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Testing parametric assumptions of trends of nonstationary time series

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SUMMARY

The paper considers testing whether the mean trend of a nonstationary time series is of certain parametric forms. A central limit theorem for the integrated squared error is derived, and with that a hypothesis-testing procedure is proposed. The method is illustrated in a simulation study, and is applied to assess the mean pattern of lifetime-maximum wind speeds of global tropical cyclones from 1981 to 2006. We also revisit the trend pattern in the central England temperature series.

Some key words: Bias correction; Central limit theorem; Integrated squared error; Local linear estimation; Locally stationary process; Nonparametric hypothesis testing.

1. INTRODUCTION

The problem of testing whether the mean trend of a time series follows certain parametric forms has attracted considerable attention; see Dette (1999), Bissantz et al. (2005), Percival & Rothrock (2005), Wu & Zhao (2007) and Pawlak & Stadtmüller (1996, 2007), among others. Parametric models have the advantage of ease of interpretation and prediction, but may suffer from misspecification, leading to erroneous conclusions. Hypothesis testing in nonparametric regression under independence has been discussed by Härdle & Mammen (1993), Hart (1997), Fan et al. (2001) and Van Keilegom et al. (2008). Other contributions can be found in Azzalini et al. (1989), Eubank & LaRiccia (1992), Aerts et al. (1999), Eubank (1999), Horowitz & Spokoiny (2001) and Fan & Jiang (2007). In this paper we adopt the following formulation: suppose we observe

$$Y_i = \mu(i/n) + e_i \quad (i = 1, \dots, n), \quad (1)$$

where $\mu(t)$, $t \in [0, 1]$, is an unknown signal or trend function and $(e_i)_{i=1}^n$ is a zero-mean noise sequence which can be nonstationary. We are interested in testing the hypothesis

$$H_0 : \mu(t) = f(\theta, t), \quad (2)$$

where the function $f(\cdot, \cdot)$ has a known form and $\theta \in \mathbb{R}^d$ is a parameter vector of f being identically zero, constant, and special cases $f \equiv 0$, $f \equiv \text{a constant}$ and $f(\theta, t) = \theta_0 + \theta_1 t$ for some $\theta = (\theta_0, \theta_1) \in \mathbb{R}^2$ correspond to testing whether a signal exists, is time-varying and nonlinear, respectively. A natural approach would be to compare a nonparametric estimator of $\mu(\cdot)$ and the fitted parametric trend $f(\hat{\theta}_n, t)$, where $\hat{\theta}_n$ might, for example, be the least squares estimator, which minimizes

$$\ell_n(\theta) = \sum_{i=1}^n \{Y_i - f(\theta, i/n)\}^2. \quad (3)$$

Nonparametric estimation of μ can be performed in a number of ways including the Priestley–Chao (1972), the Nadaraya–Watson, local polynomial, spline and wavelet methods. Here we shall use the local linear estimator, which has a good boundary performance (Fan & Gijbels, 1996),

$$\hat{\mu}_n(t) = \sum_{i=1}^n Y_i w_i(t), \quad (4)$$

where $w_i(t) = K\{(i/n - t)/b_n\}\{S_2(t) - (t - i/n)S_1(t)\}/\{S_2(t)S_0(t) - S_1^2(t)\}$ are the local linear weights, b_n is the bandwidth, $K(\cdot)$ is a kernel function and $S_j(t) = \sum_{i=1}^n (t - i/n)^j K\{(i/n - t)/b_n\}$. To test H_0 , we shall develop central limit theory for the integrated squared error (ISE)

$$\text{ISE} = \int_0^1 \{\hat{\mu}_n(t) - \mu(t)\}^2 dt.$$

Under H_0 , ISE can be estimated by Δ_2^2 , where Δ_2 is the \mathcal{L}^2 distance

$$\Delta_2 = \left[\int_0^1 \{\hat{\mu}_n(t) - f(\hat{\theta}_n, t)\}^2 dt \right]^{1/2}.$$

We reject H_0 if Δ_2 is too large. The asymptotic normality of ISE has been studied under different settings. See for example Bickel & Rosenblatt (1973), Hall (1984), Ioannides (1992) and Alcalá et al. (1999), among others. However, in those papers the errors are independent. González-Manteiga & Vilar Fernández (1995) and Biedermann & Dette (2000) considered the same problem for linear processes with independent and identically distributed innovations. The ISE is a quadratic form in the errors e_i . As commented in Pawlak & Stadtmüller (2007), existing results on quadratic forms for dependent processes are mostly confined to linear processes. It is unclear whether similar results hold for general nonlinear processes. Here we shall substantially generalize earlier results by allowing nonlinear and nonstationary error processes. Hence our central limit theory should be widely applicable.

2. PRELIMINARIES

In (1) we allow nonstationary noise processes, on which there is a huge literature. Priestley (1965, 1988) considered processes with time-varying spectral representations. Dahlhaus (1997) defined a class of locally stationary processes for which a rigorous asymptotic theory can be obtained. Mallat et al. (1998) modelled locally stationary processes with pseudo-differential operators that are time-varying convolutions. Cheng & Tong (1998) applied wavelet representations. Nason et al. (2000) proposed to use a set of discrete non-decimated wavelets rather than the Fourier complex exponentials as in Dahlhaus (1997). Giurcanu & Spokoiny (2004) treated nonstationarity by assuming that correlation functions could be well approximated by those of stationary processes. Ombao et al. (2005) generalized the framework of Dahlhaus (1997) by utilizing the smooth localized complex exponentials. Here we shall follow the framework in Draghicescu et al. (2009) and Zhou & Wu (2009) and assume that the error sequence $\{e_i\}_{i=1}^n$ is generated from the model

$$e_i = G(i/n; \mathcal{F}_i), \quad (5)$$

where $\mathcal{F}_i = (\dots, \epsilon_{i-1}, \epsilon_i)$ is a shift process of independent and identically distributed shocks ϵ_k , $k \in \mathbb{Z}$, and $G: [0, 1] \times \mathbb{R}^\infty \rightarrow \mathbb{R}$ is a measurable function such that e_i is well defined. The

framework (5) covers a wide range of nonstationary processes and naturally extends many existing stationary time series models to their nonstationary counterparts. Following Wu (2005), we interpret \mathcal{F}_i and e_i as the input and output, and G as the transform that represents the underlying physical mechanism. Let $\zeta_i(t) = G(t; \mathcal{F}_i)$. Then $G(t; \cdot)$ is the data-generating mechanism at t , possibly time-varying. If $G(t; \cdot)$ does not depend on t , then (5) provides a very general framework for stationary processes; see Priestley (1988) and Tong (1990). Under a stochastic continuity condition, cf. Condition 2, equation (5) defines a locally stationary process.

We now introduce dependence measures that will be needed for our asymptotic theory. Let ϵ'_i, ϵ_j ($i, j \in \mathbb{Z}$) be independent and identically distributed and define the coupled process $\mathcal{F}_{k,\{0\}} = (\dots, \epsilon_{-1}, \epsilon'_0, \epsilon_1, \dots, \epsilon_k)$ and $\zeta_{k,\{0\}}(t) = G(t, \mathcal{F}_{k,\{0\}})$. For $p > 0$, assume that $c_p = \sup_{t \in [0,1]} \|\zeta_0(t)\|_p < \infty$, where $\|X\|_p = E(|X|^p)^{1/p}$, and define

$$\delta_{k,p} = \sup_{t \in [0,1]} \|\zeta_k(t) - \zeta_{k,\{0\}}(t)\|_p.$$

If $G(t, \mathcal{F}_k)$ does not depend functionally on ϵ_0 , then $\zeta_k(t) - \zeta_{k,\{0\}}(t) = 0$. So $\delta_{k,p}$ measures the dependence of $G(t; \mathcal{F}_k)$ on the single input ϵ_0 over $t \in [0, 1]$. Let $p' = \min(p, 2)$ and define

$$\Theta_{m,p} = \sum_{j=m}^{\infty} \delta_{j,p}, \quad \Psi_{m,p} = \left(\sum_{j=m}^{\infty} \delta_{j,p}^{p'} \right)^{1/p'}.$$

We can interpret $\Theta_{m,p}$ as the cumulative dependence of ϵ_0 on $\{\zeta_j(t)\}_{j=m}^{\infty}$. Throughout the paper we assume that the short-range dependence condition holds for some $p \geq 2$:

$$\Theta_{0,p} = \sum_{j=0}^{\infty} \delta_{j,p} < \infty. \quad (6)$$

If (6) holds with $p = 2$, then the long-run variance function of the stationary process $\{\zeta_k(t)\}_k$ is bounded:

$$g(t) = \sum_{k \in \mathbb{Z}} \text{cov}\{\zeta_0(t), \zeta_k(t)\} < \infty. \quad (7)$$

Write $\delta_k = \delta_{k,2}$ and $\|\cdot\| = \|\cdot\|_2$. We shall impose the following regularity conditions:

Condition 1. Let $\mu \in \mathcal{C}^3[0, 1]$;

Condition 2. There exists $C > 0$ such that $\|\zeta_i(t_1) - \zeta_i(t_2)\| \leq C|t_1 - t_2|$ holds for all $t_1, t_2 \in [0, 1]$;

Condition 3. Let $g \in \mathcal{C}^2[0, 1]$.

The key, Condition (2) indicates that the underlying data-generating mechanism $\zeta_i(t) = G(t; \mathcal{F}_i)$ changes smoothly in time, thus suggesting local stationarity. More specifically, a length k subsequence $(e_j)_{j=i}^{i+k-1}$ can be approximated by the stationary process $e_j^\circ = G(i/n, \mathcal{F}_j)$, $j = i, i+1, \dots, i+k-1$, in that $\|e_j - G(i/n, \mathcal{F}_j)\| \leq Ck/n = o(1)$ if $k = o(n)$.

3. MAIN RESULTS

3.1. Asymptotic normality

Throughout the paper we assume that in (4) the kernel function $K(\cdot) \in \mathcal{K}$, the collection of symmetric, bounded functions in $\mathcal{C}^1[-1, 1]$ with $\int_{-1}^1 K(v)dv = 1$. For example, K can be the

100 Epanechnikov kernel $K(v) = 3 \max(0, 1 - v^2)/4$, the Bartlett kernel $K(v) = \max(0, 1 - |v|)$, or the rectangle kernel $K(v) = I(|v| \leq 1)/2$, where $I(\cdot)$ is the indicator function. Define

$$K^*(x) = \int_{-1}^{1-2|x|} K(v)K(v+2|x|)dv.$$

Since $K \in \mathcal{C}^1[-1, 1]$, we have $K^* \in \mathcal{K}$, which is continuous on \mathbb{R} . We present below central limit theorems for

$$I_n = \text{ISE} - E(\text{ISE}) = \int_0^1 (\{\hat{\mu}_n(t) - \mu(t)\}^2 - E[\{\hat{\mu}_n(t) - \mu(t)\}^2])dt$$

when $\mu \equiv 0$ and $\mu \not\equiv 0$, respectively. It turns out that the asymptotic distributions are different for these two cases.

105 **THEOREM 1.** *Let $\mu \equiv 0$. Assume Condition 2, Condition 3, $\Theta_{0,4} < \infty$, $K \in \mathcal{K}$, $b_n \rightarrow 0$ and $nb_n \rightarrow \infty$ as $n \rightarrow \infty$.*

(i) *Let $g_1 = \int_0^1 g(t)dt$, $\Gamma_k = \sum_{i=0}^{\infty} \delta_i \delta_{i+|k|}$ and $l_n = \lfloor 2nb_n \rfloor$. Then*

$$nb_n \int_0^1 E\{\hat{\mu}_n(t)^2\}dt - g_1 K^*(0) = \sum_{k=0}^{\infty} O\{\min(\Gamma_k, b_n) + \min(1, k/l_n)\Gamma_k\}. \quad (8)$$

(ii) *Let $K_2^* = \int_{-1}^1 \{K^*(v)\}^2 dv$ and $g_2 = \int_0^1 g^2(t)dt$. If $nb_n^{3/2} \rightarrow \infty$, then as $n \rightarrow \infty$,*

$$nb_n^{1/2} I_n \rightarrow N(0, 4g_2 K_2^*) \quad (9)$$

in distribution.

110 Theorem 1(i) deals with the mean integrated squared error $E(\text{ISE}) = \int_0^1 E\{\hat{\mu}_n(t)\}^2 dt$. Since $\sum_{k=0}^{\infty} \Gamma_k \leq (\sum_{i=0}^{\infty} \delta_i)^2 < \infty$, $b_n \rightarrow 0$ and $l_n \rightarrow \infty$, by the Lebesgue dominated convergence theorem, the right-hand side of (8) goes to zero as $n \rightarrow \infty$. The quantity g_1 is the integrated long-run variance. For stationary processes we have $g(t) \equiv g(0)$ for all t , and $g_1 = g(0)$, $g_2 = g_1^2$. Based on Theorem 1, we present in § 3.2 a simulation-based testing procedure.

115 **THEOREM 2.** *Assume Conditions 1–3 and $\Theta_{0,4} < \infty$. Also assume $K \in \mathcal{K}$, $b_n \rightarrow 0$ and $nb_n^{3/2} \rightarrow \infty$ as $n \rightarrow \infty$.*

(i) *If $nb_n^5 \rightarrow 0$, then (9) holds.*

(ii) *Let $\kappa_2 = \int K(v)v^2 dv$. If $nb_n^5 \rightarrow \infty$, then*

$$\frac{n^{1/2}}{b_n^2} I_n \rightarrow \kappa_2 N \left\{ 0, \int_0^1 g(t)\mu''(t)^2 dt \right\}$$

in distribution.

120 For both big and small bandwidths, I_n is asymptotically normal. However, for those two cases, both the normalization sequences and the asymptotic variances are different.

3.2. Implementation

We now present a testing procedure based on results in §3.1. Assume at the outset that the long-run variance function $g(\cdot)$ is known and we want to test the hypothesis of nonexistence of a signal

$$H'_0 : \mu \equiv 0. \quad (10)$$

Section 3.4 concerns testing of the hypothesis (2) with general nonzero parametric forms, and §3.6 presents an estimator for the long-run variance function $g(\cdot)$. To test (10), by Theorem 1, it is natural to use $\text{ISE} = \int_0^1 \hat{\mu}_n(t)^2 dt$ and we reject H'_0 at level α , $0 < \alpha < 1$, if

$$\text{ISE} > g_1 K^*(0)(nb_n)^{-1} + z_{1-\alpha} n^{-1} b_n^{-1/2} (4g_2 K_2^*)^{1/2},$$

where $z_{1-\alpha}$ is the $(1 - \alpha)$ th quantile of a standard normal distribution. However, our simulation study in § 4.1 shows that convergence in Theorem 1 can be quite slow. To overcome this, we propose a simulation-based procedure. Let Z_i ($i \in \mathbb{Z}$) be independent standard normal random variables, $Y_i^\diamond = e_i^\diamond = g(i/n)^{1/2} Z_i$, and let $\hat{\mu}_n^\diamond(\cdot)$ be the corresponding local linear estimator (4) with Y_i therein replaced by Y_i^\diamond . By Theorem 1, we have

$$nb_n \int_0^1 E\{\hat{\mu}_n^\diamond(t)^2\} dt - g_1 K^*(0) = O(b_n)$$

and, for $\text{ISE}^\diamond = \int_0^1 \hat{\mu}_n^\diamond(t)^2 dt$, the central limit theorem holds: as $n \rightarrow \infty$,

$$nb_n^{1/2} \{\text{ISE}^\diamond - E(\text{ISE}^\diamond)\} \rightarrow N(0, 4g_2 K_2^*) \quad (11)$$

in distribution. Hence $\text{ISE} = \int_0^1 \hat{\mu}_n(t)^2 dt$ and ISE^\diamond have the same asymptotic normal distribution, with mean $g_1 K^*(0)n^{-1}b_n^{-1}$ and variance $4g_2 K_2^* n^{-2}b_n^{-1}$ if the bound in the right-hand side of (8) is of order $o(b_n^{1/2})$. The latter observation suggests that, instead of using the central limit theorem, the cutoff value of ISE can be obtained by simulating ISE^\diamond .

3.3. Asymptotic power

We consider the power of our test for the local alternative model of the form

$$Y_i = a_n h(i/n) + e_i,$$

where h is a known nonzero \mathcal{C}^2 function on $[0, 1]$ and $(a_n)_{n \geq 1}$ is a positive sequence with $a_n \rightarrow 0$. Here a_n indicates the magnitude of departure from the null hypothesis. By Proposition 1, the power goes to 1 if $nb_n^{1/2} a_n^2 \rightarrow \infty$.

PROPOSITION 1. Assume Conditions 2 and 3 and $\Theta_{0,4} < \infty$. Also assume that $K \in \mathcal{K}$, $b_n \rightarrow 0$, $nb_n^{3/2} \rightarrow \infty$ and $nb_n^{1/2} a_n^2 \rightarrow c > 0$ as $n \rightarrow \infty$. Let Φ be the standard normal distribution function. Then the power of our testing procedure converges to $\Phi\{z_\alpha + (4g_2 K_2^*)^{-1/2} c \int_0^1 h(t)^2 dt\}$.

Proof. Let $\hat{\mu}_n^0(t) = \sum_{i=1}^n w_i(t) e_i$ and $h_n(t) = \sum_{i=1}^n w_i(t) h(i/n)$. Then

$$\int_0^1 \{\hat{\mu}_n(t)\}^2 - \{\hat{\mu}_n^0(t)\}^2 dt = 2a_n \int_0^1 \hat{\mu}_n^0(t) h_n(t) dt + a_n^2 \int_0^1 h_n(t)^2 dt. \quad (12)$$

Since $h \in \mathcal{C}^2[0, 1]$, $|\int_0^1 w_i(t)h_n(t)dt| \leq C/n$ for some $C > 0$. By Lemma 1,

$$nb_n^{1/2}a_n \left\| \int_0^1 \hat{\mu}_n^0(t)h_n(t)dt \right\| = nb_n^{1/2}a_n O(n^{-1/2}) \rightarrow 0.$$

So, Proposition 1 follows from Theorem 1 and (12) since $\int_0^1 h_n(t)^2 dt \rightarrow \int_0^1 h(t)^2 dt$. \square

3.4. Hypothesis testing with general parametric forms

Under the null hypothesis (2), let θ_0 be the true value and $\hat{\theta}_n$ be the minimizer of (3). Then a natural test statistic would be $\int_0^1 \{\hat{\mu}_n(t) - f(\hat{\theta}_n, t)\}^2 dt$. However, it is not convenient to use this directly since by Theorem 2, $\int_0^1 \{\hat{\mu}_n(t) - f(\theta_0, t)\}^2 dt$ can have different asymptotic normal distributions for different bandwidths. It is nontrivial to determine which central limit theorem to use. To solve this problem, we use the modified version

$$\text{ISE}_M = \int_0^1 \{\hat{\mu}_n(t) - \mu_M(t)\}^2 dt,$$

where $\mu_M(t) = \sum_{i=1}^n f(\theta_0, i/n)w_i(t)$ is the local linear smoothed version of $f(\theta_0, t)$. Since $\hat{\mu}_n(t) - \mu_M(t) = \sum_{i=1}^n w_i(t)e_i$, ISE_M reduces to ISE with $\mu \equiv 0$. The bias then disappears on replacing $\mu(t)$ by $\mu_M(t)$. Such a bias correction scheme was previously used in [Härdle & Mammen \(1993\)](#). Note that ISE_M is not directly usable since it depends on the unknown function μ , which under H_0 depends on the parameter θ_0 . It can be estimated by

$$\text{ISE}_{\hat{M}} = \int_0^1 \{\hat{\mu}_n(t) - \mu_{\hat{M}}(t)\}^2 dt = \int_0^1 \{\hat{\mu}_n^{(e)}(t)\}^2 dt, \quad (13)$$

where $\mu_{\hat{M}}(t) = \sum_{i=1}^n f(\hat{\theta}_n, i/n)w_i(t)$ and $\hat{\mu}_n^{(e)}(t) = \hat{\mu}_n(t) - \mu_{\hat{M}}(t)$ is the local linear smoother of the estimated residuals $\hat{e}_i = Y_i - f(\hat{\theta}_n, i/n)$.

PROPOSITION 2. Assume that $\dot{f}(\theta, t) = \partial f(\theta, t)/\partial \theta$ exists at a neighbourhood of θ_0 , that

$$\sup_{t \in [0, 1]} \sup_{|\theta - \theta_0| \leq c} |\dot{f}(\theta, t)| < \infty \quad (14)$$

holds for some $c > 0$, and that $f(\theta, t)$ admits the uniform Taylor expansion: as $\theta \rightarrow \theta_0$,

$$\sup_{t \in [0, 1]} |f(\theta, t) - f(\theta_0, t) - (\theta - \theta_0)^T \dot{f}(\theta_0, t)| = \mathcal{O}(|\theta - \theta_0|^2). \quad (15)$$

Then under conditions of Theorem 1, if $\hat{\theta}_n - \theta_0 = O_p(n^{-1/2})$, we have

$$n(\text{ISE}_{\hat{M}} - \text{ISE}_M) = O_p(1). \quad (16)$$

Proposition 2 implies that the statistic $\text{ISE}_{\hat{M}}$ can approximate ISE_M well, and it is also asymptotically normally distributed with same asymptotic mean and variance, given in Theorem 1.

3.5. Bandwidth selection

Choosing a bandwidth such that the test procedure based on $\text{ISE}_{\hat{M}}$ has a good performance, is nontrivial, and in our case it is further complicated by the presence of dependence and nonstationarity. In the regression setting with independent errors, the problem was considered

by Härdle & Marron (1985), Härdle et al. (1988), Park & Marron (1990), Ruppert et al. (1995), Wand & Jones (1995) and Xia (1998), among others. Herrmann et al. (1992) and Wu & Zhao (2007) dealt with models with stationary errors. Hall & Hart (1990), Kulasekera & Wang (1997) and Gao & Gijbels (2008) considered bandwidth selection in the context of nonparametric hypothesis testing. 175

We propose using the asymptotic mean squared error optimal bandwidth $b_n = cn^{-1/5}$, where $c > 0$ is a constant. Due to dependence and nonstationarity, it is difficult to estimate c . On the positive side, our simulation studies carried out in §4 suggests that the performance of our simulation-based test is relatively robust to the choice of c . Hence in our simulation and data analysis, we simply choose $b_n = n^{-1/5}$ in computing $\text{ISE}_{\hat{M}}$, and interestingly, this simple choice performs quite well. 180

3.6. Estimation of variance functions

In the implementation of our testing procedure, a key issue is to estimate the pointwise long-run variance function, $g(t)$, $t \in [0, 1]$. If the errors e_i were independent and identically distributed, then $g(t) \equiv \sigma_e^2 = \|e_i\|^2$, the variance of e_i . In this case we can apply difference-based variance estimators and there is a huge literature on the estimation of σ_e^2 ; see Rice (1984), Hall et al. (1990) and Dette et al. (1998), among others. In our setting, however, due to the dependence, the difference-based approach is generally invalid. For example, assuming that $\mu \in \mathcal{C}^2[0, 1]$ and e_i are stationary, as $n \rightarrow \infty$, Rice's (1984) estimator $(2n - 2)^{-1} \sum_{i=2}^n (Y_i - Y_{i-1})^2 \rightarrow \gamma_0 - \gamma_1$, by the ergodic theorem. Here $\gamma_k = \text{cov}(e_0, e_k)$ is the auto-covariance function of (e_i) . Note that $\sigma^2 = \sum_{k \in \mathbb{Z}} \gamma_k$, which is generally different from $\gamma_0 - \gamma_1$. 190

To account for dependence and nonstationarity, we estimate $g(t)$ by

$$\hat{g}(t) = \frac{\sum_{i=1}^n Q_i \mathbf{I}(|i/n - t| \leq b_n)}{\sum_{i=1}^n \mathbf{I}(|i/n - t| \leq b_n)}, \quad (17)$$

where $Q_i = e_i \sum_{|j-i| \leq m_n} e_j$. By Theorem 3, the above estimator is consistent.

THEOREM 3. Assume Conditions 2 and 3, $\Theta_{0,4} < \infty$, $b_n \rightarrow 0$ and $m_n^{-1} + m_n(nb_n)^{-1} \rightarrow 0$. Then as $n \rightarrow \infty$, 195

$$(nb_n/m_n)^{1/2} [\hat{g}(t) - E\{\hat{g}(t)\}] \rightarrow N\{0, 2g^2(t)\} \quad (18)$$

in distribution for any $t \in (0, 1)$. Also, uniformly over $t \in [b_n, 1 - b_n]$, the bias

$$E\{\hat{g}(t)\} - g(t) = O(b_n^2) + \sum_{k=0}^{m_n} O\{\min(\Gamma_k, m_n/n)\} + \sum_{k>m_n} O(\Gamma_k). \quad (19)$$

If there exists $\rho \in (0, 1)$ such that $\Gamma_k = O(\rho^k)$, letting $m_n = \lfloor \log n / \log \rho^{-1} \rfloor$, we have by (19) that $E\{\hat{g}(t)\} - g(t) = O(b_n^2) + O(m_n^2/n)$. Hence, by (18), the mean squared error of $\hat{g}(t)$ is of order $O(b_n^4 + m_n^4/n^2) + O\{m_n(nb_n)^{-1}\} = O\{(n^{-1} \log n)^{4/5}\}$ if $b_n \asymp (n^{-1} \log n)^{1/5}$. 200

To use (17), we suggest using the automatic bandwidth selector in Ruppert et al. (1995) to obtain a local linear fit $\tilde{\mu}_n(\cdot)$ of the mean function and then the estimated residuals $\tilde{e}_i = Y_i - \tilde{\mu}_n(i/n)$. Then we replace e_i in (17) by \tilde{e}_i .

3.7. A simulation-based testing procedure

We summarize the testing procedure as follows. Its validity is justified by noting that ISE and ISE° have the same asymptotic distribution, as argued in (11). Section 4 presents a simulation 205

study of its finite-sample performance.

- (i) Select a bandwidth b_n^* by the procedure in [Ruppert et al. \(1995\)](#); perform a local linear fit for $\mu(\cdot)$ and obtain the estimated residuals $\tilde{e}_i = Y_i - \hat{\mu}_n(i/n)$; compute $\hat{g}(t)$, $t \in [0, 1]$, via (17).
- (ii) Obtain an estimator $\hat{\theta}_n$ for the parameter θ ; compute $\tilde{e}_i = Y_i - f(\hat{\theta}_n, i/n)$ and $\text{ISE}_{\hat{M}}$ via (13) with the local linear estimator (4) where $b_n = n^{-1/5}$ is used.
- (iii) Generate independent $Z_1, \dots, Z_n \sim N(0, 1)$ and let $Y_i^\diamond = \hat{g}(i/n)^{1/2} Z_i$. Then compute the corresponding $\text{ISE}_{\hat{M}}^\diamond$ in the same manner as (ii).
- (iv) Let $\alpha \in (0, 1)$ be the significance level. Repeat step (iii) and obtain the estimated quantile $\hat{q}_{1-\alpha}$ of $\text{ISE}_{\hat{M}}^\diamond$, the bootstrap cutoff value.
- (v) Reject the null hypothesis at level α if $\text{ISE}_{\hat{M}} > \hat{q}_{1-\alpha}$.

4. A SIMULATION STUDY

4.1. Approximations of distributions of test statistics

Consider model (1) with $e_i = \zeta_i(i/n)$, where for any $t \in [0, 1]$, $\{\zeta_i(t)\}_{i \in \mathbb{Z}}$ follows the recursion

$$\zeta_i(t) = \rho(t)\zeta_{i-1}(t) + \sigma\epsilon_i. \quad (20)$$

Here $\epsilon_i, i \in \mathbb{Z}$, are independent random variables with $\text{pr}(\epsilon_i = -1) = \text{pr}(\epsilon_i = 1) = 1/2$. Thus, $(e_i)_{i \in \mathbb{Z}}$ is a first-order autoregressive process with time-varying coefficient. Calculations show that $E\{\zeta_i(t)\} = 0$, $g_0(t) = \text{var}\{\zeta_i(t)\} = \sigma^2/\{1 - \rho(t)^2\}$ and the long-run variance function $g(t) = \sigma^2/\{1 - \rho(t)^2\}^2$. We use the Epanechnikov kernel $K(v) = 3 \max(0, 1 - v^2)/4$. Then $K^*(0) = 3/5$ and $K_2^* = 167/770$. We consider the problem of testing $H_0' : \mu \equiv 0$. Choose $\rho(t) = 0.1 + 0.4t$, $\sigma = 1$ and $n = 500$, so the bandwidth $b_n = n^{-1/5} = 0.289$. We simulate 50 000 realizations of $\text{ISE}_{\hat{M}}$. Three different approximations of $\text{ISE}_{\hat{M}}$ are considered: the normal approximation in Theorem 1; $\text{ISE}^\sharp = \int_0^1 \hat{\mu}_n^\sharp(t)^2 dt$, where $\hat{\mu}_n^\sharp(t) = \sum_{i=1}^n Y_i^\sharp w_i(t)$, $Y_i^\sharp = g_0(i/n)^{1/2} Z_i$ and Z_i are independent standard normal variables; and $\text{ISE}^\diamond = \int_0^1 \hat{\mu}_n^\diamond(t)^2 dt$, introduced in §3.2, which differs from the second in that the long-run variance function g is used instead of the marginal variance function g_0 . In the second scheme, the dependence is ignored. We use Q-Q plot to compare the distributions. The results are presented in Fig. 1, which shows that the normal approximation does not have a satisfactory finite-sample performance. A similar phenomenon was observed by [Härdle & Mammen \(1993\)](#). If we ignore the inherited dependence structure, then one may obtain an erroneous conclusion; see Fig. 1(b) of Fig. 1. As shown in Fig. 1(c), the dependence-adjusted procedure provides a very good approximation of $\text{ISE}_{\hat{M}}$. The same conclusion applies to other parametric forms.

For a theoretical justification of the superiority of the simulation-based method, we use the Gaussian approximation principle in [Wu & Zhou \(2010\)](#). Consider the linear process $X_i = \sum_{j=0}^\infty a_j(i/n)\eta_{i-j}$, where η_i are independent and identically distributed with mean 0 and $\eta_i \in \mathcal{L}^p$ ($p > 2$), and $a_j(\cdot)$ are differentiable functions satisfying $\sum_{j=0}^\infty \sup_{t \in [0,1]} |a_j'(t)| < \infty$ and $\sum_{j=m}^\infty \sup_{t \in [0,1]} |a_j(t)| = O(m^{1/p-1/2})$. Then on a richer probability space one can construct e_1^*, \dots, e_n^* and independent standard normal random variables Z_1^*, \dots, Z_n^* such that $(e_i^*)_{i=1}^n$ and $(e_i)_{i=1}^n$ have the same distribution and, for $S_i^* = \sum_{j=1}^i e_j^*$ and $T_i^* = \sum_{j=1}^i g(i/n)^{1/2} Z_j^*$,

$$\max_{i \leq n} |S_i^* - T_i^*| = O_p(n^{1/p} \log n). \quad (21)$$

Q3

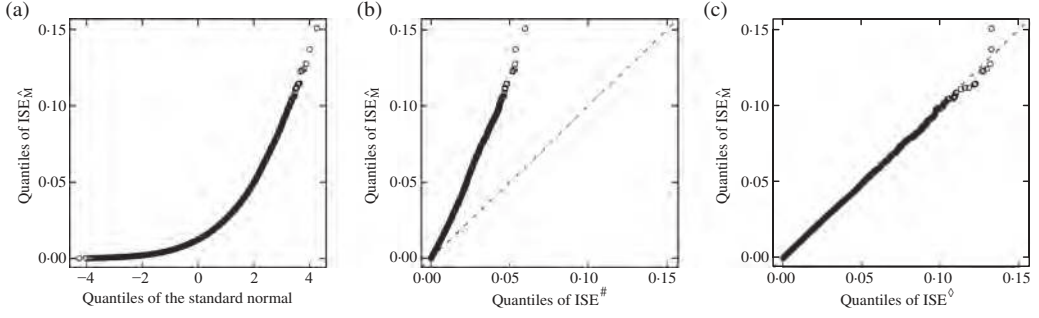


Fig. 1. Title Q-Q plots of the test statistic $\text{ISE}_{\hat{M}}$ against normal approximation (a), approximation without dependence, $\text{ISE}^\#$, (b) and approximation with dependence, ISE^\diamond , (c). The dashed lines in (b) and (c) have unit slope and zero intercept.

Q3

Let $\mu_n^*(t) = \sum_{i=1}^n w_i(t) e_i^*$ and $v_n^*(t) = \sum_{i=1}^n w_i(t) g(i/n)^{1/2} Z_i^*$. For the local linear weights $w_i(t)$ in (4), if K is Lipschitz continuous, by (21), we obtain

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$$\sup_{0 \leq t \leq 1} |\mu_n^*(t) - v_n^*(t)| = O_p\{n^{1/p} \log n / (nb_n)\} \quad (22)$$

by a summation by parts technique. By Theorem 1, $\int_0^1 \mu_n^*(t)^2 dt = O_p\{(nb_n)^{-1}\}$. So, by (22),

$$\int_0^1 |\mu_n^*(t)^2 - v_n^*(t)^2| dt = O_p\{n^{1/p} \log n (nb_n)^{-3/2}\}. \quad (23)$$

When $\mu \equiv 0$, since $(e_i^*)_{i=1}^n$ and $(e_i)_{i=1}^n$ have the same distribution, $\int_0^1 \mu_n^*(t)^2 dt$ is identically distributed as $\int_0^1 \hat{\mu}_n(t)^2 dt$. Note that $\int_0^1 v_n^*(t)^2 dt$ corresponds to $\text{ISE}_{\hat{M}}^\diamond$ in step (iii) in §3.7 if \hat{g} therein is replaced by the true g . For the mean squared error optimal bandwidth $b_n \asymp n^{-1/5}$, the error bound in (22) is $O_p(n^{\tau-6/5})$. Since τ can be arbitrarily small if p is large, $\int_0^1 \mu_n^*(t)^2 dt$ and $\int_0^1 v_n^*(t)^2 dt$ can be very close; recalling Proposition 2 that the difference between ISE_M and the realized version $\text{ISE}_{\hat{M}}$ has a larger order $O_p(n^{-1})$. So, we expect that the simulation-based method can have an excellent performance.

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4.2. Effect of bandwidths

We consider model (1) with error structure (20). With true mean function $\mu(t) = 4t^2 - 4t + 3$, $t \in [0, 1]$, we consider testing whether the mean function has a quadratic form. To study how bandwidths affect the performance of our test, we choose $\rho(t) = 0.3 - 0.5t$, $\sigma = 1$ and 10 levels of b : $b = 0.05j$ ($j = 1, \dots, 10$). For each b , we calculate $\hat{q}_{0.95}$ by repeating step (iii) in §3.7 50 000 times. Then we generate 50 000 realizations of the time-varying AR(1) process and compute the corresponding $\text{ISE}_{\hat{M}}$ values for each realization. Simulated empirical rejection proportions with $n = 100, 200$ and 500 , presented in Table 1, are reasonably close to the nominal level. If we choose $b_n = n^{-1/5}$, then $b_{100} = 0.398$, $b_{200} = 0.347$ and $b_{500} = 0.289$, and the corresponding empirical rejection probabilities are about 5.01, 4.97 and 5.03%, respectively. In addition, they become more robust to the change in bandwidths as n gets larger. So, in practice, we recommend using $b_n = n^{-1/5}$.

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Table 1. Empirical rejection percentages with different bandwidths b and sample sizes n .
The significance level is 5%.

n	b									
	0.05	0.1	0.15	0.2	0.25	0.3	0.35	0.4	0.45	0.5
100	2.8	4.0	4.5	4.7	4.7	5.0	5.2	5.0	5.1	5.1
200	3.8	4.6	4.6	5.1	4.6	5.1	5.0	5.2	5.1	5.2
500	4.6	4.7	5.0	4.9	5.1	5.0	5.0	5.1	5.2	5.1

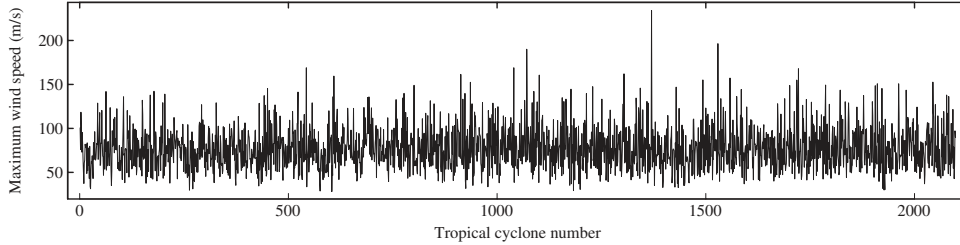


Fig. 2. Satellite-derived lifetime-maximum wind speeds of tropical cyclones during 1981–2006.

5. APPLICATIONS

5.1. Tropical cyclone data

According to Emanuel (1991), Holland (1997) and Bengtsson (2007), global warming is leading to increasing ocean temperatures and consequently to more tropical cyclones. Elsner et al. (2008) analysed patterns of tropical cyclone winds by fitting linear trends for quantiles of satellite-derived lifetime-maximum wind speeds of 2098 tropical cyclones over the globe during 1981–2006. The data set is available in the Supplementary material see Elsner et al. (2008) for a detailed description. Figure 2 shows these data.

We shall model the wind speed data by (1) and test the null hypothesis $H_0 : \mu(\cdot)$ is linear. Zhou (2010) argued that the error process (e_i) is nonstationary. Hence $g(t)$ is not a constant function. Using the procedure in Ruppert et al. (1995), we select $b_n^* = 0.081$ and estimate the long-run variance function $g(t)$ by using (17) with $m_n = (nb_n^*)^{1/3}$. For hypothesis testing, we choose $b_n = n^{-1/5} = 0.217$. The test statistic $\text{ISE}_{\hat{M}} = 1.011$ with p -value 0.12 after 50 000 repetitions of step (iii) in §3.7. So, at the 5% significance level, we accept the linear trend hypothesis. The fitted trend, with standard error, is $\mu(t) = 75.3(1.1) + 3.8(1.8)t$, $t \in [0, 1]$. For testing the hypothesis $H_0^* : \mu$ is constant, using the same method we obtain $\text{ISE}_{\hat{M}} = 2.091$ with p -value 0.04. Thus the mean constancy hypothesis is rejected at the 5% level. Zhou (2010) applied an \mathcal{L}^∞ -based method, and failed to reject H_0^* . Our \mathcal{L}^2 -based testing procedure appears to be more powerful.

5.2. Central england temperature data

We consider the annual central England temperature series from year 1659 to 2009 by using model (1). The time series is plotted in Fig. 3 and the data are available in the Supplementary material. It was first constructed by Manley (1974) and is now routinely updated by the Hadley Center, U.K. See Jones & Hulme (1997) for a more detailed description. Jones & Hulme (1997) and Jones & Bradley (1992a) fitted linear trends, while Benner (1999) and Harvey & Mills (2003) fitted quadratic curves. Realizing that the quadratic trend assumption might not be appropriate, Harvey & Mills (2003) also tried local polynomial regression. Here we shall test the

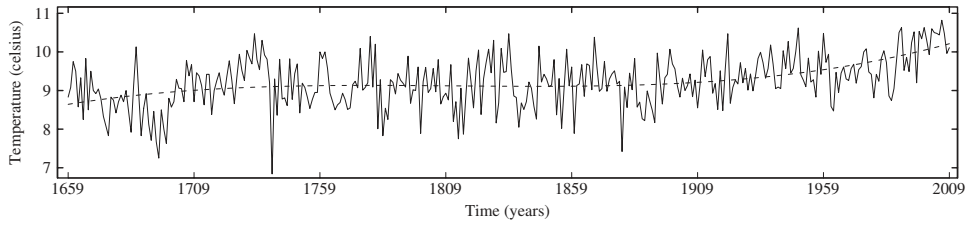


Fig. 3. Annual central England temperature series from 1659 to 2009. The dashed curve is the global cubic fit.

hypothesis $H_0 : \mu(\cdot)$ is quadratic. As in the analysis of the tropical cyclone data, we follow the procedure in §3.7 and obtain the test statistic $\text{ISE}_{\hat{M}} = 0.0059$ with $p = 0.00002$. Hence, the quadratic trend assumption is rejected at the 5% level. Interestingly, the cubic trend hypothesis is accepted: the test statistic $\text{ISE}_{\hat{M}} = 0.000096$ and the corresponding p -value is 0.47. The fitted equation with standard error in parenthesis, is $\mu(t) = 8.63(0.13) + 3.7(1.1)t - 8.9(2.5)t^2 + 6.8(1.7)t^3$, where $t \in [0, 1]$. The cubic trend fit accords well with Benner's observation that the whole series has roughly three periods: the earliest part corresponds to the coldest weather which may represent the little ice age (Jones & Bradley, 1992b), the middle part fluctuates around the mean, while the last part exhibits a warming trend. This cannot be described by a quadratic fit.

SUPPLEMENTARY MATERIAL

Supplementary Material is available at *Biometrika* online.

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APPENDIX

Recall that $\zeta_k(t) = G(t, \mathcal{F}_k)$. We assume that $E\{G(t, \mathcal{F}_k)\} = 0$ for any $t \in [0, 1]$. Define

$$\tilde{\zeta}_k(t) = E\{\zeta_k(t) \mid \mathcal{F}_{k-m,k}\}, \quad m \in \mathbb{N}.$$

Then for all $t \in [0, 1]$, $\{\tilde{\zeta}_k(t)\}_k$ is m -dependent with mean zero. Define the projection operator

$$\mathcal{P}_k \cdot = E(\cdot \mid \mathcal{F}_k) - E(\cdot \mid \mathcal{F}_{k-1}), \quad k \in \mathbb{Z}.$$

For a set $T \subseteq \mathbb{Z}$, let $\epsilon_{i,T} = \epsilon'_i$ if $i \in T$, and $\epsilon_{i,T} = \epsilon_i$ otherwise. Let $\mathcal{F}_{i,j,T} = (\epsilon_{k,T}, k = i, \dots, j)$. In the proof C denotes constants whose value may vary from place to place.

To prove Theorem 1, we shall apply the technique in Liu & Wu (2010). Lemmas 1, 2 and 3 provide bounds for m -dependent and martingale approximations for linear and quadratic forms. They can be proved by using the arguments in Lemma 1, Propositions 1 and 2 in Liu & Wu (2010), respectively. That paper deals with stationary processes, but there are no essential additional difficulties involved for generalization to nonstationary processes. Detailed proofs can be found in the online Supplementary Material.

LEMMA A1. Assume $\Theta_{0,p} < \infty$, $p \geq 2$. Let $\alpha_1, \alpha_2, \dots, \in \mathbb{R}$, $A_n = (\sum_{i=1}^n \alpha_i^2)^{1/2}$, and $C_p = 18p^{3/2}(p-1)^{-1/2}$. Then (i) $\|\sum_{i=1}^n \alpha_i e_i\|_p \leq C_p A_n \Theta_{0,p}$; and (ii) $\|\sum_{i=1}^n \alpha_i (e_i - \tilde{e}_i)\|_p \leq C_p A_n \Theta_{m+1,p}$.

LEMMA A2. Assume $\Theta_{0,4} < \infty$. Let $\alpha_j \in \mathbb{R}$,

$$L_n = \sum_{1 \leq i < j \leq n} \alpha_{j-i} e_i e_j, \quad \tilde{L}_n = \sum_{1 \leq i < j \leq n} \alpha_{j-i} \tilde{e}_i \tilde{e}_j. \quad (\text{A1})$$

Let $d_{m,4} = \sum_{k=0}^{\infty} \min(\delta_{k,4}, \Psi_{m+1,4})$ and $A_n = (\sum_{i=1}^{n-1} \alpha_i^2)^{1/2}$. Then

$$\|\{L_n - E(L_n)\} - \{\tilde{L}_n - E(\tilde{L}_n)\}\| \leq C_4 \Theta_{0,4} d_{m,4} n^{1/2} A_n.$$

LEMMA A3. Let $\alpha_1, \dots, \alpha_{n-1} \in \mathbb{R}$ and $V_m(\alpha) = \max_{i < n} \alpha_i^2 + m \sum_{i=1}^{n-1} |\alpha_i - \alpha_{i-1}|^2$; let

$$\tilde{D}_{k,n} = \sum_{l=0}^{\infty} \mathcal{P}_k \tilde{e}_{k+l}, \quad \tilde{M}_n = \sum_{1 \leq i < j \leq n} \alpha_{j-i} \tilde{D}_{i,n} \tilde{D}_{j,n}.$$

Assume $\Theta_{0,4} < \infty$. Then $\|\tilde{L}_n - E(\tilde{L}_n) - \tilde{M}_n\|^2 \leq C m^3 n V_m(\alpha)$.

THEOREM A1. Assume Condition 2 and Condition 3, (A3), $\Theta_{0,4} < \infty$, $K \in \mathcal{K}$, $b_n \rightarrow 0$ and $nb_n^{3/2} \rightarrow \infty$. Recall (A1) for L_n and let $\alpha_j = n^{-1} b_n^{-1/2} K^* \{j/(2nb_n)\}$. Then as $n \rightarrow \infty$, $L_n - E(L_n) \rightarrow N(0, g_2 K_2^*)$ in distribution.

Proof. Recall Lemma A2 for A_n and Lemma A3 for $V_m(\alpha)$. Since $K^* \in \mathcal{K}$, $V_m(\alpha) = O(mn^3 b_n^2)$ and $A_n = O(n^{-1/2})$. By Lemmas A2 and A3,

$$\lim_{m \rightarrow \infty} \limsup_{n \rightarrow \infty} \|L_n - E(L_n) - \tilde{M}_n\| = 0. \quad (\text{A2})$$

In (7) we replace $\zeta_i(t)$ by $\tilde{\zeta}_i(t)$ and let $\tilde{g}(t)$ be the long-run variance function for the latter m -dependent process. Set $\tilde{g}_2 = \int_0^1 \tilde{g}^2(t) dt$, by (A2) it suffices to verify that $\tilde{M}_n \rightarrow N(0, \tilde{g}_2 K_2^*)$ in distribution. This can be proved by using the argument of Theorem 6 in Liu & Wu (2010), where the case of stationary processes is dealt with. Here we shall only detail the step, cf. (A3) below, that requires special attention of nonstationarity since all other steps similarly follow. A complete proof is available as supplementary material. We shall show that

$$\sum_{j=2m+1}^n E(J_{1,j,n}^2) E(\tilde{D}_{j,n}^2) \rightarrow \tilde{g}_2 K_2^*, \quad (\text{A3})$$

where $J_{1,j,n} = \sum_{i=1}^{j-2m} \alpha_{j-i} \tilde{D}_{i,n}$. Let $\tilde{D}_{k,n}^* = \sum_{l=0}^{\infty} \mathcal{P}_k \tilde{\zeta}_{k+l}(k/n)$. By Condition (2),

$$|\|\tilde{D}_{k,n}\|^2 - \|\tilde{D}_{k,n}^*\|^2| \leq (\|\tilde{D}_{k,n}\| + \|\tilde{D}_{k,n}^*\|) \|\tilde{D}_{k,n} - \tilde{D}_{k,n}^*\| \leq C m^3 n^{-1} = o(1).$$

Observe that $\|\tilde{D}_{k,n}^*\|^2 = \tilde{g}(k/n)$. Then (A3) follows from

$$\begin{aligned} \sum_{j=2m+1}^n \sum_{i=1}^{j-2m} \alpha_{j-i}^2 \|\tilde{D}_{i,n}^*\|^2 \|\tilde{D}_{j,n}^*\|^2 &= \sum_{1 \leq i < j \leq n} \alpha_{j-i}^2 \tilde{g}(i/n) \tilde{g}(j/n) + o(1) \\ &= \sum_{i=1}^{n-1} \tilde{g}(i/n) \sum_{j=1+i}^n \alpha_{j-i}^2 \tilde{g}(j/n) + o(1) \\ &= \sum_{i=1}^{n-1} \tilde{g}^2(i/n) \sum_{j=1+i}^n \alpha_{j-i}^2 + o(1) = \tilde{g}_2 K_2^* + o(1), \end{aligned}$$

since $\sum_{j=1+i}^n \alpha_{j-i}^2 = n^{-1} K_2^* \{1 + o(1)\}$ and $\sum_{j=1+i}^n \alpha_{j-i}^2 \{\tilde{g}(j/n) - \tilde{g}(i/n)\} = o(n^{-1})$. \square

Proof of Theorem 1. Let $\check{\mu}_n(t) = \sum_{i=1}^n v_i(t) Y_i$ be the Priestley–Chao estimator, where $v_i(t) = (nb_n)^{-1} K\{(i/n - t)/b_n\}$. Note that $w_i(t) = v_i(t) + \mathcal{O}\{(nb_n)^{-2}\}$ uniformly over $(b_n, 1 - b_n)$. Both $\int_0^{b_n} E\{\hat{\mu}_n(t)^2\}dt$ and $\int_{1-b_n}^1 E\{\hat{\mu}_n(t)^2\}dt$ are of order $O(n^{-1})$. Then

$$\left\| \int_0^1 \hat{\mu}_n(t)^2 dt - \int_0^1 \check{\mu}_n(t)^2 dt \right\|_1 \leq C\{n^{-1} + (nb_n)^{-2}\}.$$

Hence it suffices to prove the same results for the Priestley–Chao estimator $\check{\mu}_n(t)$. For (i), write $l = l_n$. If $|i - j| \leq l$, by Condition (2) and the Cauchy–Schwarz inequality, $|E[e_i e_j - \zeta_j(i/n)]| \leq \|e_i\| \|\zeta_j(j/n) - \zeta_j(i/n)\| = O(l/n)$. Let $\gamma_k(t) = E\{\zeta_0(t)\zeta_k(t)\}$. Then $E(e_i e_j) - \gamma_{j-i}(i/n) = O(l/n)$. Since $\mathcal{P}_s, s \in \mathbb{Z}$, are orthogonal,

$$\begin{aligned} |E(e_i e_j)| &= \left| E \left(\sum_{s \in \mathbb{Z}} \mathcal{P}_s e_i \sum_{s' \in \mathbb{Z}} \mathcal{P}_{s'} e_j \right) \right| \leq \sum_{s \in \mathbb{Z}} |E\{(\mathcal{P}_s e_i)(\mathcal{P}_s e_j)\}| \\ &\leq \sum_{s \in \mathbb{Z}} \|\mathcal{P}_s e_i\| \|\mathcal{P}_s e_j\| \leq \sum_{s \in \mathbb{Z}} \delta_{i-s} \delta_{j-s} = \Gamma_{i-j}. \end{aligned} \quad (\text{A4})$$

Similarly, $|\gamma_{j-i}(i/n)| \leq \Gamma_{j-i}$. Then

$$\frac{1}{n} \sum_{i=1}^n \sum_{k=-l}^l \left| K^* \left(\frac{k}{2nb_n} \right) \{E(e_i e_{i+k}) - \gamma_k(i/n)\} \right| = \sum_{k=0}^l O\{\min(\Gamma_k, b_n)\}. \quad (\text{A5})$$

Let $\bar{\gamma}_k = \int_0^1 \gamma_k(t)dt$. Since $\gamma_k(\cdot)$ is Lipschitz continuous and $K^* \in \mathcal{C}[-1, 1]$,

$$\begin{aligned} &\frac{1}{n} \sum_{i=1}^n \sum_{k=-l}^l K^* \left(\frac{k}{2nb_n} \right) \gamma_k(i/n) - K^*(0) \sum_{k=-l}^l \bar{\gamma}_k \\ &= \sum_{k=-l}^l K^* \left(\frac{k}{2nb_n} \right) \{\bar{\gamma}_k + O(n^{-1})\} - K^*(0) \sum_{k=-l}^l \bar{\gamma}_k = O(l/n) + \sum_{k=0}^l O(k/l) \Gamma_k. \end{aligned} \quad (\text{A6})$$

By Lemma A1(i), $\sup_t \|\hat{\mu}_n(t)\|^2 = O\{(nb_n)^{-1}\}$. Note that $\int_{-b}^{1+b} K(x - t/b)K(y - t/b)dt = bK^*(y/2 - x/2)$ if $0 \leq x \leq y \leq b^{-1}$. Hence

$$\begin{aligned} nb_n \int_0^1 E\{\hat{\mu}_n(t)^2\}dt &= nb_n \int_{-b_n}^{1+b_n} E\{\hat{\mu}_n(t)^2\}dt + O(b_n) \\ &= \frac{1}{n} \sum_{1 \leq i, j \leq n} K^* \left(\frac{i-j}{2nb_n} \right) E(e_i e_j) + O(b_n). \end{aligned} \quad (\text{A7})$$

Since $g_1 = \sum_{k \in \mathbb{Z}} \bar{\gamma}_k$, by (A5), (A6) and (A7), (8) follows in view of

$$\frac{1}{n} \sum_{i=1}^n \left\{ \sum_{k=-l}^l - \sum_{\substack{\min(n-i, l) \\ \max(1-i, l)}} \right\} K^* \left(\frac{k}{2nb_n} \right) E(e_i e_{i+k}) = \frac{1}{n} O(l) = O(b_n).$$

For (ii), by Theorem A1, we have

$$nb_n^{1/2} \int_{-b_n}^{1+b_n} [\hat{\mu}_n(t)^2 - E\{\hat{\mu}_n(t)^2\}]dt \rightarrow N(0, 4g_2 K_2^*)$$

in distribution. So, (ii) follows in view of $nb_n^{1/2} \int_{-b_n}^0 E\{\hat{\mu}_n(t)^2\}dt = nb_n^{1/2} O\{b_n(nb_n)^{-1}\} \rightarrow 0$ and similarly, for the right tail, $nb_n^{1/2} \int_1^{1+b_n} E\{\hat{\mu}_n(t)^2\}dt \rightarrow 0$. \square

Proof of Theorem 2. Observe that $I_n = II_n + III_n$, where

$$II_n = \int_0^1 \{[\hat{\mu}_n(t) - E\{\hat{\mu}_n(t)\}]^2 - E([\hat{\mu}_n(t) - E\{\hat{\mu}_n(t)\}]^2)\} dt,$$

$$III_n = \int_0^1 2[\hat{\mu}_n(t) - E\{\hat{\mu}_n(t)\}][E\{\hat{\mu}_n(t)\} - \mu(t)] dt.$$

365 By Theorem 1(ii), $nb_n^{1/2}III_n \rightarrow N(0, 4g_2K_2^*)$ in distribution. Then it suffices to show that

$$\frac{n^{1/2}}{b_n^2}III_n = \sum_{i=1}^n q_{n,i} e_i \rightarrow \kappa_2 N(0, \sigma^2) \quad (\text{A8})$$

in distribution, where $\sigma^2 = \int_0^1 g(t)\mu''(t)^2 dt$ and

$$q_{n,i} = \frac{2n^{1/2}}{b_n^2} \int_0^1 w_i(t)[E\{\hat{\mu}_n(t)\} - \mu(t)] dt.$$

Under Condition 1, the bias $E\{\hat{\mu}_n(t)\} - \mu(t) = b_n^2 \kappa_2 \mu''(t)/2 + o(b_n^2)$. Since $K \in \mathcal{K}$, by elementary calculations, $r_{n,i} := q_{n,i} - n^{-1/2} \kappa_2 \mu''(i/n)$ satisfies $\sum_{i=1}^n |r_{n,i}|^2 = o(1)$. By Lemma A1(i), $\|\sum_{i=1}^n r_{n,i} e_i\| = o(1)$, and (A8) follows if

$$\sum_{i=1}^n n^{-1/2} \mu''(i/n) e_i \rightarrow N(0, \sigma^2) \quad (\text{A9})$$

370 in distribution. To prove (A9) we apply the m -dependence approximation method. By Lemma A1(ii),

$$\lim_{m \rightarrow \infty} \limsup_{n \rightarrow \infty} \left\| \sum_{i=1}^n n^{-1/2} \mu''(i/n) (e_i - \tilde{e}_i) \right\| = 0.$$

Note that $\tilde{e}_i = E(e_i | \mathcal{F}_{i-m,i})$ are m -dependent. Let $\tilde{\sigma}^2 = \int_0^1 \tilde{g}(t)\mu''(t)^2 dt$. Hence (A9) follows from Hoeffding & Robbins' (1948) central limit theorem for $\sum_{i=1}^n n^{-1/2} \mu''(i/n) \tilde{e}_i$ for m -dependent random variables \tilde{e}_i in view of

$$E \left\{ \sum_{i=1}^n n^{-1/2} \mu''(i/n) \tilde{e}_i \right\}^2 = \frac{1}{n} \sum_{|i-j| \leq m} \mu''(i/n) \mu''(j/n) E(\tilde{e}_i \tilde{e}_j) \rightarrow \tilde{\sigma}^2.$$

To see the above relation, we note that $E(\tilde{e}_i \tilde{e}_j) - \tilde{\gamma}_{i-j}(i/n) = o(1)$ if $|i-j| \leq m$, and
 375 $n^{-1} \sum_{i=1}^n \mu''(i/n)^2 \tilde{\gamma}_{i-j}(i/n) \rightarrow \int_0^1 \mu''(t)^2 \tilde{\gamma}_{i-j}(t) dt$, by the continuity of $\tilde{\gamma}_{i-j}(\cdot)$ and $\mu''(\cdot)$. \square

Proof of Proposition 2. Write

$$\text{ISE}_{\hat{M}} - \text{ISE}_M = 2 \int_0^1 \{\hat{\mu}_n(t) - \mu_M(t)\} \{\mu_M(t) - \mu_{\hat{M}}(t)\} dt$$

$$+ \int_0^1 \{\mu_M(t) - \mu_{\hat{M}}(t)\}^2 dt = 2A_n + B_n.$$

So, (16) follows if both A_n and B_n are of order n^{-1} . To this end, by (15),

$$\mu_M(t) - \mu_{\hat{M}}(t) = \sum_{i=1}^n w_i(t) \{f(\theta_0, t) - f(\hat{\theta}_n, t)\}$$

$$= (\theta_0 - \hat{\theta}_n)^\top \sum_{i=1}^n w_i(t) \dot{f}(\theta_0, t) + O_p(n^{-1}) \quad (\text{A10})$$

holds uniformly over $t \in [0, 1]$ since $\hat{\theta}_n$ is $n^{1/2}$ -consistent. Then by (14), we have $B_n = O_p(n^{-1})$. For A_n , note that $\hat{\mu}_n(t) - \mu_M(t) = \sum_{j=1}^n w_j(t)e_j$. By (A10), since $\|\hat{\mu}_n(t) - \mu_M(t)\|n^{-1} = O(n^{-3/2}b_n^{-1/2})$ and, for some constant $C > 0$, by (14),

$$\left| \int_0^1 w_j(t) \sum_{i=1}^n w_i(t) \dot{f}(\theta_0, t) dt \right| \leq C \int_0^1 |w_j(t)| dt \leq \frac{C}{n},$$

then we also have $A_n = O_p(n^{-1})$ in view of Lemma A1(i). □ 385

Proof of Theorem 3. Let $\mathcal{I}_n(t) = \{i : |i/n - t| \leq b_n\}$. Observe that

$$\sum_{i=1}^n Q_i \mathbf{I}(|i/n - t| \leq b_n) = \sum_{i,j \in \mathcal{I}_n(t)} e_i e_j \mathbf{I}(|i - j| \leq m_n) + R_n,$$

where by (A4),

$$R_n = \sum_{i \in \mathcal{I}_n(t), j \notin \mathcal{I}_n(t)} e_i e_j \mathbf{I}(|i - j| \leq m_n) = O_p(m_n).$$

With elementary manipulations, (18) follows by applying the argument of Theorem 1 to $(e_i)_{i \in \mathcal{I}_n(t)}$ with $\alpha_{i-j} = \mathbf{I}(|i - j| \leq m_n)$. For (19), write

$$E(Q_i) - g(i/n) = \sum_{j=i-m_n}^{i+m_n} E[\zeta_i(i/n)\{\zeta_j(j/n) - \zeta_j(i/n)\}] + \sum_{j:|j-i|>m_n} E\{\zeta_i(i/n)\zeta_j(i/n)\}.$$

By (A6) and (Condition 2), the first and second terms above are of order $\sum_{k \leq m_n} O\{\min(\Gamma_k, m_n/n)\}$ and $\sum_{k > m_n} O(\Gamma_k)$, respectively. Then (19) follows from Condition 3. □ 390

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