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TESTS FOR THE MARGINAL PROBABILITIES IN THE TWO-WAY CONTINGENCY TABLE UNDER RESTRICTED ALTERNATIVES

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Abstract. Testing hypotheses on the marginal probabilities of a two-way contingency table is discussed. Three statistics are considered for testing the hypothesis of specified probabilities in the margins against alternatives with certain kind of order restriction. The properties of these statistics are discussed and their asymptotic behaviors are compared in depth. An application which motivated the consideration of the original testing problem is illustrated with a practical data.

Key words and phrases: Asymptotic power, categorical data, restricted alternative, most stringent somewhere most powerful test.

1. Introduction

Denote by p_{ab} the (a, b)-cell probability of a$(r+1)\times(c+1)$ contingency table. Put

$$p_{a\cdot} = \sum_b p_{ab}\ , \quad p_{\cdot b} = \sum_a p_{ab}\ , \quad a = 1,\dots, r+1; \quad b = 1,\dots, c+1\ ,$$

where

$$\sum_a \sum_b p_{ab} = \sum_a p_{a\cdot} = \sum_b p_{\cdot b} = 1\ .$$

The purpose of this paper is to consider testing problem of the hypothesis,

$$H: p_{a\cdot} = p^{\circ}_{a\cdot}\ , \quad p_{\cdot b} = p^{\circ}_{\cdot b}\ , \quad a = 1,\dots, r; \quad b = 1,\dots, c\ ,$$

against

$$K : \sum_{a=1}^{l} p_{a\cdot} \leq \sum_{a=1}^{l} p_{a\cdot}^{\circ} , \quad \sum_{b=1}^{m} p_{\cdot b} \leq \sum_{b=1}^{m} p_{\cdot b}^{\circ} , \qquad l = 1,\ldots, r; \quad m = 1,\ldots, c ,$$

with at least one inequality strict, where $p_{a\cdot}^{\circ}$ and $p_{\cdot b}^{\circ}$, $a=1,\ldots, r$; $b=1,\ldots, c$, are given constants.

Testing the hypothesis $p_{a\cdot} = p_{a\cdot}^{\circ}$ for all a against $\sum_{a=1}^{l} p_{a\cdot} \leq \sum_{a=1}^{l} p_{a\cdot}^{\circ}$, $l=1,\ldots, r$, that is the hypotheses regarding one of the two margins of a contingency table, has been considered by Schaafsma (1966). The problem in this paper is concerned with the two margins of a contingency table.

Three statistics are considered in Section 2 which lead to asymptotic tests for testing the hypotheses. The first statistic is constructed directly by applying the most stringent somewhere most powerful (MSSMP-) principle discussed by Schaafsma and Smid (1966) for a general class of multivariate one-sided test. On the other hand, the second and third statistics are constructed by combining two MSSMP-test statistics for the hypotheses regarding each of two margins: the first one simply adds the two statistics, but the second one uses the likelihood ratio (LR-) principle for the combination. In Section 3 we consider an example from a multiply matched case-control study. This supplies a ground for considering the above hypotheses. The approximate p-values of these three tests are obtained for the purpose of illustration by using the one-to-three matched data from a case-control study for studying the association of stomach cancer and neutritious pattern. The asymptotic efficiency of the three tests is considered in Section 4 by employing the Pitman efficiency or comparing their asymptotic powers.

2. The construction of tests

We consider first in Subsection 2.1 the MSSMP-test statistic for testing hypotheses with restricted alternatives under a general framework of multivariate normal distribution, and then introduce "approximately MSSMP-test" for H vs. K in Subsection 2.2. The T-test and R-test are introduced in Subsections 2.3 and 2.4.

2.1 *Preliminary*

Let $X=(X_1,\ldots, X_h)'$ be a random vector distributed as an h-variate normal distribution with mean $\boldsymbol{\mu}$ and known covariance matrix $\Lambda=(\lambda_{lm})$. Consider testing the hypothesis

$$H' : \boldsymbol{\mu} = \mathbf{0} \quad \text{against} \quad K' : \boldsymbol{\mu} \geq \mathbf{0} \quad (\boldsymbol{\mu} \neq \mathbf{0}) ,$$

where $\boldsymbol{\mu} \geq \mathbf{0}$ means that all components of $\boldsymbol{\mu}$ are non-negative.

For the purpose of constructing explicitly the statistic for testing H' against K', we employ the MSSMP-principle. The general form of the

MSSMP-test given by Schaafsma and Smid (1966) is represented by; Reject H' iff $<\xi, X> \geq$ constant, where $<\cdot\,,\cdot>$ is the inner product defined by $<\boldsymbol{u}, \boldsymbol{v}> = \boldsymbol{u}'\Lambda^{-1}\boldsymbol{v}$ for $\boldsymbol{u}, \boldsymbol{v} \in R^h$, and ξ is a non-negative vector which minimizes the maximum angles between ξ, $\xi \geq 0$, and each axis $\boldsymbol{e}_1 = (1, 0,\ldots, 0)',\ldots, \boldsymbol{e}_h = (0,\ldots, 0, 1)'$.

Generally it is difficult to express the MSSMP-test statistics explicitly. However, we may show:

THEOREM 2.1. *If* $\Lambda(\sqrt{\lambda^{11}},\ldots, \sqrt{\lambda^{hh}})' \geq 0$, *the MSSMP-test for H' against K' is expressed by*

$$\text{(2.1)} \qquad \textit{Reject } H' \textit{ iff } \quad \frac{\sum_{l=1}^{h} \sqrt{\lambda^{ll}}\, X_l}{\sqrt{\sum_{l=1}^{h} \sum_{m=1}^{h} \lambda_{lm} \sqrt{\lambda^{ll}} \sqrt{\lambda^{mm}}}} > u_\alpha\,,$$

where λ^{ll} is the (l, l)-element of Λ^{-1} and u_α is the upper α-quantile of the standard normal distribution.

PROOF. It is clear that if there exists $\xi \geq \boldsymbol{0}$ which satisfies

$$\text{(2.2)} \qquad \frac{< \xi, \boldsymbol{e}_1 >}{\|\xi\|\,\|\boldsymbol{e}_1\|} = \frac{< \xi, \boldsymbol{e}_2 >}{\|\xi\|\,\|\boldsymbol{e}_2\|} = \cdots = \frac{< \xi, \boldsymbol{e}_h >}{\|\xi\|\,\|\boldsymbol{e}_h\|}\,,$$

then the ξ minimizes the maximum angles between ξ and each axis $\boldsymbol{e}_1, \boldsymbol{e}_2,\ldots, \boldsymbol{e}_h$. Thus the MSSMP-test is given by this ξ. We now obtain the ξ satisfying (2.2). Put $\xi'\Lambda^{-1} = (\gamma_1,\ldots, \gamma_h)$, then $<\xi, \boldsymbol{e}_l> = \gamma_l$, and since $\|\boldsymbol{e}_l\| = \sqrt{\lambda^{ll}}$, it follows that $\xi = c\Lambda(\sqrt{\lambda^{11}},\ldots, \sqrt{\lambda^{hh}})'$ for a constant c. When $c>0$, $\xi \geq \boldsymbol{0}$ from the assumption of the theorem. Substituting this ξ to the general form we have (2.1).

Note that the vector $c\Lambda(\sqrt{\lambda^{11}},\ldots, \sqrt{\lambda^{hh}})'$ is a normal vector of the elliptic quadric $\boldsymbol{x}'\Lambda\boldsymbol{x} = c$ (>0), in $\boldsymbol{R}^h$, at the point $k(\sqrt{\lambda^{11}},\ldots, \sqrt{\lambda^{hh}})'$ with $\sqrt{k} = c/\sum_{i,j} \sqrt{\lambda^{ii}\lambda^{jj}}\,\lambda_{ij}$. The condition $\Lambda\,(\sqrt{\lambda^{11}},\ldots, \sqrt{\lambda^{hh}})' \geq \boldsymbol{0}$ is satisfied, for example, when all random variables X's are positively correlated, or equally correlated (also see the application below).

2.2 *S-test*

Applying the above result, we construct a statistic for testing H against K. Put

$$Z_l = n^{-1/2} \sum_{a=1}^{l} (np^{\circ}_{a\cdot} - n_{a\cdot})\,, \qquad l = 1,\ldots, r\,,$$

$$Z_{r+m} = n^{-1/2} \sum_{b=1}^{m} (np^{\circ}_{\cdot b} - n_{\cdot b})\,, \qquad m = 1,\ldots, c\,,$$

$$\boldsymbol{Z} = (Z_1, \ldots, Z_h)' , \qquad h = r + c ,$$

$$\theta_l = \sum_{a=1}^{l} (p_{a\cdot}^{\circ} - p_{a\cdot}) , \qquad l = 1, \ldots, r ,$$

$$\theta_{r+m} = \sum_{b=1}^{m} (p_{\cdot b}^{\circ} - p_{\cdot b}) , \qquad m = 1, \ldots, c ,$$

$$\boldsymbol{\theta} = (\theta_1, \ldots, \theta_h)' , \qquad h = r + c .$$

It is straightforward to show that $\boldsymbol{Z} - n^{1/2}\boldsymbol{\theta}$ is asymptotically distributed as an h-variate normal distribution with mean $\mathbf{0}$ and covariance matrix $\Sigma = (\sigma_{lm})$, where

$$\sigma_{lm} = \begin{cases} \left(\sum_{a=1}^{l} p_{a\cdot}\right)\left(1 - \sum_{b=1}^{m} p_{b\cdot}\right) & \text{for} \quad 1 \le l \le m \le r , \\ \sum_{a=1}^{l} \sum_{b=r+1}^{m} (p_{a,b-r} - p_{a\cdot}\, p_{\cdot b-r}) & \text{for} \quad l = 1, \ldots, r; \quad m = r+1, \ldots, h , \\ \left(\sum_{a=r+1}^{l} p_{\cdot a-r}\right)\left(1 - \sum_{b=r+1}^{m} p_{\cdot b-r}\right) & \text{for} \quad r+1 \le l \le m \le h . \end{cases}$$

Since the hypotheses H and K are equivalent to H'': $\boldsymbol{\theta} = \mathbf{0}$ and K'': $\boldsymbol{\theta} \geq \mathbf{0}$, we may use from Theorem 2.1 the following test statistic for testing H against K:

$$S = \frac{\sum_{l=1}^{h} \sqrt{\sigma_*^{ll}}\, Z_l}{\sqrt{\sum_{l=1}^{h} \sum_{m=1}^{h} \sigma_{lm}^{*} \sqrt{\sigma_*^{ll}} \sqrt{\sigma_*^{mm}}}} ,$$

when the condition

$$\text{(2.3)} \qquad \Sigma_* (\sqrt{\sigma_*^{11}}, \cdots, \sqrt{\sigma_*^{hh}})' \geq 0 ,$$

is satisfied, where $\Sigma_* = (\sigma_{lm}^{*})$ is a consistent estimator of Σ under H and $\Sigma_*^{-1} = (\sigma_*^{lm})$ is the inverse of Σ_*. We call the test based on S the S-test. The test is an approximately MSSMP-test. It follows that the S-test is asymptotically the uniformly most powerful test for the alternatives in the direction specified by $\boldsymbol{\theta} = \boldsymbol{\theta}^{\circ} = c\Sigma(\sqrt{\sigma_{\circ}^{11}}, \ldots, \sqrt{\sigma_{\circ}^{hh}})'$, $c > 0$, where $\sigma_{\circ}^{ii}$ is the (i, i)-element of Σ^{-1}.

2.3 *T-test*

The approximately MSSMP-test for the corresponding hypotheses regarding one of the margins of the contingency table has been considered by Schaafsma (1966). The test statistic is given by

$$T_1 = \sum_{l=1}^{r} \left(\frac{w_{1l}}{Q_1}\right)^{1/2} Z_l ,$$

where

$$w_{1l} = p_{l.}^{\circ\,-1} + p_{l+1.}^{\circ\,-1}\,, \qquad l = 1,\ldots, r\,,$$

and

$$Q_1 = \sum_{l=1}^{r} p_{l.}^{\circ}\left(\sum_{a=1}^{r+1-l}\sqrt{w_{1,r+1-a}}\right)^2 - \left(\sum_{l=1}^{r}\sum_{a=1}^{r+1-l} p_{l.}^{\circ}\sqrt{w_{1,r+1-a}}\right)^2 .$$

Let T_2, w_{2m}, $m=1,\ldots,c$ and Q_2 be corresponding ones for the other margin. We consider the following test statistic T for testing H vs. K:

$$T = \frac{T_1 + T_2}{\sqrt{\boldsymbol{d}'\Sigma_*\boldsymbol{d}}}\,,$$

where

$$\boldsymbol{d} = \left(\left(\frac{w_{11}}{Q_1}\right)^{1/2},\ldots,\left(\frac{w_{1r}}{Q_1}\right)^{1/2},\left(\frac{w_{21}}{Q_2}\right)^{1/2},\ldots,\left(\frac{w_{2c}}{Q_2}\right)^{1/2}\right)'.$$

T is simpler than S and has no restriction. We call the test based on T the T-test. T is asymptotically distributed as the standard normal distribution under H. It is easily seen that $S=T$, when $\boldsymbol{d}=c(\sqrt{\sigma_*^{11}},\ldots,\sqrt{\sigma_*^{hh}})'$ for some positive constant c.

It follows that the T-test is asymptotically the uniformly most powerful test for the alternatives in the direction specified by $\boldsymbol{\theta}=\boldsymbol{\theta}^1=c\Sigma\boldsymbol{d}$, $c>0$.

2.4 *R-test*

Kudô (1963) considered the LR-principle for testing hypotheses with restricted alternatives under multivariate normal distribution. Although the test by the principle is not easily available when $h\geq 4$, we may apply the principle to the asymptotic distribution of (T_1, T_2). Consider the statistic,

$$R = \begin{cases} (1-\rho_*^2)^{-1/2}(T_1^2 + T_2^2 - 2\rho_* T_1 T_2)^{1/2} & \text{if} \quad T_1 \geq 0, \quad T_2 \geq 0\,, \\ (1-\rho_*^2)^{-1/2}(T_1 - \rho_* T_2) & \text{if} \quad T_2 < 0, \quad T_1 \geq T_2\,, \\ (1-\rho_*^2)^{-1/2}(T_2 - \rho_* T_1) & \text{if} \quad T_1 < 0, \quad T_2 \geq T_1\,, \end{cases}$$

where ρ_* is a consistent estimator of ρ, the correlation coefficient of T_1 and T_2;

$$\rho = \frac{1}{\sqrt{Q_1 Q_2}}\sum_{l=1}^{r}\sum_{m=1}^{c}\sqrt{w_{1l}}\sqrt{w_{2m}}\,(p_{lm} - p_{l.}^{\circ}\,p_{.m}^{\circ}) \tag{2.4}$$

under H. If (T_1, T_2) were distributed exactly as the bivariate normal distribution $N(\mu_1, \mu_2, 1, 1, \rho_*)$, the statistic R is the LR-test statistic for the null hypothesis, H'': $\mu_1=\mu_2=0$, against the alternative, K'': $\mu_1\geq 0$, $\mu_2\geq 0$. The test rejects H iff $R\geq c_\alpha$, where c_α is the critical point at the significance level α. Following Chatterjee and De (1972) we may obtain an approximate value of c_α by solving

$$\alpha = \{1 - F_{\chi_2^2}(c^2)\}\left\{\frac{1}{2} - (2\pi)^{-1}\cos^{-1}\rho_*\right\} + 1 - \Phi(c) ,$$

where $F_{\chi_2^2}$ and Φ are the distribution functions of the central χ^2-distribution with 2 degrees of freedom and the standard normal distribution.

Before comparing the asymptotic efficiency of the tests based on statistics S, T and R we look at an application of the tests briefly in the next section.

3. An application

Several controls are matched frequently to a case in a comparative study by means of extraneous variables. Let X_i be a random sample from the case, Y_{ij} be a random sample from the j-th control matched to the case, $j=1,\ldots, k$, and V_i be the vector of extraneous variables used for the matching. Suppose that X_i and $Y_{i1},\ldots, Y_{ik}$ are conditionally independent and that $Y_{i1},\ldots, Y_{ik}$ are identically distributed when conditioned on V_i. Let $F(x|v_i)$ and $G(y|v_i)$ be conditional distribution functions of X_i and Y_{i1} conditioned on V_i. We assume that X's and Y's are two-dimensional random vectors and that F and G are continuous. Let $F_s(\cdot|v)$ and $G_s(\cdot|v)$, $s=1, 2$, be the marginal distribution functions of $F(\cdot|v)$ and $G(\cdot|v)$. We discuss the situation where $F_s(x|v)\leq G_s(x|v)$ is presumed. We consider testing the hypothesis

$$H_0\colon F_s(x|v_i) = G_s(x|v_i) \quad \text{for all} \quad x, \quad i = 1,\ldots, n; \quad s = 1, 2 ,$$

against

$$K_0\colon F_s(x|v_i) \leq G_s(x|v_i), \qquad i = 1,\ldots, n; \quad s = 1, 2 ,$$

for all x with strict inequality at least one x, based on ranks of the observations.

Denote the components of X_i and Y_{ij} by (X_{1i}, X_{2i}) and (Y_{1ij}, Y_{2ij}), and R_{si} be the rank of X_{si} among X_{si}, $Y_{si1},\ldots, Y_{sik}$, for $s=1, 2$ and $i=1,\ldots, n$. We may summarize the paired ranks (R_{1i}, R_{2i}), $i=1,\ldots, n$, in a $(k+1)\times(k+1)$ contingency table. Let n_{ab} be the number of i's satisfying $R_{1i}=a$ and $R_{2i}=b$, and put

$$n_{a\cdot} = \sum_{b=1}^{k+1} n_{ab} , \qquad n_{\cdot b} = \sum_{a=1}^{k+1} n_{ab} .$$

We assume (R_{1i}, R_{2i}), $i=1,\dots,n$, are identically distributed. This assumption is satisfied, for example, when

$$F(x, y|\nu) = F(x - \psi_1(\beta; \nu), y - \psi_2(\beta; \nu)) ,$$

$$G(x, y|\nu) = G(x - \psi_1(\beta; \nu), y - \psi_2(\beta; \nu)) .$$

Denote the cell probabilities by p_{ab}, and put $p_{a\cdot}=\sum_b p_{ab}$ and $p_{\cdot b}=\sum_a p_{ab}$. Then

$$p_{a\cdot} = P(R_1 = a), \quad p_{\cdot b} = P(R_2 = b), \quad a, b = 1,\dots, k + 1 .$$

Further, for $s=1, 2$,

$$\begin{aligned} P(R_s \leq a) &= \frac{k!}{(a-1)!(k-a)!} \int_{-\infty}^{\infty} \{G_s(x|\nu)\}^{a-1}\{1 - G_s(x|\nu)\}^{k-a} \\ &\quad \times F_s(x|\nu) dG_s(x|\nu), \qquad a = 1,\dots, k , \\ P(R_s \leq k + 1) &= 1 , \end{aligned}$$

which are independent of ν from the assumption.

It follows from these formulae that the hypotheses H_0 and K_0 are equivalently represented in the contingency table as follows:

$$\begin{aligned} H_1\colon \ & \sum_{a=1}^{l} p_{a\cdot} = l/(k+1) , && l = 1,\dots, k , \\ & \sum_{b=1}^{m} p_{\cdot b} = m/(k+1) , && m = 1,\dots, k , \\ K_1\colon \ & \sum_{a=1}^{l} p_{a\cdot} \leq l/(k+1) , && l = 1,\dots, k , \\ & \sum_{b=1}^{m} p_{\cdot b} \leq m/(k+1) , && m = 1,\dots, k , \end{aligned}$$

with either first or last k inequalities strict. We shall apply the tests developed in this paper for testing H_1 against K_1. It is easily seen that the condition (2.3) in Theorem 2.1 is satisfied if

$$P(R_1 \leq l, R_2 \leq m) \geq P(R_1 \leq l)P(R_2 \leq m), \quad l, m = 1,\dots, k ,$$

that is, if R_1 and R_2 are positively dependent.

Example. A case-control study was conducted in a district of Japan to study the relationship of stomach cancer and nutritious pattern. Three controls are matched to a case based on sex, location and age. For an illustrative purpose we use here the data of the total intake of protein and fat from 55 cases and 55×3 controls in the study. Naturally, two factors are positively correlated and it is seen that the joint distribution of the two factors is skewed and far away from normal distributions. The ranked data of the two factors are summarized in Table 1.

A set of first order efficient estimators of the cell probabilities is obtained by minimizing

$$\sum_{i,j} \frac{(n_{ij} - np_{ij})^2}{n_{ij}} ,$$

under the restrictions $p_{i\cdot} = p_{\cdot j} = 1/4$, $i, j = 1, \ldots, 4$ (Rao (1973)). These estimates are listed in Table 2.

The values of $z_1, z_2, \ldots, z_6$, T_1, T_2, ρ_* and d are calculated as follows:

$$\begin{aligned} z_1 &= 0.236, & z_2 &= 0.472, & z_3 &= 0.438 , \\ z_4 &= 0.371, & z_5 &= 0.472, & z_6 &= 0.573 , \\ T_1 &= 1.025, & T_2 &= 1.266, & \rho_* &= 0.547 , \\ \boldsymbol{d} &= \left(\frac{4}{5}\right)^{1/2} (1, 1, 1, 1, 1, 1)' . \end{aligned}$$

The values of the statistics S, T and R and the approximate p-values of the tests based on these statistics are given in Table 3.

Table 1. The ranked data of the total intake of protein and fat from 55 cases and 55×3 controls in a district of Japan.

R_1 \ R_2	1	2	3	4	
1	7	2	2	1	12
2	2	6	2	2	12
3	1	4	4	5	14
4	1	1	5	10	17
	11	13	13	18	55

Table 2. Estimates of cell probabilities of Table 1 under the null hypothesis.

	1	2	3	4	
1	0.158	0.037	0.040	0.015	0.25
2	0.049	0.123	0.044	0.034	0.25
3	0.022	0.074	0.079	0.075	0.25
4	0.020	0.016	0.087	0.126	0.25
	0.25	0.25	0.25	0.25	

Table 3. The values and approximate p-values of the three test statistics.

Statistic	Values	p-values of the tests
S	1.293	0.097
T	1.301	0.098
R	1.325	0.235

The table shows that the p-values of the S-test and T-test are almost equal. Whereas the p-value of the R-test is considerably larger than those of the other tests. These findings can be explained as follows: (1) Put $\boldsymbol{n}=(\sqrt{\sigma_*^{11}}, \dots, \sqrt{\sigma_*^{66}})'$. Then, in this example, it follows by simple calculation that

$$\frac{\boldsymbol{n}'\boldsymbol{l}}{\sqrt{\boldsymbol{n}'\boldsymbol{n}}\sqrt{\boldsymbol{l}'\boldsymbol{l}}} \doteqdot 1 .$$

Therefore we have $\boldsymbol{d} \doteqdot c(\sqrt{\sigma_*^{11}}, \dots, \sqrt{\sigma_*^{66}})'$ for some positive constant c. This leads to $S = T$ as we discussed in the text; (2) Figure 1 gives a sketch of the 10% rejection region of the T-test and R-test: the shaded area for the T-test and the dotted area for the R-test. The broken line is the 20% contour of the R-test. P shows the sample point of the present data. The figure clearly shows the cause

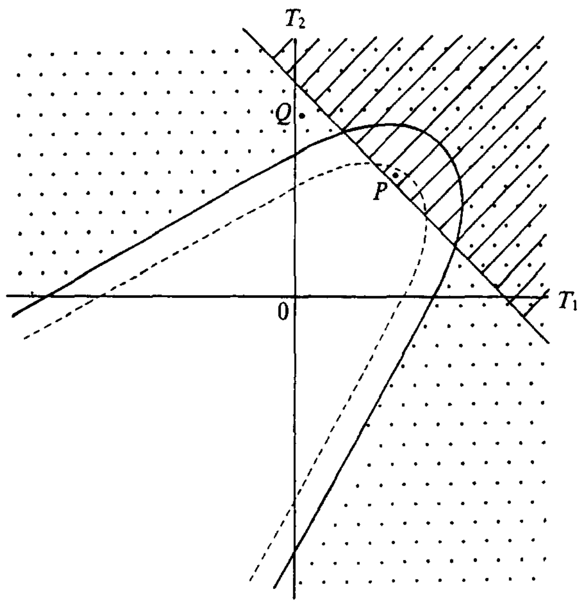

Fig. 1.

of the considerable difference of the p-values between the tests, namely P is almost in the (1, 1) direction which makes the R-test most conservative compared to the T-test. If the sample point were, for example, at Q in the figure, the results would be reversed. In general, the T-test provides smaller p-value than the R-test in the region around the straight line of the (1, 1) direction.

4. Asymptotic comparison of the tests

We first compare the S-test and T-test in Subsection 4.1 by using the Pitman efficiency, then compare the asymptotic powers of the T-test and R-test in Subsection 4.2.

4.1 *Comparison of the S-test and T-test*

For arbitrary fixed $\{p'_{ab}\}$ such that

$$p'_{a\cdot} = p^{\circ}_{a\cdot}\ , \quad p'_{\cdot b} = p^{\circ}_{\cdot b}\ , \qquad a = 1,\ldots, r+1; \quad b = 1,\ldots, c+1\ ,$$

we consider the following sequence of alternatives:

$$H_{1n}\colon p^{(n)}_{ab} = p'_{ab} + n^{-1/2}\delta_{ab}\ , \qquad a = 1,\ldots, r+1; \quad b = 1,\ldots, c+1\ ,$$

where $\{\delta_{ab}\}$ is a set of real numbers such that

$$\sum_{a=1}^{r+1}\sum_{b=1}^{c+1} \delta_{ab} = 0\ ,$$

$$\sum_{a=1}^{l} \delta_{a\cdot} \le 0, \quad \sum_{b=1}^{m} \delta_{\cdot b} \le 0, \qquad l = 1,\ldots, r; \quad m = 1,\ldots, c\ ,$$

with at least one inequality strict.

We shall obtain the Pitman efficiency of the S-test with respect to the T-test under H_{1n}.

It is easy to see that $\sigma^{*}_{lm} \to \sigma^{\circ}_{lm}$ in probability as $n \to \infty$, under H_{1n}, where σ°_{lm} is the (l, m)-element of the covariance matrix Σ generated by the $\{p'_{ab}\}$. We denote by $\sigma_{\circ}^{lm}$ the (l, m)-element of Σ^{-1}. Also it may be easy to show that, under H_{1n}, S and T are asymptotically distributed as normal distributions with unit variances and means Δ_1 and Δ_2 respectively as $n \to \infty$, where

$$\Delta_1 = -\frac{\displaystyle\sum_{s=1}^{r} \sqrt{\sigma_{\circ}^{ss}}\left(\sum_{i=1}^{s} \delta_{i\cdot}\right) + \sum_{t=1}^{c} \sqrt{\sigma_{\circ}^{r+t,r+t}}\left(\sum_{j=1}^{t} \delta_{\cdot j}\right)}{\sqrt{\displaystyle\sum_{s=1}^{r+c}\sum_{t=1}^{r+c} \sigma^{\circ}_{st}\sqrt{\sigma_{\circ}^{ss}}\sqrt{\sigma_{\circ}^{tt}}}}\ ,$$

$$\Delta_2 = -\frac{Q_1^{-1/2} \sum_{s=1}^{r} \sqrt{w_{1s}} \left(\sum_{i=1}^{s} \delta_{i\cdot} \right) + Q_2^{-1/2} \sum_{t=1}^{c} \sqrt{w_{2t}} \left(\sum_{j=1}^{t} \delta_{\cdot j} \right)}{\sqrt{\boldsymbol{d}'\Sigma \boldsymbol{d}}} ,$$

$$\boldsymbol{d} = \left(\left(\frac{w_{11}}{Q_1} \right)^{1/2}, \ldots, \left(\frac{w_{1r}}{Q_1} \right)^{1/2}, \left(\frac{w_{21}}{Q_2} \right)^{1/2}, \ldots, \left(\frac{w_{2c}}{Q_2} \right)^{1/2} \right)' .$$

Then the Pitman efficiency of the S-test relative to the T-test is given by

$$e_P(S, T) = \left(\frac{\Delta_1}{\Delta_2} \right)^2 ,$$

(see Mitra (1958)).

We evaluate $e_P(S, T)$ in detail when $r=c=2$ and $p_{a\cdot}=p_{\cdot b}=1/3$, $a, b=1, 2, 3$. Since 8 parameters are involved in Δ_1 and Δ_2, the numerical comparison would be voluminous unless the parameters are restricted in some way. We shall consider a class of $\{p'_{ab}; a, b=1, 2, 3\}$ generated by the bivariate normal distributions $N(\boldsymbol{0}, 1, 1, \rho)$, $-1<\rho<1$, as follows: Let (U_1, U_2) be a random vector from this distribution. Put $q_1(\rho)=P(U_1>u_{1/3}, U_2>u_{1/3})$, $q_2(\rho)=P(|U_1|<u_{1/3}, U_2>u_{1/3})$, $q_3(\rho)=P(U_1<-u_{1/3}, U_2>u_{1/3})$ and $q_4(\rho)=P(|U_1|<u_{1/3}, |U_2|<u_{1/3})$ where $u_{1/3}$ is the upper $1/3$-quantile of the standard normal distribution. Then one of $\{p'_{ab}; a, b=1, 2, 3\}$ satisfying $p'_{a\cdot}=p'_{\cdot b}=1/3$ is given by

$$Q = \begin{pmatrix} q_1(\rho) & q_2(\rho) & q_3(\rho) \\ q_2(\rho) & q_4(\rho) & q_2(\rho) \\ q_3(\rho) & q_2(\rho) & q_1(\rho) \end{pmatrix} .$$

We consider matrices $\{p'_{ab}; a, b=1, 2, 3\}$ generated from Q by repeating the following operation;

(01) interchanging two rows,
(02) interchanging two columns ,
(03) interchanging two rows and then two columns .

All of the matrices $\{p'_{ab}\}$ generated satisfy the constraint $p'_{a\cdot}=p'_{\cdot b}=1/3$. Since $q_1(\rho)=q_3(-\rho)$, $q_2(\rho)=q_2(-\rho)$ and $q_4(\rho)=q_4(-\rho)$, such matrices for all $-1<\rho<1$ may be classified into the following 9 types: Putting $q_i=q_i(\rho)$,

Type 1.	$(q_1, q_2, q_3; q_2, q_4, q_2; q_3, q_2, q_1)$,	$-1<\rho<1$,
Type 2.	$(q_1, q_2, q_3; q_3, q_2, q_1; q_2, q_4, q_2)$,	$-1<\rho<1$,
Type 3.	$(q_2, q_4, q_2; q_1, q_2, q_3; q_3, q_2, q_1)$,	$-1<\rho<1$,
Type 4.	$(q_1, q_3, q_2; q_2, q_2, q_4; q_3, q_1, q_2)$,	$-1<\rho<1$,
Type 5.	$(q_2, q_1, q_3; q_4, q_2, q_2; q_2, q_3, q_1)$,	$-1<\rho<1$,

Type 6.	$(q_1, q_3, q_2; q_3, q_1, q_2; q_2, q_2, q_4)$,	$-1<\rho<1$,
Type 7.	$(q_4, q_2, q_2; q_2, q_1, q_3; q_2, q_3, q_1)$,	$-1<\rho<1$,
Type 8.	$(q_2, q_1, q_3; q_2, q_3, q_1; q_4, q_2, q_2)$,	$-1<\rho<1$,
Type 9.	$(q_2, q_2, q_4; q_1, q_3, q_2; q_3, q_1, q_2)$,	$-1<\rho<1$.

Here the entries in the parentheses correspond to $p_{11}, p_{12}, p_{13}; p_{21}, p_{22}, p_{23}; p_{31}, p_{32}, p_{33}$.

Note that the 9 types of $\{p'_{ab}\}$ with $\rho=-0.9(0.1)0.9$ generate altogether 9×19 sets of $\{p'_{ab}\}$. Next we select $\{\delta_{ab}\}$. The $e_P(S, T)$ depends only on the marginals, i.e., $(\delta_{1\cdot}, \delta_{2\cdot}, \delta_{3\cdot}; \delta_{\cdot 1}, \delta_{\cdot 2}, \delta_{\cdot 3})$. We specified the following 20 types of $(\delta_{1\cdot}, \delta_{2\cdot}, \delta_{3\cdot}; \delta_{\cdot 1}, \delta_{\cdot 2}, \delta_{\cdot 3})$ in the calculation.

(1)	$(0, -1, 1; 0, 0, 0)$,	(11)	$(-1, -1, 2; 0, -1, 1)$,
(2)	$(0, -1, 1; 0, -1, 1)$,	(12)	$(-1, -1, 2; -1, 0, 1)$,
(3)	$(-1, 0, 1; 0, 0, 0)$,	(13)	$(-1, -1, 2; -1, 1, 0)$,
(4)	$(-1, 0, 1; 0, -1, 1)$,	(14)	$(-1, -1, 2; -1, -1, 2)$,
(5)	$(-1, 0, 1; -1, 0, 1)$,	(15)	$(-2, 1, 1; 0, 0, 0)$,
(6)	$(-1, 1, 0; 0, 0, 0)$,	(16)	$(-2, 1, 1; 0, -1, 1)$,
(7)	$(-1, 1, 0; 0, -1, 1)$,	(17)	$(-2, 1, 1; -1, 0, 1)$,
(8)	$(-1, 1, 0; -1, 0, 1)$,	(18)	$(-2, 1, 1; -1, 1, 0)$,
(9)	$(-1, 1, 0; -1, 1, 0)$,	(19)	$(-2, 1, 1; -1, -1, 2)$,
(10)	$(-1, -1, 2; 0, 0, 0)$,	(20)	$(-2, 1, 1; -2, 1, 1)$.

Thus, altogether $9\times 19\times 20=3420$ sets of $\{p_{ab}^{(n)}; a, b=1, 2, 3\}$ are generated. It was found that among these 3420 sets, 540 sets led to $e_P(S, T)=1$ and 1432 sets led to $e_P(S, T)>1$. Table 4 summarizes the values of $e_P(S, T)$. The table shows that the T-test competes well with the S-test. We found by calculation that when the sample size is large enough, the S-test satisfies the condition (2.3) for all set of $\{p'_{ab}\}$ generated.

Table 4. The distribution of $e_P(S, T)$.

$e_P(S, T)$	0.75–0.85	0.85–0.95	0.95–1.05	1.05–1.15	1.15–1.25	Total
Frequency	6	142	3131	134	7	3420

4.2 *Comparison of the T-test and R-test*

We next compare the T-test and R-test. Under the sequence of the alternative hypothesis H_{1n} described in the last section it follows that the random vector (T_1, T_2) converges in law to (U_1, U_2) which is distributed as a bivariate normal distribution $N_2(\tau_1, \tau_2, 1, 1, \rho)$ where

$$\tau_1=\sum_{l=1}^{r}\sum_{a=1}^{l}\left(\frac{w_{1l}}{Q_1}\right)^{1/2}\delta_{a\cdot}\ ,$$

$$\tau_2 = \sum_{m=1}^{c}\sum_{b=1}^{m}\left(\frac{w_{2m}}{Q_2}\right)^{1/2}\delta_{.b} ,$$

and ρ is given in (2.4). From this the asymptotic power of the T-test, asymptotically with level α, is given by

$$1 - \Phi\left(u_\alpha - \frac{\tau_1 + \tau_2}{\sqrt{2(1+\rho)}}\right) .$$

On the other hand, following Bartholomew (1961) and Chatterjee and De (1972), the corresponding asymptotic power of the R-test is given by

$$\begin{aligned}&[1 - \Phi(c_\alpha - \lambda\cos\xi)]\Phi(-\lambda\sin\xi)\\&+[1 - \Phi(c_\alpha - \lambda\cos(\psi - \xi))]\Phi(-\lambda\sin(\psi-\xi))\\&+\frac{1}{2\pi}\int_0^{\psi-\xi}\int_{c_\alpha}^{\infty}\exp\left\{-\frac{1}{2}(r^2+\lambda^2-2r\lambda\cos\theta)\right\}drd\theta ,\end{aligned}$$

where

$$\begin{aligned}&\lambda = (1-\rho^2)^{-1/2}(\tau_1^2+\tau_2^2-2\rho\tau_1\tau_2)^{1/2} ,\\&\psi = \cos^{-1}(-\rho) ,\\&\xi = \cos^{-1}[(\tau_1 - \rho\tau_2)/(\tau_1^2+\tau_2^2-2\rho\tau_1\tau_2)^{1/2}] .\end{aligned}$$

Fixing α, ρ and λ, and denoting the asymptotic powers of the T-test and R-test by $\beta_T(\xi)$ and $\beta_R(\xi)$, we consider the powers as functions of ξ. It is straightforward to see that both $\beta_T(\xi)$ and $\beta_R(\xi)$ attain their maximum values at $\xi=\psi/2$, symmetrically decrease as $|\xi-\psi/2|$ increases and attain their minimum values at $\xi=0$ or ψ. Note that $\xi=0$, $\psi/2$, ψ correspond to $\tau_2=0$, $\tau_1=\tau_2$ and $\tau_1=0$ respectively, and that the vector $\boldsymbol{\theta}^1=c\Sigma\boldsymbol{d}\,(c>0)$, to which the T-test is a uniformly most powerful in an asymptotic sense, implies $\tau_1=\tau_2$. We studied the behavior of $\beta_T(\xi)$ and $\beta_R(\xi)$ for selected values of α, λ and ρ. Figure 2 illustrates the case when $\alpha=0.05$; $\lambda=2$; $\rho=-0.5, 0.0, 0.5$. The figure shows that the T-test is superior to R-test around $\tau_1=\tau_2$, but inferior around $\tau_1=0$ or $\tau_2=0$. Incidentally, this reinforces the finding in the example in the previous section. The figure also shows that the superiority of the T-test around $\tau_1=\tau_2$ increases as the value of ρ decreases; that the power of the R-test is fairly stable for various directions of the alternatives. These findings are unalterd for different values of λ and α.

5. Concluding remarks

In this paper we have discussed the problem of testing $p_{a.}=p_{a.}^{\circ}$, $p_{.b}=p_{.b}^{\circ}$

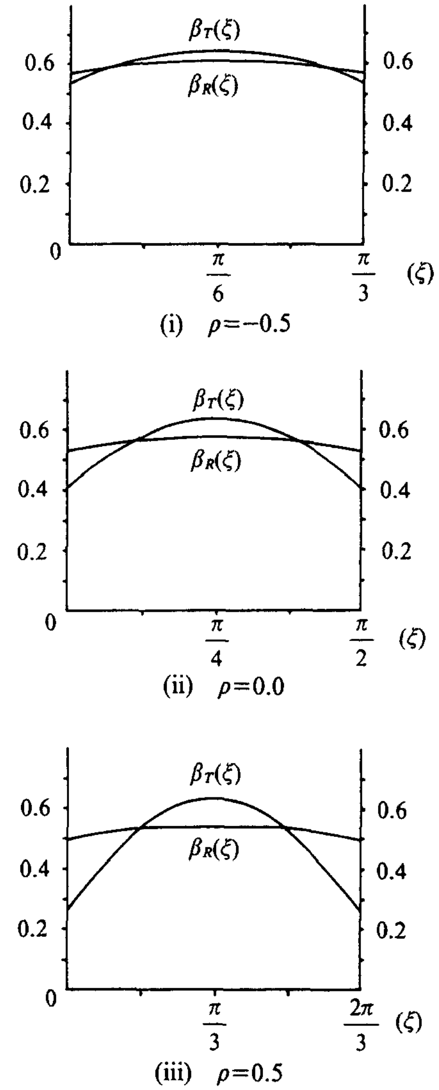

Fig. 2. Asymptotic powers $\beta_T(\xi)$ and $\beta_R(\xi)$ for $\alpha=0.05$; $\lambda=2$; $\rho=-0.5, 0.0, 0.5$.

for $a=1,\ldots, r$, $b=1,\ldots, c$ against alternatives $\sum_{a=1}^{l} p_{a\cdot} \leq \sum_{a=1}^{l} p^{\circ}_{a\cdot}$, $\sum_{b=1}^{m} p_{\cdot b} \leq \sum_{b=1}^{m} p^{\circ}_{\cdot b}$, for $l=1,\ldots, r$, $m=1,\ldots,c$, with at least one inequality strict. The problem is not only interesting by itself as a testing hypothesis in a contingency table, but also it has been shown in this paper that the alternative hypothesis is related to the one-sided alternative in a comparative study under a bivariate nonparametric formulation.

We have considered the three tests, S-test, T-test and R-test. The S-test is an approximately most stringent somewhere most powerful test. The T-test and R-test combine approximately most stringent somewhere most powerful tests obtained from each marginal of the contingency table. Whereas the T-test simply adds, R-test employs the likelihood ratio criterion for the combination.

The alternative hypothesis is composite with restriction and it is difficult to compare the three tests in general. We have considered the restricted family of alternative hypothesis which is generated by a bivariate normal distribution for the comparison of the S-test and T-test. Also we have directly compared the asymptotic powers of the T-test and R-test. Under these setups it has been shown that the three tests are competitive regarding their asymptotic powers, in particular

i) T-test is highly competitive with the S-test.

ii) T-test is superior to the R-test around $E(T_1)=E(T_2)$, but inferior to the R-test around $E(T_1)=0$ or $E(T_2)=0$.

iii) The superiority of the T-test around $E(T_1)=E(T_2)$ increases as the correlation of T_1 and T_2 decreases.

iv) The power of the R-test is fairly stable for various directions of the alternatives.

We could not compare the powers of the S-test and T-test directly because of the involvement of too many parameters.

The usefulness of the tests has been shown by the practical data from a case-control study. It has been shown that the T-test has smaller p-values than the R-test in the region around the straight line, $T_1=T_2$.

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REFERENCES

Bartholomew, D. J. (1961). A test of homogeneity of means under restricted alternatives (with discussion), *J. Roy. Statist. Soc. Ser. B*, **23**, 239–281.

Chatterjee, S. K. and De, N. K. (1972). Bivariate nonparametric location tests against restricted alternatives, *Calcutta Statist. Assoc. Bull.*, **21**, 1–20.

Kudô, A. (1963). A multivariate analogue of the one-sided test, *Biometrika*, **50**, 403–418.

Mitra, S. K. (1958). On the limiting power function of the frequency chi-square test, *Ann. Math. Statist.*, **29**, 1221–1233.

Rao, C. R. (1973). *Linear Statistical Inference and Its Applications*, 2nd ed., Wiley, New York.

Schaafsma, W. (1966). *Hypothesis Testing Problems with the Alternative Restricted by a Number of Inequalities*, Noordhoff, Groningen.

Schaafsma, W. and Smid, L. J. (1966). Most stringent somewhere most powerful tests against alternatives restricted by a number of linear inequalities, *Ann. Math. Statist.*, **37**, 1161–1172.