TESTS OF COMPOSITE HYPOTHESES FOR THE MULTIVARIATE EXPONENTIAL FAMILY[1]

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0. Summary. This paper is concerned with testing the hypothesis that the parameter in a multivariate exponential distribution lies in a linear subspace of the natural parameter space. Our main result characterizes a complete class of tests which is independent of the particular exponential distribution. This class is, in fact, complete relative to the stronger ordering among tests which compares conditional power, given a certain statistic, pointwise. The conclusion holds without any restriction on the exponential distribution. Many of the tests are admissible, but examples show that although the class is essentially the smallest class complete relative to all exponential distributions, it is not in general minimally complete. Some special cases where the class is minimally complete are discussed.

1. Introduction. Suppose that $X_1, X_2, \cdots, X_n$ is a random sample of k-dimensional vectors each having the distribution

$$P_\theta(A) = c(\theta) \int_A e^{\theta x} \lambda(dx) \tag{1.1}$$

where λ is a finite measure on R^k, $\theta = (\theta_1, \theta_2, \cdots, \theta_k)$, $\theta x = \sum_{i=1}^k \theta_i x_i$. As is well known, $\sum_{i=1}^n X_i$ is a sufficient statistic and also has an exponential distribution. Therefore, for testing hypotheses concerning θ, it will be sufficient to consider only samples of size one. Let Ω denote the "natural" parameter space: $\Omega = \{\theta \varepsilon R^k: \int e^{\theta x}\lambda(dx) < \infty\}$. Ω is clearly convex. Let Ω_0 be an r-dimensional linear subspace $(r < k)$ of R^k. We will be concerned with tests of the null hypothesis $\theta \varepsilon \Omega_0 \cap \Omega$. Notice that if T is any linear transformation on R^k, TX again has an exponential distribution. It is clear that by choosing T to be an appropriate orthogonal transformation the problem can be put into the canonical form

$$H_0 : \theta_{r+1} = \theta_{r+2} = \cdots = \theta_k = 0.$$

Note that the linear transformation may change λ and hence $c(\theta)$. In order to simplify notation we will write the sample space as $\mathbf{X} \times \mathbf{Y}$, where $\mathbf{X} = R^r$, $\mathbf{Y} = R^{k-r}$, and a sample point as (x, y). Similarly the parameter point will be expressed as the pair (θ, ω) where θ is an r-vector, and ω is a $(k - r)$-vector. Thus, the hypothesis, in canonical form, is $H_0 : \omega = 0$. Throughout this paper it will be assumed that the origin is an interior point of Ω. Furthermore, there is no loss in generality in assuming that λ is normalized to be a probability measure on the Borel sets of R^k.

The case $r = 0$—the case of no nuisance parameters—was first investigated

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by A. Birnbaum [1]. He showed that the class of tests which accept the hypothesis when Y is in some convex set in $\mathbf{Y}$ is a complete class, although his statement involved the restriction that the probability distributions be absolutely continuous. Birnbaum's proof, however, contains an essential error which was corrected by J. Sacks [8] in an unpublished paper. Since a corrected proof has not appeared in print we give in Section 2 a short proof which also extends the theorem to the case of general measures λ.

The case $r = k - 1$, that is, when the parameter ω is real, has been studied extensively by Lehmann [6] who finds the complete class of tests when the alternative is one sided. Completeness in the two sided case appears to be untreated.

In Section 3 the main result characterizes a complete class of tests of $\omega = 0$ as those tests whose acceptance regions have convex x-sections. The proof of this result utilizes the Birnbaum theorem of Section 2, and in fact the theorem is clearly motivated by the Birnbaum theorem. If one denotes by μ_x the conditional measure on $\mathbf{Y}$ determined by λ for a fixed $x \in \mathbf{X}$, then given a test ψ one can find, by Birnbaum's theorem, a test $\phi_x(y)$ with convex acceptance region which conditionally dominates ψ. The crux of the matter is the fact that considered as a function on $\mathbf{X} \times \mathbf{Y}$, ϕ may not be jointly measurable. In fact, due to the non-uniqueness of the dominating test for the simple hypothesis it is easy to construct, in this manner, many non-measurable ϕ which dominate ψ for each fixed x. Our construction in Section 3 is essentially to insure that we end up with a measurable test.

Another possible approach to proving the main theorem would seem to be what was essentially Birnbaum's approach. Under Wald's topology of regular convergence which in our case is equivalent to weak * convergence of functions in L_∞, the closure of the class $\mathcal{B}$ of Bayes tests is essentially complete. It is easily established that the Bayes tests have acceptance regions with convex x-sections (henceforth the class of such tests will be denoted by $\Phi_{\mathfrak{D}}$), i.e. $\mathcal{B} \subset \Phi_{\mathfrak{D}}$. Even though $\mathcal{B}$ is extremely difficult to characterize, it would be enough to show, as Birnbaum does in the simple hypothesis case that $\bar{\Phi}_{\mathfrak{D}} = \Phi_{\mathfrak{D}}$, where the closure is in the sense of weak * convergence. This, in fact, is not true if $r > 0$, and it is quite easy to produce a sequence in $\Phi_{\mathfrak{D}}$ whose weak * limit is completely randomized. We hope that this discussion will provide some sort of justification for the somewhat technical nature of our proof, and the measure theoretical difficulties which must be circumvented.

Some results on admissibility of tests are given in Section 4. Admissibility of tests with convex acceptance regions has been well studied and established in certain circumstances. Birnbaum was the first to give sufficient conditions, using a technique generalized by Stein [9] in order to prove admissibility of Hotelling's T^2 test. A Bayes approach was used by Kiefer and Schwartz [5] in order to show that many of the classical tests used in multivariate analysis are admissible. The latter paper also contains an extensive bibliography pertaining to admissibility. Actually, the tests Kiefer and Schwartz consider appear to have, like the T^2 test, convex acceptance regions, and not merely acceptance regions with convex x-sections. The difficulties inherent to admissibility are great in the multivariate case, and it is not surprising that our knowledge of admissibility of tests in the larger class $\Phi_{\mathfrak{D}}$ is unsatisfactory. From a more practical point of view, undue emphasis on the admissibility question is unrewarding inasmuch as one is not concerned with the power of a test at all alternatives, but only at certain more important points. In this connection we feel that likelihood ratio tests (which do indeed belong to $\Phi_{\mathfrak{D}}$) for multivariate exponential problems should possess desirable properties in general, as demonstrated by Hoeffding [4] in the multinomial case.

Finally, we have relegated the proof oi a crucial but technical Main Lemma 3.2 to an appendix.

2. Testing a simple hypothesis. Before taking up the main result it is necessary to consider the case of testing a simple hypothesis. We observe a random vector Y in R^k and suppose Y has density

$$p_\omega(y) = c(\omega)e^{\omega' y}$$

with respect to a dominating measure μ in $\mathbf{Y}$. The hypothesis to be tested is $\omega = 0$, the alternative $\omega \neq 0$.

Let $\mathfrak{C}$ denote the class of closed convex sets in $\mathbf{Y}$. A test ϕ is said to have a convex acceptance region if for some $C \in \mathfrak{C}$,

$$\begin{aligned} \phi(y) &= 0, && y \in \operatorname{int} C, \\ &= \gamma(y), && y \in \partial C, \\ &= 1, && y \in C'. \end{aligned}$$

Here, $0 \leqq \gamma \leqq 1$ is a (measurable) randomization, ∂C denotes the boundary of C, and the prime on a set denotes complement. Let us denote the class of all tests with convex acceptance regions by $\Phi_{\mathfrak{C}}$.

It is shown in [1] that every Bayes test belongs to $\Phi_{\mathfrak{C}}$, so to prove that $\Phi_{\mathfrak{C}}$ is essentially complete it is enough to prove that $\Phi_{\mathfrak{C}}$ is closed in the topology of weak $*$ convergence, i.e., if $\phi_n \in \Phi_{\mathfrak{C}}$ and $\lim_{n\to\infty} \int \phi_n f\, d\mu = \int \phi f\, d\mu$ for all integrable f, then $\phi \in \Phi_{\mathfrak{C}}$. This will be shown in Theorem 2.1 with the aid of the Blaschke selection theorem.

If S_r denotes the solid sphere of radius r in R^k, and A and B are closed subsets of S_r, the Hausdorff distance between A and B is

$$d(A, B) = \inf \{\epsilon : A \subset N_\epsilon(B), B \subset N_\epsilon(A)\}$$

where $N_\epsilon(A)$ denotes the ϵ-neighborhood of A. The class of all closed subsets of S_r becomes a metric space with this distance function.

LEMMA 2.1. (Blaschke Selection Theorem, [11].) *Given any sequence of closed convex subsets of S_r, say $\{C_n\}$, there exists a subsequence $\{C_{n_k}\}$ and a closed convex set $C \subset S_r$ such that* $\lim_{k\to\infty} d(C_{n_k}, C) = 0$.

THEOREM 2.1. *For any σ-finite measure μ, $\Phi_{\mathfrak{C}}$ is closed in the weak $*$ topology.*

PROOF. Let $\phi \varepsilon \bar{\Phi}_{\mathfrak{C}}$. There exists a sequence of tests $\phi_n \varepsilon \Phi_{\mathfrak{C}}$ such that $\lim_{n\to\infty} \phi_n =^* \phi$. Each ϕ_n is associated with a set $C_n \varepsilon \mathfrak{C}$.

We outline an extension of Lemma 2.1 that gives a subsequence n_i and a closed convex set C such that for all sufficiently large r $d(C_{n_i} \cap S_r, C \cap S_r) \to 0$ as $i \to \infty$. Two cases may arise. If for every positive r, $C_n \cap S_r$ is non-empty for only finitely many n, we take C to be empty. Otherwise, there is some r such that $C_n \cap S_r$ is non-empty for infinitely many n. We may as well assume $C_n \cap S_r$ is non-empty for all n. Lemma 2.1 gives a subsequence n_{1i} and a closed non-empty convex set $C^1 \subset S_r$ for which $d(C_{n_{1i}} \cap S_r, C^1) \to 0$ as $i \to \infty$. There is a further subsequence n_{2i} and a closed non-empty convex set $C^2 \subset S_{r+1}$ such that $d(C_{n_{2i}} \subset S_r, C^2) \to 0$. It is not difficult to show that $C^2 \cap S_r = C^1$. In fact, this would follow from Theorem 31 of [3] in case C^2 meets S_r at an interior point of S_r. This restriction is not needed in the present case, however. Proceeding inductively, one obtains further subsequences, and finally a diagonal sequence n_i and closed non-empty convex sets C^p such that $C^p \cap S_q = C^q$ whenever $p \geqq q \geqq r$, and $d(C_{n_i} \cap S_p, C^p) \to 0$ as $i \to \infty$. Define $C = \bigcup_{p=r}^{\infty} C^p$. It is seen easily that C is closed and convex and that for all $p \geqq r$, $d(C_{n_i} \cap S_p, C \cap S_p) \to 0$.

The following argument applies when C is non-empty. Otherwise, if for each r, $C_n \cap S_r$ is empty for all but finitely many n, the result below gives $\phi = 1$ $[\mu]$, which is an element of $\Phi_{\mathfrak{C}}$.

Let A be a compact set disjoint from C. Then there exists an $\epsilon > 0$ such that $N_\epsilon(A)$ is disjoint from C. Choose r such that $N_\epsilon(A) \subset S_r$. Then if

$$d(C_{n_i} \cap S_r, C \cap S_r) < \epsilon,$$

it follows that C_{n_i} and A are disjoint, or $A \subset C'_{n_i}$. Hence,

$$\int_A \phi f \, d\mu = \lim_{i\to\infty} \int_A \phi_{n_i} f \, d\mu = \int_A f \, d\mu$$

for all integrable f because $\phi_{n_i} \to^* \phi$ and $\phi_{n_i} = 1$ on C'_{n_i}. Therefore, ϕ must be one on A $[\mu]$. But C' can be covered by a denumerable number of compact sets A, so $\phi = 1$ on C' $[\mu]$. A similar argument shows $\phi = 0$ on int C $[\mu]$. Thus, ϕ is equal almost everywhere to an element of $\Phi_{\mathfrak{C}}$.

3. Testing composite hypotheses. We return now to the original problem of testing $H_0 : \omega = 0$ where θ is unknown. A test $0 \leqq \phi \leqq 1$ is now a measurable function on $\mathbf{X} \times \mathbf{Y}$. The test ϕ is said to have convex acceptance sections if there exists a measurable set $C \subset \mathbf{X} \times \mathbf{Y}$, each of whose x-sections are closed and convex in $\mathbf{Y}$, and

$$\phi(x, y) = 0, \quad y \varepsilon \operatorname{int} C(x), \qquad = 1, \quad y \varepsilon C'(x). \tag{3.1}$$

On the boundaries of $C(x)$ ϕ may be randomized. The family of all tests with convex acceptance sections will be denoted by $\Phi_{\mathfrak{D}}$.

Our main theorem below states that $\Phi_{\mathfrak{D}}$ is a complete class of tests. Moreover, it is shown that $\Phi_{\mathfrak{D}}$ is a complete class relative to the ordering which at each x

compares pointwise the conditional power $E_\omega\{\phi(X, Y) \mid X = x\}$. The obvious methods of attack which one might consider to prove completeness were discussed in the introduction and are inadequate to handle the measure theoretical difficulties.

In order to circumvent these difficulties we first discretize $\mathbf{X}$ in such a way that only a denumerable number of sections need be considered. The Birnbaum theorem is applied to each of these sections to obtain a test in $\Phi_{\mathfrak{C}}$ which dominates a given test ψ. The piecing together of all these conditional tests then presents no difficulty. Thus, relative to the nth discretization there is a test in $\Phi_{\mathfrak{D}}$ which dominates the discretized version of ψ. Now, a further difficulty arises in extracting, in some sense, a limit point of these $C_n(x)$. It is inadequate to consider, for example, weak * convergence in $\mathbf{X}$ of a denumerable number of hyperplanes characterizing $C_n(x)$ because in general this kind of convergence does not yield convergence of the power function, which is what we want. We are instead forced to consider a rather unusual limit point of $\{C_n(x)\}$ in Lemma 3.2. This lemma is a non-trivial generalization of the Blaschke theorem to the case of a sequence of convex sets which depend measurably on an indexing variable x.

As a preliminary some notation is introduced relating to the discretization. As we mentioned in the introduction the dominating measure λ on $\mathbf{X} \times \mathbf{Y}$ may be taken to be a probability distribution. The marginal distribution of X and the conditional distribution of Y given $X = x$ determined by λ will be denoted by ν and $\mu(dy; x)$ respectively. Thus, for any $A \subset \mathbf{X} \times \mathbf{Y}$,

$$\lambda(A) = \int_{\mathbf{X}} \mu(A(x); x)\, d\nu(x),$$

where $A(x)$ is the section of A at x.

When (X, Y) has density (1.1), the marginal density of X with respect to ν is easily seen to be

$$p_{\theta,\omega}(x) = c(\theta, \omega)e^{\theta x} \int_{\mathbf{Y}} e^{\omega y}\mu(dy; x);$$

the conditional density of Y given $X = x$, with respect to $\mu(dy; x)$ is

$$p_\omega(y \mid x) = e^{\omega y}/\int_{\mathbf{Y}} e^{\omega u}\mu(du; x). \tag{3.2}$$

Consider any partition of $\mathbf{X}$ into a finite or denumerable number of Borel sets, and let $\mathcal{B}$ denote the σ-field which such a partition generates. In fact, we shall need a sequence of partitions which are successive refinements, with associated σ-fields $\mathcal{B}_n$, $n = 1, 2, \cdots$, such that the smallest σ-field $\mathcal{B}_\infty$ containing them all is the class of Borel sets of $\mathbf{X}$.

Associated with $\mathcal{B}_n$ is the σ-field in $\mathbf{X} \times \mathbf{Y}$

$$\mathfrak{F}_n = \{(B \times \mathbf{Y}) : B \in \mathcal{B}_n\}, \qquad 1 \leqq n \leqq \infty.$$

In terms of $\mathfrak{F}_n$, define the conditional discretized probability measures $\mu(dy; x, n)$ in $\mathbf{Y}$:

$$\begin{aligned} \mu(C; x, n) &= \lambda(\mathbf{X} \times C \mid \mathfrak{F}_n) \\ &= \lambda(B \times C)/\nu(B) \end{aligned} \tag{3.3}$$

if x is in an atom B of $\mathcal{B}_n$, C a Borel set in $\mathbf{Y}$, $\nu(B) > 0$. If $\nu(B) = 0$ define $\mu(\cdot\,; x, n)$ arbitrarily.

In the next lemma E_0 denotes expectation when $(\theta, \omega) = 0$, i.e., with respect to λ.

LEMMA 3.1. *Let ψ be any test and define ψ_n by*

$$\psi_n = E_0\{\psi \mid \mathfrak{F}_n, Y\}. \tag{3.4}$$

Then, whenever $(0, \omega) \varepsilon \Omega$,

$$\lim_{n\to\infty} \int e^{\omega y}\psi_n(x, y)\mu(dy; x, n) = \int e^{\omega y}\psi(x, y)\mu(dy; x) \ [\nu]. \tag{3.5}$$

PROOF. Let f be any function in $\mathbf{X} \times \mathbf{Y}$, integrable with respect to λ. The sequence $E_0(f \mid \mathfrak{F}_n) : 1 \leqq n \leqq \infty$ is a martingale. From the martingale convergence theorem one obtains

$$\lim_{n\to\infty} E_0(f \mid \mathfrak{F}_n) = E_0(f \mid \mathfrak{F}_\infty) \ [\nu].$$

If, in addition, f is measurable with respect to $\mathfrak{F}_n$ and Y an easy calculation with definition (3.3) shows that $E_0(f \mid \mathfrak{F}_n) = \int f(x, y)\mu(dy; x, n)$ if x is in an atom B of $\mathcal{B}_n$. By assumption the function whose value is $e^{\omega y}\psi(x, y)$ is integrable. Consequently,

$$\begin{aligned}\int e^{\omega y}\psi_n(x, y)\mu(dy; x, n) &= E_0[e^{\omega Y}\psi_n(X, Y) \mid \mathfrak{F}_n] \\ &= E_0\{E_0[e^{\omega Y}\psi_n(X, Y) \mid \mathfrak{F}_n, Y) \mid \mathfrak{F}_n\} \\ &= E_0\{e^{\omega Y}\psi(X, Y) \mid \mathfrak{F}_n\} \\ &\to E_0\{e^{\omega Y}\psi(X, Y) \mid \mathfrak{F}_\infty\} \ [\nu] \\ &= \int e^{\omega y}\psi(x, y)\mu(dy; x).\end{aligned}$$

This convergence is (3.5) and proves the lemma.

We can now state the principal result. Set $\Omega_1 = \{\omega: \int e^{\omega y}\lambda(dy) < \infty\}$.

THEOREM 3.1. *Let ψ be any test of $H_0 : \omega = 0$. Then there exists a test $\phi \varepsilon \Phi_{\mathfrak{D}}$ with the property that for each $\omega \varepsilon \Omega_1$*

$$\int \phi(x, y)e^{\omega y}\mu(dy; x) \geqq \int \psi(x, y)e^{\omega y}\mu(dy; x) \ [\nu], \tag{3.6}$$

with equality in case $\omega = 0$.

Observe that the integrals in (3.6) are the conditional powers of the two tests up to the common factor $[\int e^{\omega y}\mu(dy; x)]^{-1}$, as the conditional density (3.2) shows. It follows immediately from (3.6) that ϕ is at least as good as ψ in the usual sense, implying that $\Phi_{\mathfrak{D}}$ is complete.

The proof of this theorem, which is essentially constructive, is given in several steps.

1°. *Application of the Birnbaum Theorem.* Let ψ be an arbitrary test and define ψ_n as in (3.4) for each n. Relative to each one of the measures $\mu(dy; x, n)$ the functions $e^{\omega y}$ determine an exponential family of distributions as ω ranges in Ω_1.

For each $x \varepsilon \mathbf{X}$ and each n, application of the generalized Birnbaum theorem yields a conditional test $\phi_n(x, \cdot)$ which accepts H_0 on a closed convex set and satisfies

$$\int e^{\omega y}\phi_n(x, y)\mu(dy; x, n) \geqq \int e^{\omega y}\psi_n(x, y)\mu(dy; x, n), \qquad \omega \varepsilon \Omega_1, \tag{3.7}$$

with equality when $\omega = 0$. More precisely,

$$\begin{aligned}\phi_n(x, y) &= 0, \qquad y \varepsilon \operatorname{int} C_n(x), \\ &= 1, \qquad y \varepsilon C_n'(x),\end{aligned} \tag{3.8}$$

with randomization possible on the boundary of $C_n(x)$. Since ψ_n is measurable with respect to the σ-field generated by $\mathfrak{F}_n$ and Y, and the measure $\mu(\cdot; x, n)$ depends only on that atom of $\mathfrak{B}_n$ containing x, it is clear that we may take $C_n(x)$ to be the same for each x in an atom of $\mathfrak{B}_n$. Thus, at most a denumerable number of sections are involved, and the resulting ϕ_n is a test.

2°. (MAIN) LEMMA 3.2. *There exists a Borel set $C \subset \mathbf{X} \times \mathbf{Y}$, each of whose sections $C(x)$ is closed and convex in* $\mathbf{Y}$, *satisfying* (i) *for each x, there is a subsequence $\{n_i(x)\}_{i=1}^{\infty}$ for which*

$$\lim_{i\to\infty} d(C_{n_i(x)}(x) \cap S_r, C(x) \cap S_r) = 0, \tag{3.9}$$

for r sufficiently large; and (ii) *$n_i(x)$ is Borel measurable for each i.*

There is no difficulty obtaining a limit point of $\{C_n(x)\}$ for each x. Generally, there may exist many limit points. The essential content of this lemma is the measurability of C. Regarding the subsequence $n_i(x)$, the fact that no fixed subsequence exists, independent of x, suggests that the lemma is non-trivial. The proof of Lemma 3.2 is postponed to an appendix in view of its technicality.

We remark that the proof of Theorem 3.1 could be briefly concluded were it assumed that λ is absolutely continuous, or merely that the measures $\mu(\cdot; x)$ are continuous on boundaries of convex sets in $\mathbf{Y}$. In that case all tests could be non-randomized. Lemma 3.1 together with an argument as in [7] would show that the test ϕ which accepts H_0 on C would dominate ψ.

Continuing with the proof in the general case, it is clear from the construction that the function $d(C_n(x) \cap S_r, C_m(x) \cap S_r)$ is measurable for each m and r. The same is true of $d(C_m(x) \cap S_r, C_{n_i(x)}(x) \cap S_r)$ and $d(C_m(x) \cap S_r, C(x) \cap S_r)$, by Lemma 3.2. Define the functions

$$g_n(x, \omega) = \int e^{\omega y}\mu(dy; x, n) \qquad g(x, \omega) = \int e^{\omega y}\mu(dy; x). \tag{3.10}$$

It follows from Lemma 3.1 that for $\omega \varepsilon \Omega_1$,

$$\lim_{n\to\infty} g_n(x) = g(x) \; [\nu]. \tag{3.11}$$

3°. *Choice of a measurable subsequence.* We now redefine the subsequence in Lemma 3.2 as follows, although using the same notation for it for economy's sake. First of all, let $\Omega_0 = \{\omega_i\}_{i=1}^{\infty}$ be dense in Ω_1, and containing $\omega = 0$. For

$i = 1, 2, \cdots$ we define $n_i(x)$ to be the first integer $n \geqq i$ for which[2]

$$(3.12) \quad \text{(i)} \quad \sup_{m \geqq n} |g_m(x, \omega) - g(x, \omega)| \leqq 1, \quad \text{for } \omega = \omega_1, \omega_2, \cdots, \omega_i;$$

$$\text{(ii)} \quad d(C_n(x) \cap S_i, C(x) \cap S_i) \leqq 1/i.$$

In view of its countable description and earlier remarks, $n_i(x)$ here defined is still measurable.

Define the measurable functions

$$f_i(x, \omega) = \int e^{\omega y} \mu(dy; x, n_i(x)), \qquad \omega \varepsilon \Omega_1 .$$

LEMMA 3.3. *For each* $\omega \varepsilon \Omega_0$, $\lim_{i \to \infty} f_i(x, \omega) = g(x, \omega)$ $[\nu]$, *and the convergence is dominated.*

PROOF. Convergence, pointwise, follows from (3.11). Domination follows from (3.12i) in the definition of $n_i(x)$, in view of

$$f_i(x, \omega) = g_{n_i(x)}(x, \omega) \leqq 1 + g(x, \omega)$$

for sufficiently large i, and the fact that the dominating function is integrable.

4°. *Denumerable approximating convex sections.* In order to capture the effect of randomization, consider a denumerable subclass $\mathcal{K}$ of $\mathcal{C}$, and dense in the class of all bounded convex sets (in the sense of the Hausdorff metric). Our aim is to approximate $C_{n_i(x)}(x)$ closely from within and without by a member of $\mathcal{K}$. Let $\mathcal{K}$ be ordered in some way. Then, define $K_{1i}(x)$ and $K_{2i}(x)$ to be the first elements K_1 and K_2, respectively, in $\mathcal{K}$, for which

$$(3.13) \qquad N_{2/i}(K_1) \subset C_{n_i(x)}(x) \cap S_i \subset N_{3/i}(K_1),$$

$$N_{2/i}(C_{n_i(x)}(x) \cap S_i) \subset K_2 \subset N_{3/i}(C_{n_i(x)}(x) \cap S_i).$$

If there is no K_1 with the required property, define $K_{1i}(x)$ to be the empty set. Because $n_i(x)$ is measurable it is clear that the set of x for which $K_{1i}(x)$ equals a particular set in $\mathcal{K}$ is measurable, and likewise for $K_{2i}(x)$. Also, it is easily verified that the set in $\mathbf{X} \times \mathbf{Y}$, whose x-section is $K_{1i}(x)$ (or $K_{2i}(x)$) is measurable. The neighborhoods in (3.13) are chosen so that $K_{1i}(x)$ and $K_{2i}(x)$ approximate $C(x)$ from within and without:

$$(3.14) \qquad N_{1/i}(K_1) \subset C(x) \cap S_i \subset N_{4/i}(K_1),$$

$$N_{1/i}(C(x) \cap S_i) \subset K_2 \subset N_{4/i}(C(x) \cap S_i).$$

COROLLARY 3.3.1. *For each* $\omega \varepsilon \omega_0$, *each* $r = 1, 2, \cdots$, *and* $k = 1, 2,$

$$(3.15) \quad \lim_{i \to \infty} \int_{K_{kr}(x)} e^{\omega y} \mu(dy; x, n_i(x)) = \int_{K_{kr}(x)} e^{\omega y} \mu(dy; x) \; [\nu],$$

and the convergence is dominated.

This follows in case of integration over a fixed set K from Lemma 3.1. In the present case (3.15) still holds because $K_{kr}(x)$ is one of at most denumerably many sets from $\mathcal{K}$.

[2] If either $C_n(x) \cap S_i$ or $C(x) \cap S_i$ is empty, let $n_i(x)$ be defined by condition (i). By the construction, these are measurable conditions.

5°. *Evaluation of some limits.* Replacing n by $n_i(x)$ in (3.7) gives

$$(3.16)\quad \int e^{\omega y}\phi_{n_i(x)}(x, y)\mu(dy; x, n_i(x)) \geqq \int e^{\omega y}\psi_{n_i(x)}(x, y)\mu(dy; x; n_i(x))$$

for all $\omega \varepsilon \Omega_1$, with equality when $\omega = 0$. By Lemma 3.1,

$$(3.17)\quad \lim_{i\to\infty} \int e^{\omega y}\psi_{n_i(x)}(x, y)\mu(dy; x, n_i(x)) = \int e^{\omega y}\psi(x, y)\mu(dy; x)\ [\nu].$$

Let us break the left hand integral in (3.16) into the three pieces

$$(3.18)\quad \begin{aligned} I_{1i}(x, \omega, r) &= \int_{K_{1r}(x)} \phi_{n_i(x)}(x, y)e^{\omega y}\mu(dy; x, n_i(x)), \\ I_{2i}(x, \omega, r) &= \int_{K_{2r}'(x)} \phi_{n_i(x)}(x, y)e^{\omega y}\mu(dy; x, n_i(x)), \\ I_{3i}(x, \omega, r) &= \int_{B_r(x)} \phi_{n_i(x)}(x, y)e^{\omega y}\mu(dy; x, n_i(x), \end{aligned}$$

where $B_r(x) = K_{2r}(x)\backslash K_{1r}(x)$.

Considering the first integral in (3.18), if $i > r$ a simple set inclusion argument, taking into account (3.12) and (3.13) shows that if $y \varepsilon K_{1r}(x)$ then $y \varepsilon \operatorname{int} C_{n_i(x)}(x)$, so that $\phi_{n_i(x)}(x, y) = 0$. It follows trivially that

$$(3.19)\quad \lim_{r\to\infty} \lim_{i\to\infty} I_{1i}(x, \omega, r) = 0\ [\nu].$$

Consider briefly the second integral in (3.18). Another application of Lemma 3.1, Corollary 3.3.1, and (3.14) shows in similar fashion

$$(3.20)\quad \lim_{r\to\infty} \lim_{i\to\infty} I_{2i}(x, \omega, r) = \int_{C'(x)} e^{\omega y}\mu(dy; x)\ [\nu].$$

The details are omitted.

6°. *Representation of the randomization.* It remains to consider the limit of the third term in (3.18). For each $\omega \varepsilon \Omega_0$ the functions $\{I_{3i}(\cdot, \omega, r): i = 1, 2, \cdots; r = 1, 2, \cdots\}$ are dominated in $L_1(\mathbf{X}, \nu)$ according to Lemma 3.3. It follows ([2], p. 292) that these functions have a weak sequential limit point. To avoid introducing still another subsequence, let us suppose it has already been extracted so that for each $\omega \varepsilon \Omega_0$,

$$(3.21)\quad \lim_{r\to\infty} \lim_{i\to\infty} I_{3i}(x, \omega, r) = L(x, \omega)$$

in the weak sense.

In order to represent $L(x, \omega)$, consider the functional

$$(3.22)\quad F(f, \omega) = \int_{\mathbf{X}} f(x)L(x, \omega)\nu(dx),$$

where $f \varepsilon L_\infty(\mathbf{X}, \nu)$, $\omega \varepsilon \Omega_0$. Substituting in (3.21),

$$F(f, \omega) = \lim_{r\to\infty} \lim_{i\to\infty} \int_{\mathbf{X}} \int_{B\ (x)} f(x)e^{\omega y}\phi_{n_i(x)}(x, y)\mu(dy; x, n_i(x))\nu(dx).$$

Denote the linear subspace of $L_1(\mathbf{X} \times \mathbf{Y}, \lambda)$ spanned by functions of the form $f(x)e^{\omega y}$ by $\mathfrak{L}$, where f is chosen from $L_\infty(\mathbf{X}, \nu)$ and $\omega \varepsilon \Omega_0$.

Let $\{\beta_\alpha\}$ be a finite set of real numbers associated with the functions $\{f_\alpha(x)\}$ and parameters $\{\omega_\alpha\}$. Because of the linearity of the integral, setting $g(x, y) = \sum \beta_\alpha f_\alpha(x)e^{\omega_\alpha y}$, we get

$$(3.23)\quad \left|\sum \beta_\alpha F(f_\alpha, \omega_\alpha)\right| \leqq \limsup_r \limsup_i \int_{\mathbf{X}} \int_{B_r(x)} |g(x, y)|\ \mu(dy; x, n_i(x)\nu(dx).$$

For any finite set of constants $\{c_\alpha\}$,

$$(3.24)\quad \lim_{i\to\infty} \int_{B_r(x)} \left|\sum c_\alpha e^{\omega_\alpha y}\right| \mu(dy;\, x,\, n_i(x)) = \int_{B_r(x)} \left|\sum c_\alpha e^{\omega_\alpha y}\right| \mu(dy;\, x)\ [\nu].$$

Because of continuity of $|\sum c_\alpha e^{\omega_\alpha y}|$ in the c_α, it follows from (3.24) that

$$\lim_{i\to\infty} \int_{B_r(x)} |g(x, y)|\, \mu(dy;\, x,\, n_i(x)) = \int_{B_r(x)} |g(x, y)|\, \mu(dy;\, x)\ [\nu].$$

Since the last functions are dominated, we may integrate with respect to ν and then interchange the order of limit and integration to obtain

$$(3.25)\quad \lim_{i\to\infty} \int_{\mathbf{X}} \int_{B_r(x)} |g(x, y)|\, \mu(dy;\, x,\, n_i(x))\nu(dx) = \int_{\mathbf{X}} \int_{B_r(x)} |g(x, y)|\, \mu(dy;\, x)\nu(dx).$$

As seen from (3.14), $B_r(x)$ converges to $B(x)$, the boundary of $C(x)$ as $r \to \infty$. Let B be the set with sections $B(x)$. That B is measurable follows from the remarks made prior to Corollary 3.3.1.

Taking the limit as $r \to \infty$ in (3.25) then yields

$$\lim_{r\to\infty} \int_{\mathbf{X}} \int_{B_r(x)} |g(x, y)|\, \mu(dy;\, x)\nu(dx) = \int_{\mathbf{X}} \int_{B(x)} |g(x, y)|\, \mu(dy;\, x)\nu(dx) = \int_B |g|\, d\lambda.$$

Consequently, the limits taken successively in (3.23) show that

$$(3.26)\quad \left|\sum \beta_\alpha F(f_\alpha\,,\, \omega_\alpha)\right| \leqq \int_B |g|\, d\lambda.$$

Inequality (3.26) guarantees that F, defined by (3.22), can be unambiguously linearly extended to $\mathfrak{L}$ without increasing its norm. Moreover, the value of F extended at $g \in \mathfrak{L}$ is determined by the restriction of g to B, as (3.26) shows. Therefore, one may regard F as a bounded linear functional defined on the restriction of $\mathfrak{L}$ to B, a linear subspace in $L_1(B, \lambda)$. In this event the Hahn Banach Theorem states that F can be extended to all of $L_1(B, \lambda)$ without increase in norm. Exploiting the representation of the dual of $L_1(B, \lambda)$, there exists a measurable function ϕ_B with $|\phi_B| \leqq 1$, giving the representation

$$(3.27)\quad F(f, \omega) = \int_B \phi_B(x, y) f(x) e^{\omega y}\, d\lambda.$$

It does not follow directly that $\phi_B \geqq 0$, but a minor circumlocution would show that there is at least one non-negative ϕ_B.

The equality of (3.22) and (3.27) for each $f \in L_\infty(\mathbf{X}, \nu)$ yields immediately for $\omega \in \Omega_0$,

$$(3.28)\quad L(x, \omega) = \int_{B(x)} \phi_B(x, y) e^{\omega y} \mu(dy;\, x)\ [\nu].$$

The proof of Theorem 3.1 can now be concluded. Limits of the three terms in (3.18) are evaluated in (3.19), (3.20), and (3.21) and (3.28) together, first as i, and then r tend to infinity. Define a test $\phi \in \Phi_{\mathfrak{D}}$ by

$$\begin{aligned} \phi(x, y) &= 0, && y \in \operatorname{int} C(x), \\ &= \phi_B(x, y), && y \in B(x), \\ &= 1, && y \in C'(x). \end{aligned}$$

The evaluations above show that

$$\lim_{i\to\infty} \int e^{\omega y}\phi_{n_i(x)}(x, y)\mu(dy; x, n_i(x)) = \sum_{k=1}^{3} \lim_r \lim_i I_{ki}(x, \omega, r) = \int \phi(x, y)e^{\omega y}\mu(dy; x) \; [\nu],$$

for $\omega \varepsilon \Omega_0$. Recalling the limit (3.17) already found for the right hand side of (3.16), there results

$$\int \phi(x, y)e^{\omega y}\mu(dy; x) \geqq \int \psi(x, y)e^{\omega y}\mu(dy; x) \; [\nu], \tag{3.29}$$

whenever $\omega \varepsilon \omega_0$, with equality when $\omega = 0$. Because of continuity in ω the inequality in (3.29) must hold for all $\omega \varepsilon \Omega_1$. This completes the proof of the theorem.

4. Admissibility. Admissibility of tests in $\Phi_{\mathfrak{D}}$ with convex sections is naturally related to admissibility of tests whose acceptance regions are convex. Stein [9] has given an important sufficient condition for admissibility of tests with closed convex acceptance regions. Essentially, the condition is that for each supporting hyperplane, there exist parameter points in the set of alternatives arbitrarily far out on some perpendicular. Thus, in the event that the parameter space is the full Euclidean k-space, and the null hypothesis is a bounded set (for example, a simple hypothesis), any test with closed convex acceptance region is admissible. This is true for all dominating measures satisfying the condition on the parameter space. But, if the measure does not assign measure zero to the boundaries of convex sets, Stein's theorem does not apply to convex acceptance regions with randomization on the boundary.

Turning to the composite hypothesis of Section 3, we find that the question of admissibility of tests in $\Phi_{\mathfrak{D}}$ is quite complicated, and intimately tied up with the dominating measure λ. First of all, we remark that $\Phi_{\mathfrak{D}}$ is essentially the smallest class of tests which is complete for every dominating measure λ. That is, if $\phi \varepsilon \Phi_{\mathfrak{D}}$, there is some measure λ such that $\nu\{x_0\} = 1$ for some x_0, and $\mu(\cdot; x_0)$ is absolutely continuous and such that the natural parameter space associated with $\mu(\cdot; x_0)$ is the full Euclidean $k - r$ space. Stein's theorem applies to the conditional test at x_0, and shows that ϕ cannot be dominated at this section. Since this is the only relevant section, ϕ must be admissible.

There is one special case in which the admissibility question can be satisfactorily answered. Suppose that ω is real ($r = k - 1$). We can then assert that all tests in $\Phi_{\mathfrak{D}}$ are admissible. In order to demonstrate this we first show that if a test ψ is at least as good as a test ϕ then ψ and ϕ must have the same size and "center of gravity" with respect to each of the conditional distributions $\mu(\cdot; x)$. That is

$$\int_{\mathrm{Y}} (\psi(x, y) - \phi(x, y))\mu(dy; x) = 0 \; [\nu], \tag{4.1}$$

and

$$\int_{\mathrm{Y}} y(\psi(x, y) - \phi(x, y))\mu(dy; x) = 0 \; [\nu]. \tag{4.2}$$

If ψ is at least as good as ϕ, then $\int (\psi - \phi)e^{\theta x+\omega y}\, d\lambda \geqq 0$ for all $(\theta, \omega) \varepsilon \Omega$, with equality whenever $\omega = 0$. A completeness argument shows (4.1). Then, (4.1) together with the validity of the above inequality for all ω in a neighborhood of the origin show that for each θ, $\int y(\psi - \phi)e^{\theta x}\, d\lambda = 0$. Again, completeness yields (4.2). Now, if $\phi \,\varepsilon\, \Phi_{\mathfrak{D}}$ were inadmissible, there would exist an essentially distinct test $\psi \,\varepsilon\, \Phi_{\mathfrak{D}}$ satisfying (4.1), (4.2). But now the conditional tests accept on intervals of the real line. An interval, together with the randomization at the end points, is determined by the size and "center of gravity." Hence, the conditions mentioned imply $\psi \equiv \phi$, so that ϕ cannot be dominated.

In the following paragraphs we list a few examples of admissible tests in $\Phi_{\mathfrak{D}}$, and finally, an example of a non-admissible test. Perhaps these examples are not very important from a practical point of view, but they do, at least illustrate the complicated nature of admissibility.

(a) Suppose that the measure $\mu(\cdot\,; x)$ is independent of x (and hereafter denoted by μ) and has finite support. We suppose, also, that x is real, and that $1 - \phi(x, y) = I_{[x \geqq 0]}(x)I_{C_1}(y) + I_{[x<0]}(x)I_{C_2}(y)$, where C_1 and C_2 are closed convex sets in $\mathbf{Y}$. We will show that ϕ is admissible. The reader can easily extend this to the case of any finite number of convex sections. We suppose that there is a test ψ which is at least as good as ϕ. Since μ has finite support there are only a finite number of subsets of the support of μ which we enumerate as $C_1, C_2, \cdots, C_N$. First of all, one can represent ψ as $1 - \psi(x, y) = \sum_{i=1}^{N} \lambda_i(x)I_{C_i}(y)$, where $0 \leqq \lambda_i(x)$, $\sum_{i=1}^{N} \lambda_i(x) = 1$. This is merely a consequence of the fact that the set of all tests is convex, and the extreme points are $I_{C_i}(x)$. Now, set $g_i(\omega) = \int_{C_i} e^{\omega y}\, d\mu$. The fact that ψ is at least as good as ϕ can be expressed as

$$(4.3) \qquad g_1(\omega) \int_0^\infty e^{\theta x}\, d\nu + g_2(\omega) \int_{-\infty}^{0^-} e^{\theta x}\, d\nu \geqq \sum_{i=1}^{N} g_i(\omega) \int \lambda_i(x)e^{\theta x}\, d\nu$$

for all θ, ω, with equality holding when $\omega = 0$. In turn, this becomes

$$(4.4) \quad g_1(\omega) \int_0^\infty (1 - \lambda_1(x))e^{\theta x}\, d\nu \geqq -g_2(\omega) \int_{-\infty}^{0^-} e^{\theta x}\, d\nu + g_1(\omega) \int_{-\infty}^{0^-} \lambda_1(x)e^{\theta x}\, d\nu + \sum_{i=2}^{N} g_i(\omega) \int \lambda_i(x)e^{\theta x}\, d\nu.$$

If λ_1 is not equal to one $[\nu]$ for $x \geqq 0$ divide both sides of (4.4) by $\int_0^\infty (1 - \lambda_1(x))e^{\theta x}\, d\nu$. The terms $(\int_0^\infty \lambda_i(x)e^{\theta x}\, d\nu)/(\int_0^\infty (1 - \lambda_1(x))e^{\theta x}\, d\nu)$ are bounded and by choosing an appropriate sequence θ_n and letting $\theta_n \to \infty$ have a limit γ_i, $0 \leqq \gamma_i$, $\sum_{i=2}^{N} \gamma_i = 1$. All other terms will have limit zero, so

$$(4.5) \qquad g_1(\omega) \geqq \sum_{i=2}^{N} \gamma_i g_i(\omega)$$

for all ω with equality when $\omega = 0$. If one were to consider testing the simple hypothesis $\omega = 0$ on the basis of observing y, (4.5) says that the test which accepts with probability $\sum_{i=2}^{N} \gamma_i I_{C_i}(y)$ is at least as good as the test which accepts when y is in the closed convex set C_1. According to Stein's theorem equality must hod for all ω in (4.5), and completeness yields $I_{C_i} = \sum_{i=2}^{N} \gamma_i I_{C_i}[\mu]$. This is impossible since $C_1, C_2, \cdots, C_N$ are distinct sets in the support of μ. Consequently, on $x \geqq 0$ we have $\lambda_1(x) = 1$ $[\nu]$. Then (4.3) becomes

$$g_2(\omega) \int_{-\infty}^{0^-} e^{\theta x}\, d\nu \geqq \sum_{i=1}^{N} g_i(\omega) \int_{-\infty}^{0^-} \lambda_i(x)e^{\theta x}\, d\nu$$

for all θ, ω with equality when $\omega = 0$. Again, Stein's theorem yields equality for all θ, ω and completeness gives $\psi = \phi$ [λ].

(b) In this example we suppose that ν has finite support, and show that if every x-section of ϕ is an admissible test for testing the simple hypothesis $\omega = 0$, then ϕ is admissible for the composite hypothesis. Let us denote the support of ν by $\mathcal{S}$, and let x_0 be an extreme point of the convex hull of $\mathcal{S}$. If a test ψ is at least as good as ϕ then

$$\sum_{x \varepsilon \mathcal{S}} e^{\theta x} \int e^{\omega y}(\psi(x, y) - \phi(x, y))\mu(dy; x)\nu\{x\} \geqq 0 \tag{4.6}$$

for all θ, ω with equality when $\omega = 0$. Consider a hyperplane $a(x - x_0) = 0$ which supports the convex hull of $\mathcal{S}$ at x_0 and such that $a(x - x_0) < 0$ for all $x \varepsilon \mathcal{S}$, $x \neq x_0$. In (4.6) let $\theta = na$, and multiply (4.6) by e^{-nax_0}. Letting $n \to \infty$, we obtain

$$\int e^{\omega y}(\psi(x_0, y) - \phi(x_0, y))\mu(dy; x_0) \geqq 0 \tag{4.7}$$

for all ω, with equality when $\omega = 0$. If the x_0-section of ϕ is admissible, this gives a contradiction unless equality holds for all ω, and then completeness gives $\phi(x_0, \cdot) = \psi(x_0, \cdot)[\mu(\cdot; x_0)]$. In the latter case we can replace $\mathcal{S}$ by $\mathcal{S} - \{x_0\}$ in (4.6) and repeat the argument for an extreme point of $\mathcal{S} - \{x_0\}$. After finitely many steps we must either arrive at a contradiction, or conclude $\psi = \phi$ [λ].

(c) In order to apply the result of (b) to the case where λ has finite support (say, in the multinomial case) one needs conditions to guarantee that the sections are admissible for testing the simple hypothesis $\omega = 0$ when the dominating measure μ has finite support. For testing such a simple hypothesis on the basis of Y, we will show that *a test ϕ is admissible if and only if the set $C = \{y \varepsilon \mathbf{Y}: \phi(y) < 1\}$ is convex, and for every y which is not an extreme point of C, $\phi(y) = 0$*. First, suppose that the conditions on ϕ hold. Then, if ϕ is dominated by a test ψ, there must exist some point y_0 in the support of μ such that $\phi(y_0) > \psi(y_0)$. Otherwise, the tests could not have the same size. Because of the conditions on ϕ we see that y_0 must either belong to the complement of C, or be an extreme point of C. In either case there is a hyperplane $a(y - y_0) = 0$ containing y_0 with $a(y - y_0) < 0$ for all $y \varepsilon C$, $y \neq y_0$. Letting $\omega = na$, the fact that ψ dominates ϕ gives

$$0 \leqq e^{-nay_0}E_{na}(\psi - \phi) = \int (\psi - \phi)e^{na(y-y_0)}\, d\mu$$

which converges to $(\psi(y_0) - \phi(y_0))\mu\{y_0\}$ as $n \to \infty$. Thus, $\psi(y_0) \geqq \phi(y_0)$, so that y_0 would have to be a non-extreme point of C. Again, this is impossible because $\phi = 0$ at all non-extreme points. Hence, ϕ is admissible.

On the other hand, if ϕ is admissible, it must belong to the complete class $\Phi_{\mathcal{C}}$ so there is some closed convex set C such that $\phi = 0$ on the interior of C, and $\phi = 1$ on the complement of C. We can clearly suppose that $\phi(y) < 1$ at all extreme points of C, for otherwise we could simply delete that extreme point from C and consider a new convex set C_1 having the same properties. If there were some $y_0 \varepsilon C$, not an extreme point, with $\phi(y_0) > 0$, we will show that ϕ could be strictly dominated. Such a y_0 in C can be expressed as $y_0 = \sum_{i=1}^{n} \lambda_i y_i$, where $0 < \lambda_i$, $\sum_{i=1}^{n} \lambda_i = 1$, and $y_1, y_2, \cdots, y_n$ are extreme points of C. Define $\epsilon_0 = \min\{\phi(y_0), \min_{1 \leqq i \leqq n}(1 - \phi(y_i)\mu_i)/\mu_0\lambda_i\}$, where $\mu_i = \mu\{y_i\}, i = 0, 1, \cdots, n$. Then set $\epsilon_i = \epsilon_0\mu_0\lambda_i/\mu_i$. Define $\phi^*(y_i) = \phi(y_i) + \epsilon_i$, $i = 1, 2, \cdots, n$, $\phi^*(y_0) = \phi(y_0) - \epsilon_0$, and $\phi^* = \phi$ otherwise. Note that $\phi(y_i) < \phi^*(y_i) \leqq 1$, and $\phi(y_0) > \phi^*(y_0) \geqq 0$. To show that ϕ^* dominates ϕ consider

$$e^{-\omega y_0}E_\omega(\phi^* - \phi) = \sum_{i=1}^n \epsilon_i\mu_i e^{\omega(y_1 - y_0)} - \epsilon_0\mu_0 .$$

Note that by our construction $\sum_{i=1}^n \epsilon_i\mu_i = \epsilon_0\mu_0$, and using the fact that $e^{\omega y}$ is convex we get

$$\begin{aligned} e^{-\omega y_0}E_\omega(\phi^* - \phi) &\geqq \epsilon_0\mu_0(\exp[\omega \textstyle\sum (\epsilon_i\mu_i/\epsilon_0\mu_0)(y_i - y_0)] - 1) \\ &= \epsilon_0\mu_0(\exp[\omega \textstyle\sum \lambda_i(y_i - y_0)] - 1) = 0 \end{aligned}$$

for all ω. In fact, there would be strict inequality for some ω, showing that ϕ could be dominated. Thus, if ϕ is to be admissible ϕ must be zero at all non-extreme points of C.

(d) Returning to the case of composite hypotheses, we suppose λ has finite support. Probably the most important special case of this is the multinomial case. Let ϕ be a test and let $C = \{(x, y) : \phi(x, y) < 1\}$. Then *$\phi$ is admissible if and only if*, $[\lambda]$, *every x-section of C is convex* $[\mu(\cdot\,; x)]$ *and* $\phi(\cdot, x) = 0$ *at all non-extreme points of the section* C_x. We showed in (c) that the above condition implies that each section is admissible for testing the simple hypothesis, and (b) says ϕ is admissible. Conversely, if ϕ is admissible and there is an x_0 in the support of λ such that C_{x_0} does not satisfy the condition, then there is a y_0 in the convex hull of C_{x_0} which is not an extreme point such that $\phi(x_0, y_0) > 0$. The construction in (c) allows us to find a new test ϕ^* which strictly dominates ϕ at the x_0-section, and whose conditional power is the same at all other sections. Since x_0 is in the support of λ, ϕ^* would then strictly dominate ϕ.

(e) Our final example shows that $\Phi_{\mathfrak{D}}$ contains tests which are not admissible, at least for some choices of λ. Let $\mathbf{X} = R^1$, $\mathbf{Y} = R^2$, and define the sets $C_1 = \{(\pm 1, 0), (\pm 2, 0)\}$, $C_2 = \{(0, \pm 1), (0, \pm 2)\}$, and $C_0 = \{(\pm 1, 0), (0, \pm 1)\}$. For each x let $\mu(\cdot\,; x)$ be uniform on $C_1 \cup C_2$, and let λ be a product measure on $\mathbf{X} \times \mathbf{Y}$. We compare the test ϕ_0 which accepts when $y \in C_0$, regardless of x, with the test ϕ_A which accepts when $y \in C_1$, $x \in A$, and when $y \in C_2$, $x \notin A$. An easy calculation shows that ϕ_A is dominated by ϕ_0 provided, for each θ, $\frac{1}{5} < P_\theta(A) < \frac{4}{5}$. Hence, in order for ϕ_A to be inadmissible one need only mix its sections sufficiently rapidly to meet this condition. This is always possible if θ is a translation parameter for the distribution of X.

5. Likelihood ratio tests. For testing a simple hypothesis it was shown in [1] that likelihood ratio tests have convex acceptance regions. Likewise, we shall show for composite hypotheses of the type we have treated that likelihood ratio tests are in $\Phi_{\mathfrak{D}}$. To see this, consider a likelihood ratio test which accepts $H_0'': \omega = 0$ when

$$\sup_{\theta'} p(x, y; \theta', 0)/\sup_{\{(\theta,\omega):\omega\neq 0\}} p(x, y; \theta, \omega) \geqq k. \tag{5.1}$$

For a given x, the acceptance set in (5.1) may be expressed as

$$\bigcap \{B(\theta, \omega) : (\theta, \omega) \in \Omega, \omega \neq 0\}.$$

where $B(\theta, \omega) = \{y : \sup_{\theta'} p(x, y; \theta', 0) \geqq kp(x, y; \theta, \omega)\}$. Here, $\sup_{\theta'} p(x, y; \theta', 0)$ is constant as far as y is concerned, while $p(x, y; \theta, \omega)$ is a convex function of y. It follows that the sets $B(\theta, \omega)$ are closed and convex in $\mathbf{Y}$ so that their intersection is also closed and convex.

6. Appendix. In this section we give a proof of the Main Lemma 3.2. This proof is patterned after one of the standard proofs of the Blaschke convergence theorem which can be found in [11]. First partition $\mathbf{Y}$ into cubes of side $1/2^k$ in such a manner that the $(k + 1)$st partition is a refinement of the kth partition $k = 1, 2, \cdots$. Consider the set of all finite unions of cells in the kth partition. Such a finite union will be denoted by D with appropriate affixes. We will suppose that for each k the set of finite unions of cells is given some definite ordering: $D_1^k, D_2^k, \cdots$. Note that each D_i^k will always appear as some D_j^{k+1}.

At first, we will fix a sphere of radius r about the origin, S_r, and suppose that for all n and x, $C_n(x) \subset S_r$. We will say that D_i^k is a minimal cover for $C_n(x)$ if $C_n(x) \subset D_i^k$ and $C_n(x)$ intersects every cell of D_i^k. There must be some D_i^k which is a minimal cover for infinitely many $C_n(x)$, since there are only finitely many minimal covers and infinitely many $C_n(x)$ in S_r. Such a D_i^k will simply be called *minimal at x*.

Define $D^1(x)$ to be the first (in our given ordering) D_i^1 which is minimal at x. Suppose that $D^1(x), D^2(x), \cdots, D^{k-1}(x)$ have already been defined. Then there is some D_i^k contained in $D^{k-1}(x)$ which is minimal at x. Define $D^k(x)$ to be the first such D_i^k with this property. Finally, let $C(x) = \bigcap_{k=1}^{\infty} D^k(x)$.

We now show that $C = \{(x, y) : y \in C(x)\}$ is a Borel set. Since $C = \bigcap_{k=1}^{\infty} \{(x, y) : y \in D^k(x)\}$, it will be sufficient to show $\{(x, y) : y \in D^k(x)\}$ is a Borel set for each k. But, $\{(x, y) : y \in D^k(x)\} = \bigcup_i \{(x, y) : y \in D^k(x), D^k(x) = D_i^k\} = \bigcup_i \{x : D^k(x) = D_i^k\} \times D_i^k$, so it is sufficient to show that $\{x : D^k(x) = D_i^k\}$ is a Borel set for every i and k. We prove this by induction on k. If $k = 1$, $\{x : D^1(x) = D_i^1\} = \{x : D_i^1 \text{ is minimal at } x\} \cap \bigcap_{j=1}^{i-1} \{x : D_j^1 \text{ is not minimal at } x\}$, so it need only be shown that the first set is a Borel set. But $\{x : D_i^1 \text{ is minimal at } x\} = \limsup \{x : D_i^1 \text{ is a minimal cover for } C_n(x)\}$, and $\{x : D_i^1 \text{ is a minimal cover for } C_n(x)\}$ is a union of atoms of $\mathcal{B}_n$ according to the manner in which the $C_n(x)$ were defined. Thus, the assertion is true for $k = 1$. Suppose that the assertion is true for $k - 1$. Then, $\{x : D_i^k = D^k(x)\} = \{x : D^k \text{ is minimal at } x, D_i^k \subset D^{k-1}(x)\} \cap \bigcap_{j=1}^{i-1} \{x : D_j^k \text{ is minimal at } x, D_j^k \subset D^{k-1}(x)\}'$. We need only show that the first set is a Borel set, and this in turn is the intersection of two sets. $\{x : D_i^k \text{ is minimal at } x\}$ is a Borel set by the argument given for $k = 1$.

$$\{x : D_i^k \subset D^{k-1}(x)\} = \bigcup_{\{j : D_i^k \subset D_j^{k-1}\}} \{x : D_j^{k-1} = D^{k-1}(x)\},$$

and this is a Borel set by the induction hypothesis. This completes the induction.

Now we must show (i) $C(x)$ is closed and convex; (ii) for each x there exists subsequence $\{n_k(x)\}$ such that $C_{n_k(x)}(x) \to C(x)$; (iii) $n_k(x)$ is Borel measurable.

By definition of $D^1(x)$, there are infinitely many n such that $D^1(x)$ is a minimal cover for $C_n(x)$. Let $n_1(x)$ be the first such n. Also, there are infinitely many n such that all the sets $D^1(x), D^2(x), \cdots, D^k(x)$ are minimal covers for $C_n(x)$. After $n_1(x), n_2(x), \cdots, n_{k-1}(x)$ have been defined, let $n_k(x)$ be the first such n which is greater than $n_{k-1}(x)$. Then $\{x: n_1(x) = j\} = \{x: D^1(x)$ is a minimal cover for $C_j(x)\} \cap \bigcap_{i=1}^{j-1} \{x: D^1(x)$ is not a minimal cover for $C_i(x)\}$. Arguments as before show that this is a Borel set. It is an easy induction argument to show that $n_k(x)$ is Borel measurable for every k.

In the kth refinement, let ϵ_k be the cell diameter. We have $\lim_{k\to\infty} \epsilon_k = 0$. Moreover,

$$C_{n_k(x)}(x) \subset D^k(x) \subset N(C_{n_k(x)}(x), \epsilon_k),$$

because $D^k(x)$ is a minimal cover for $C_{n_k(x)}(x)$. Hence, $d(C_{n_k(x)}(x), D^k(x)) \leqq \epsilon_k$ and has limit zero as $k \to \infty$. Also,

$$d(C_{n_k(x)}(x), C(x)) \leqq d(C_{n_k(x)}, D^k(x)) + d(D^k(x), C(x)) \to 0$$

as $k \to \infty$. The convexity and closure now follow from the Blaschke convergence theorem.

The next step is to modify the above construction to take care of the case when the $C_n(x)$ are not all contained in a fixed sphere. Define $C_n^1(x) = C_n(x) \cap S_1$. According to the above construction, there exists a measurable set $C^1 \subset \mathbf{X} \times \mathbf{Y}$ such that for each x, $C^1(x)$ is closed and convex, and a measurable sequence $n_k^1(x)$ such that $C^1_{n_k^1(x)}(x) \to C^1(\mathrm{x})$. Define $C_k^2(x) = C_{n_k^1(x)}(x) \cap S_2$. We can again apply the construction to the sequence $C_k^2 = \{(x, y): y \varepsilon C_k^2(x)\}$ to find C^2 whose sections are closed and convex. The measurability of C^2 depends, as in the original construction, only upon the fact that sets of the form $\{x: D_i^k$ is a minimal cover for $C_k^2(x)\}$ are measurable, and this is clear from the definition of $C_k^2(x)$. We also have a measurable sequence $n_k^2(x)$ such that $C^2_{n_k^2(x)}(x) \to C^2(x)$. Note that for each fixed x, $\{n_k^2(x)\}$ is a subsequence of $\{n_k^1(x)\}$. According to [3], Theorem 31, if the interior of $\mathcal{S}_1$ meets $C^2(x)$, then $C^2_{n_k^2(x)}(x) \cap \mathcal{S}_1 \to C^2(x) \cap \mathcal{S}_1$ so that in this case $C^1(x) = C^2(x) \cap \mathcal{S}_1$. In any case it is easy to verify that $C^1(x) \subset C^2(x) \cap \mathcal{S}_1$. By induction we can obtain a sequence C^r of measurable sets whose sections are closed and convex subsets of $\mathcal{S}^r$, and $C^r \subset C^{r+1}$, $r = 1, 2, \cdots$. Define $C = \bigcup_{r=1}^{\infty} C^r$. Then C is measurable and has convex sections. For sufficiently large k, $C(x) \cap \mathcal{S}_k = C^k(x)$ as a consequence of Theorem 31 [3], and it is clear from this that $C(x)$ is closed for each x. Now, we notice that for each n, $d(C_n(x) \cap \mathcal{S}_r, C^r(x))$ is a measurable function of x. This can be seen by expressing the function as $\lim_{k\to\infty} d(C_n(x) \cap \mathcal{S}_r, C_{n_k^r(x)}(x) \cap \mathcal{S}_r)$ and using the definition of $C_n(x)$ and the measurability of the subsequence. Define $n_1(x)$ as the first n such that $d(C_n(x) \cap \mathcal{S}_1, C^1(x)) < 1$, and $n_k(x)$ as the first $n > n_{k-1}(x)$ such that $d(C_n(x) \cap \mathcal{S}_k, C^k(x)) < 1/k$. Then the $n_k(x)$ are measurable. The same proof used to prove Theorem 31 in [3] shows that if $d(C_{n_k(x)}(x) \cap \mathcal{S}_k, C^k(x)) \to 0$ then $d(C_{n_k(x)}(x) \cap \mathcal{S}_r, C(x) \cap \mathcal{S}_r) \to 0$ provided the interior of $\mathcal{S}_r$ meets $C(x)$. Since this is the case for r sufficiently large, we have the theorem. (We have here omitted the trivial case when $C(x)$ is empty.)

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