H_1).

(4) If there is a point X' in s_{II} such that $r(X') = u$, we have $P_1[X$ in s_I and $Q(X, m-1) \leqq u] = P_1[X$ in s_{II} and $r(X) \leqq u]$, by (1) and (3). By continuity, the same thing is true if there is a sequence of points $(X_1, X_2, \cdots)$ in s_{II} such that $\lim_{i \to \infty} r(X_i) = u$.

(5) For any u not of the type discussed in (4), we define $B(u) =$ l.u.b. $r(X)$ for all X with $r(X) < u$, $b(u) =$ g.l.b. $r(X)$ for all X with $r(X) > u$. We have

$$P_1[X \text{ in } s_I \text{ and } Q(X, m-1) \leqq B(u)] = P_1[X \text{ in } s_{II} \text{ and } r(X) \leqq B(u)],$$

and

$$P_1[X \text{ in } s_I \text{ and } Q(X, m-1) \leqq b(u)] = P_1[X \text{ in } s_{II} \text{ and } r(X) \leqq b(u)].$$

But

$$P_1[X \text{ in } s_{II} \text{ and } r(X) \leqq B(u)] = P_1[X \text{ in } s_{II} \text{ and } r(X) \leqq b(u)],$$

and therefore our lemma is proved.

First we examine what has occurred when H_1 is true. Since $T''_j(m)$ and $T'_j(m)$ coincide until the $(m-1)^{st}$ observation is taken, we start our investigation at the $(m-1)^{st}$ observation. Also, $T''_j(m):A_1(m-1)$ is the same set as $T'_j(m):A_1(m-1)$. Since $P_1(s_I) = P_1(s_{II})$, we have $T''_j(m):P_1(A_2(m-1)) = T'_j(m):P_1(A_2(m-1))$. Now choose any number k between m and j inclusive. We shall show that $T''_j(m):P_1(A_i(k)) = T'_j(m):P_1(A_i(k))$, $i = 1, 2$. This will complete the proof that $T''_j(m)^*D^*T'_j(m)$ when H_1 is true. For any set S, we denote the complement by $\bar{S}$. We have

$$T''_j(m):P_1(A_i(k)) \\ = P_1(\bar{s}_{II}\cdot C(m-1))P_1(A_i(k) \mid \bar{s}_{II}\cdot C(m-1)) + P_1(s_{II}\cdot C(m-1)\cdot A_i(k)),$$

and

$$T'_j(m):P_1(A_i(k)) \\ = P_1(\bar{s}_I\cdot C(m-1))P_1(A_i(k) \mid \bar{s}_I\cdot C(m-1)) + P_1(s_I\cdot C(m-1)\cdot A_i(k)).$$

But corresponding terms in the expressions on the right of the two equations are equal to each other, therefore the two left sides are equal. The only terms on the right for which equality is not obvious are $T'_j(m):P_1(s_I\cdot C(m-1)\cdot A_i(k))$ and $T''_j(m):P_1(s_{II}\cdot C(m-1)\cdot A_i(k))$. These are equal to $T'_j(m):P_1(A_i(k)\cdot s_I)$ and $T''_j(m):P_1(A_i(k)\cdot s_{II})$ respectively. To show that these two latter expressions are equal to each other, we define $Y_1[u, A_i(k), V]$ to be the probability when H_1 is true of falling in $V:A_i(k)$ when we arbitrarily assume that $Q(X, m-1) = u$ and then start sampling, using the test V as though the first observation were the mth, the second were the $(m+1)^{st}$, etc. Clearly, $Y_1[u, A_i(k), T'_j(m)]$ and $Y_1[u, A_i(k), T''_j(m)]$ are equal to each other for all u in the open interval $(0, \infty)$, and are continuous on this interval. We also define $F_1(u, s_I)$ as $P_1[X$ in s_I and

$Q(X, m - 1) \leqq u]$, and $G_1(u, s_{II})$ as $P_1[X$ in s_{II} and $r(X) \leqq u]$. We know from the lemma that $F_1(u, s_I) = G_1(u, s_{II})$ identically in u. We can assume that $T_j'(m)$ is such that the sets $[X$ in s_I and $Q(X, m - 1) = 0]$ and $[X$ in s_I and $Q(X, m - 1) = \infty]$ are empty. Then $F_1(u, s_I)$ is continuous at 0 and ∞. We have

$$T_j'(m):P_1(A_i(k)\cdot s_I) = \int_0^\infty Y_1[u, A_i(k), T_j'(m)]\, dF_1(u, s_I),$$

and

$$T_j''(m):P_1(A_i(k)\cdot s_{II}) = \int_0^\infty Y_1[u, A_i(k), T_j''(m)]\, dG_1(u, s_{II}).$$

From the considerations above, we know that these Stieltjes integrals exist and are equal to each other. Thus we have shown that $T_j''(m):P_1(A_i(k))$ is equal to $T_j'(m):P_1(A_i(k))$ for $i = 1, 2$ and for any k between m and j inclusive.

Now we examine the situation when H_2 is true. Once again, we can start our investigation at the $(m - 1)^{st}$ observation. We have $T_j'(m):P_2(C(m - 1)) = P_2(s_I) + P_2(\bar{s}_I\cdot C(m - 1))$, and $T_j''(m):P_2(C(m - 1)) = P_2(s_{II}) + P_2(\bar{s}_{II}\cdot C(m - 1))$. But the second terms on the right of these two equalities are equal, while $P_2(s_I) > P_2(s_{II})$, since $P_1(s_I) = P_1(s_{II})$, and in s_I, $Q(X, m - 1) > R$, while in s_{II}, $Q(X, m - 1) \leqq R$. Now we take any k between m and j inclusive, and examine the expressions

$$T_j'(m):P_2(C(k)) = P_2(s_I\cdot C(k)) + P_2(\bar{s}_I\cdot C(m - 1))P_2(C(k) \mid \bar{s}_I\cdot C(m - 1)),$$

$$T_j''(m):P_2(C(k)) = P_2(s_{II}\cdot C(k)) + P_2(\bar{s}_{II}\cdot C(m - 1))P_2(C(k) \mid \bar{s}_{II}\cdot C(m - 1)).$$

The second terms on the right of these two equalities are equal to each other. We investigate the first terms on the right. In a notation that will be recognized by analogy with that already used, we have

$$T_j'(m):P_2(s_I\cdot C(k)) = \int_0^\infty Y_2[u, C(k), T_j'(m)]\, dF_2(u, s_I),$$

$$T_j''(m):P_2(s_{II}\cdot C(k)) = \int_0^\infty Y_2[u, C(k), T_j''(m)]\, dG_2(u, s_{II}).$$

But $dF_2(u, s_I) > dG_2(u, s_{II})$ for all u, because $dF_1(u, s_I) = dG_1(u, s_{II})$, while $dF_2(u, s_I) > RdF_1(u, s_I)$ and $dG_2(u, s_{II}) \leqq RdG_1(u, s_{II})$ for all u. Also, $Y_2[u, C(k), T_j'(m)] = Y_2[u, C(k), T_j''(m)]$ for all u. Therefore we find that $T_j'(m):P_2(C(k)) > T_j''(m):P_2(C(k))$. To complete the proof that $T_j''(m)^*D^*T_j'(m)$, we have to show that $T_j''(m):P_2(A_2) \geqq T_j'(m):P_2(A_2)$, or that

$$T_j''(m):P_2(\bar{s}_I\cdot\bar{s}_{II}\cdot A_2) + P_2(s_I) + P_2(s_{II}\cdot A_2)$$

$$\geqq T'_j(m):P_2(\bar{s}_I \cdot \bar{s}_{II} \cdot A_2) + P_2(s_{II}) + P_2(s_I \cdot A_2)$$

or, since the first terms on the two sides of the inequality are equal, that $P_2(s_I) - T'_j(m):P_2(s_I \cdot A_2) \geqq P_2(s_{II}) - T''_j(m):P_2(s_{II} \cdot A_2)$, or

$$\int_0^\infty dF_2(u, s_I) - \int_0^\infty Y_2[u, A_2, T'_j(m)]\, dF_2(u, s_I)$$

$$\geqq \int_0^\infty dG_2(u, s_{II}) - \int_0^\infty Y_2[u, A_2, T''_j(m)]\, dG_2(u, s_{II}),$$

or that

$$\int_0^\infty (1 - Y_2[u, A_2, T'_j(m)])\, dF_2(u, s_I) \geqq \int_0^\infty (1 - Y_2[u, A_2, T''_j(m)])\, dG_2(u, s_{II}),$$

and this last inequality is immediately seen to hold.

By the assumption made above, there is a test $\hat{T}_j(m)$ coinciding with $T''_j(m)$ before the mth observation, having the property $W(m)$, and such that $\hat{T}_j(m)^*D^*T''_j(m)$. Now we transfer points between $\hat{T}_j(m):A_1(m - 1)$ and $\hat{T}_j(m):C(m - 1)$ so that after the transfer, for any X left in $\hat{T}_j(m):A_1(m - 1)$ we have $Q(X, m - 1) \leqq S$, and for any X left in $\hat{T}_j(m):C(m - 1)$ we have $Q(X, m - 1) > S$, where $0 < S < c$. Then, when S is properly chosen, we can show exactly as above that we can define a test $\hat{T}'_j(m)$, coinciding with $\hat{T}_j(m)$ before the mth observation, such that $\hat{T}'_j(m)^*D^*\hat{T}_j(m)$. Using the assumption made above again, there is a test $\bar{T}_j(m)$ having the property $W(m)$, coinciding with $\hat{T}'_j(m)$ before the mth observation, and such that $\bar{T}_j(m)^*D^*\hat{T}'_j(m)$. But then we have $\bar{T}_j(m)^*D^*T'_j(m)$, and also $\bar{T}_j(m):A_1(m - 1)$ is of the form $Q(X, m - 1) \leqq S$, $\bar{T}_j(m):A_2(m - 1)$ is of the form $Q(X, m - 1) \geqq R$, and $\bar{T}_j(m):C(m - 1)$ is of the form $S < Q(X, m - 1) < R$. Thus the existence of $\bar{T}_j(m)$ shows that if our assumption holds starting from the mth observation, it also holds starting from the $(m - 1)^{st}$. Since it holds at the jth observation, the theorem is proved. (Note that we were able to carry out the proof no matter what the acceptance and continuation regions were before the $(m - 1)^{st}$ observation).

4. Existence of a generalized sequential probability ratio test uniformly better than a given test.

THEOREM 2. *If T is any test of H_1 against H_2 satisfying the assumptions of Sections 2 and 3, then there is a generalized sequential probability ratio test G such that G^*D^*T.*

PROOF. We start with the sequence of generalized sequential probability ratio tests $(G_1, G_2, \cdots)$ of Theorem 1. From this sequence we can choose a subsequence so that the sequence of $\bar{A}_1$ associated with the subsequence of tests converges (convergence to ∞ is allowed throughout this proof). From this subsequence of tests we choose a second subsequence so that the associated sequence of $\bar{B}_1$ converges. From this second subsequence of tests we choose a third subsequence so that the associated sequence of $\bar{A}_2$ converges. We continue this way in an obvious manner. Then we form a new sequence of tests consisting of the first test in the first subsequence, the second test in the second subsequence, $\cdots$, the ith test in the ith subsequence, $\cdots$. Denote this new sequence by $(S_1, S_2, \cdots)$. Define G to be the generalized sequential probability ratio test given by the two sequences of bounds $(A_1^*, A_2^*, \cdots)$, $(B_1^*, B_2^*, \cdots)$, where $A_i^* = \lim_{j\to\infty} (\bar{A}_i$ associated with $S_j)$, $B_i^* = \lim_{j\to\infty} (\bar{B}_i$ associated with $S_j)$. By our construction, these limits exist. To see that G^*D^*T, it suffices to note that $S_j{:}D_i(n) \geqq T{:}D_i(n)$ for all n, all j, and $i = 1, 2$; and $\lim_{j\to\infty} S_j{:}P_i(A_i) \geqq T{:}P_i(A_i)$ for $i = 1, 2$; and also that for any generalized sequential probability ratio test, the probabilities of falling in the various acceptance and continuation regions under either hypothesis are continuous functions of the associated bounds in the two sequences which characterize the test. (Note that any generalized sequential probability ratio test accepts H_1 as soon as $Q(X, n)$ becomes zero, accepts H_2 as soon as $Q(X, n)$ becomes infinite).

5. Relation of results to decision theory. The relation of the results of this paper to general decision theory is fairly clear. In decision theory we are given a loss function, which we shall assume depends only on the true hypothesis, the hypothesis chosen as correct, and the number of observations required to come to a decision. We shall write this loss function as $W(H, D, N)$, where H is the true situation and can equal either 1 or 2, D is the decision as to which hypothesis is correct and can also equal either 1 or 2, and N is the number of observations required to come to a decision. We also make the following reasonable assumptions about the loss function: $W(1, 2, N) \geqq W(1, 1, N)$ for all N, $W(2, 1, N) \geqq W(2, 2, N)$ for all N, and $W(i, j, N)$ is nondecreasing in N for any fixed i and j. Then the discussion of the previous sections shows that if T is any test, there is a generalized sequential probability ratio test G such that

$$G{:}P_i(W(i, D, N) \leqq w) \geqq T{:}P_i(W(i, D, N) \leqq w) \qquad \text{for all } w \text{ and for } i = 1, 2.$$

6. Concluding remarks. The restriction to tests not using randomization that we made above is not necessary. For suppose R is any test, with or without randomization, such that $\lim_{n\to\infty} R{:}D_i(n) = 1$ for $i = 1, 2$. Truncating R at the mth observation in the usual way, we get a test $R(m)$ such that

$$R(j){:}D_i(n) \geqq R{:}D_i(n) \qquad \text{for all } n, \text{ all } j, \text{ and } i = 1, 2; \text{ and}$$

$$\lim_{j\to\infty} R(j){:}P_i(A_i) = R{:}P_i(A_i) \qquad \text{for } i = 1, 2.$$

Theorem 5.1 of [1] tells us that there exists a nonrandomized test $T(j)$ such that $T(j){:}P_k(A_i(n)) = R(j){:}P_k(A_i(n))$ for all n and for $i = 1, 2$, $k = 1, 2$. From this, it is easy to see that Theorems 1 and 2 hold if we consider randomized tests.

Also, the restriction that the density functions be bounded can be dropped, and the results still hold.

Finally, similar results hold in those cases where the observations are not taken one at a time, but in groups of predetermined size.

7. Acknowledgment. The author wishes to thank Dr. Milton Sobel for several suggestions which made this paper more readable.

REFERENCE

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