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# Texture synthesis by reaction diffusion process

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Abstract. This work is devoted to the mathematical study of nonlinear and anisotropic reaction-diffusion equation. This type of equation appears in texture synthesis and computer vision. The proposed model utilizes a diffusion tensor which may be adapted to the image structure. For this reason, New techniques are needed to show the existence of weak solution with the initial data is in $L^2(\Omega)$ of this model.

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## 1. Introduction

The matrix field of the structure tensor, introduced by Forstner and Gulch [9] plays a fundamental role in image processing and computer vision, as it allows both orientation estimation and image structure analysis. It has proven its usefulness in many application fields such as corner detection [9], texture analysis [10] and optical flow [9].

The structure tensor offers three advantages. Firstly, the matrix representation of the image gradient allows the integration of information from a local neighborhood without cancelation effects. Such effects would appear if gradients with opposite orientation were integrated directly. Secondly, smoothing the resulting matrix field yields robustness under noise by introducing an integration scale. This scale determines the local neighborhood over which an orientation estimation at a certain pixel is performed. Thirdly, the integration of local orientation creates additional information, as it becomes possible to distinguish areas where structures are oriented uniformly, like in regions with edges, from areas where structures have different orientations, like in corner regions.

Among the authors whose propose the anisotropic nonlinear diffusion models [1, 4, 6, 15, 17] for image processing, we can find that Cottet and Germain [12] proposed the following model

$$\begin{cases} \frac{\partial u}{\partial t} - \sigma\epsilon^2 \operatorname{div}(A_\epsilon(u)\nabla u) = f(u) & \text{in } ]0,T[\times\Omega \\ u(0,x) = u_0(x) & \text{in } \Omega \\ u = 0 & \text{on } ]0,T[\times\partial\Omega \end{cases} \tag{1}$$

Where $A_\epsilon(u)$ is the orthogonal projection onto the direction which is perpendicular to the gradient of $u_\epsilon$. Their model diffuses only in one direction, it is clear that its

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result depends very much on the smoothing direction.
In 1994, Weickert [11] proposed an new model based on the structure tensor

$$\begin{cases} \frac{\partial u}{\partial t} - \operatorname{div}(D(J_\rho(\nabla u_\sigma))\nabla u) = 0 & \text{in } ]0,T[\times\Omega \\ u(0,x) = u_0(x) & \text{in } \Omega \\ < D(J_\rho(\nabla u_\sigma))\nabla u, \xi >= 0 & \text{on } ]0,T[\times\partial\Omega \end{cases} \tag{2}$$

where diffusion tensor $D$ is a matrix depending on the eigenvalues and on the eigenvectors of the structure tensor $J = \nabla u \otimes \nabla u$. This tensor product contains merely the same information as the gradient itself, it has the big advantage that it can be smoothed without cancelation effects for areas where gradients have opposite signs. This smoothing stabilizes the orientation information.

In this work, we discuss the following problem:

$$\begin{cases} \frac{\partial u}{\partial t} - \operatorname{div}(D(J_\rho(\nabla u_\sigma))\nabla u) = f(t,x,u) & \text{in } ]0,T[\times\Omega \\ u(0,x) = u_0(x) & \text{in } \Omega \\ < D(J_\rho(\nabla u_\sigma))\nabla u, \xi >= 0 & \text{on } ]0,T[\times\partial\Omega \end{cases} \tag{3}$$

where $u_0$ is an observed image, $\Omega$ is a regular bounded open set of $\mathbb{R}^N$ with smooth boundary $\partial\Omega$, $Q_T = ]0,T[\times\Omega$, $\sum_T = ]0,T[\times\partial\Omega$ and $\xi$ is the unit outward normal to $\Omega$. Let $\sigma > 0$, $K_\sigma$ is the Gaussian filter where:

$$K_\sigma(x) = \frac{1}{(2\pi\sigma)^{\frac{N}{2}}} e^{(-\frac{|x|^2}{4\sigma})}, x \in \mathbb{R}^N$$

We consider the gradient norm of $w$ as $|\nabla w| = \sqrt{\sum_{i=1}^N (\frac{\partial w}{\partial x_i})^2}$, $\nabla w_\sigma$ is the smoothed version of gradient norm: $\nabla w_\sigma := \nabla(w * K_\sigma) = w * \nabla K_\sigma$. The matrix $J_0$ resulting from the tensor product

$$J_0 := \nabla u_\sigma \otimes \nabla u_\sigma := \nabla u_\sigma \nabla u_\sigma^T$$

has an orthonormal basis of eigenvectors $v_1, v_2$ with $v_1 || \nabla u_\sigma$ and $v_2 \perp \nabla u_\sigma$. The corresponding eigenvalues $|\nabla u_\sigma|^2$ and 0 give just the contrast in the eigendirections. By convolving $J_0(\nabla u_\sigma)$ with a Gaussian $K_\rho$ we obtain the structure tensor

$$J_\rho(\nabla u_\sigma) = K_\rho * (\nabla u_\sigma \otimes \nabla u_\sigma)$$

The matrix $J_\rho = \begin{pmatrix} j_{11} & j_{12} \\ j_{21} & j_{22} \end{pmatrix}$ is symmetric, positive semidefinite and possesses orthonormal eigenvectors $w_1, w_2$ with

$$w_1 = \begin{pmatrix} \frac{2j_{12}}{\sqrt{(j_{22}-j_{11}+\sqrt{(j_{11}-j_{22})^2+4j_{12}^2})^2+4j_{12}^2}} \\ \frac{j_{22}-j_{11}+\sqrt{(j_{11}-j_{22})^2+4j_{12}^2}}{\sqrt{(j_{22}-j_{11}+\sqrt{(j_{11}-j_{22})^2+4j_{12}^2})^2+4j_{12}^2}} \end{pmatrix}$$

if $j_{11} \neq j_{22}$ or $j_{12} \neq 0$. The corresponding eigenvalues are given by

$$\mu_1 = \frac{1}{2}\left(j_{11} + j_{22} + \sqrt{(j_{11} - j_{22})^2 + 4j_{12}^2}\right)$$

$$\mu_2 = \frac{1}{2}\left(j_{11} + j_{22} - \sqrt{(j_{11} - j_{22})^2 + 4j_{12}^2}\right)$$

The eigenvalues describe the contrast in the eigendirections. Furthermore, the diffusion tensor $D = (d_{i,j})$ satisfies the following properties:

- $(C_1)$ Smoothness: $D \in \mathcal{C}^\infty(\mathbb{R}^{2\times2}; \mathbb{R}^{2\times2})$.
- $(C_2)$ Symmetry: $d_{12}(J) = d_{21}(J)$ for all symmetric matrices $J \in \mathbb{R}^{2\times2}$.

$(C_3)$ Uniform positive definiteness: For all $w \in L^\infty(\Omega, \mathbb{R}^2)$ with $|w(x)| \leq K$ on $\Omega$, there exists a positive lower bound $\nu(K)$ for the eigenvalues of $D(J_\rho(w))$.

The regularization by convolving with a Gaussian kernel makes the edge detection insensitive to noise at scale smaller than $\sigma$ and helps to ensure the existence results.

To adapt the diffusion tensor $D$ to the local structure, one may prescribe that it should possess the same eigenvectors as the structure tensor $J_\rho$. The corresponding eigenvalues $\lambda_1$ and $\lambda_2$ of $D$ are chosen as in Weickert [13] and that's gives

$$D = S\Lambda S^T$$

where $S = \begin{pmatrix} w_1 & w_2 \end{pmatrix}$ and $\Lambda = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$.

Moreover, these following main properties hold:

- The positivity of the solution is preserved with time, which is ensured by

$$\text{for almost } (t,x) \in Q_T,\ f(t,x,0) \geq 0 \tag{4}$$

- The total mass of the components is controlled with time, which is given by

$$\forall u \in \mathbb{R}^+,\ \text{for all } (t,x) \in Q_T, uf(t,x,u) \leq 0 \tag{5}$$

Let introduce for $f$ the hypotheses

$$f : Q_T \to \mathbb{R} \text{ is measurable}$$

$$f(t,x,.) : \mathbb{R} \to \mathbb{R} \text{ is Locally Lipshitz, namely : } |f(t,x,u) - f(t,x,\hat{u})| \leq K(r)|u - \hat{u}| \tag{6}$$

for a.e $(t,x) \in Q_T$ and for all $0 \leq |u|, |\hat{u}| \leq r$.

We suppose that there exists a positive constant $M$ such that

$$\forall (t,x,r) \in Q_T \times \mathbb{R},\ |f(t,x,r)| \leq M$$

And that for all $R \geq 0$, $sup_{|u|\leq R}(|f(t,x,u)|) \in L^1(Q_T)$.

## 2. Preliminaries and main result

Firstly, we precise in which sense we want to solve the problem (3).

**Definition 2.1.** A function $u$ is a *weak solution* of (3) if:

$$\begin{cases} u \in C(0,T;L^2(\Omega)) \cap L^2(0,T;H^1(\Omega)) \\ f(t,x,u) \in L^1(Q_T) \\ \text{for every } \varphi \in C^1(Q_T) \text{ such that } \varphi(T,.) = 0 \\ \int\limits_{Q_T} -u\frac{\partial \varphi}{\partial t} + \int\limits_{Q_T} D(J_\rho(\nabla u_\sigma))\nabla u \nabla \varphi = \int\limits_{Q_T} f(t,x,u)\varphi + \int\limits_{\Omega} u_0(x)\varphi(0,x)dx \end{cases} \tag{7}$$

The main result of this work is the following theorem:

**Theorem 2.1.** *Assume that* $(4) - (6)$ *hold. If* $u_0$ *in* $L^2(\Omega)$*, then there exists a weak positive solution* $u$ *of the problem* (3).

## 3. Approximating Scheme

We consider the function of truncation $\eta_n \in C_0^\infty(\mathbb{R})$ such that: $0 \leq \eta_n \leq 1$ and

$$\eta_n(r) = \begin{cases} 1 & \text{if } |r| \leq n \quad (8) \\ 0 & \text{if } |r| \geq n+1 \quad (9) \end{cases}$$

We truncate the nonlinearity by $\eta_n$,

$$f_n(t,x,u) = \eta_n(|u|)f(t,x,u)$$

Note that the function $f_n$ satisfy the same properties as $f$ with $M = M_{n,T}$ is a constant depending on $n$ and $T$.

We consider the following truncated problem:

$$\begin{cases} u_n \in \mathcal{W}(H^1) \\ \int\limits_\Omega \frac{\partial u_n}{\partial t}\varphi + \int\limits_\Omega D(J_\rho(\nabla u_{n\sigma}))\nabla u_n \nabla\varphi = \int\limits_\Omega f_n(t,x,u_n)\varphi \qquad \forall\varphi \in H^1(\Omega) \\ u_n(0) = u_0^n \end{cases} \tag{10}$$

where

$$\mathcal{W}(H^1) = \{w \in L^2(0,T;H^1(\Omega)); \frac{\partial w}{\partial t} \in L^2(0,T;(H^1(\Omega))^{'})\}.$$

We have $\mathcal{W}(H^1) \hookrightarrow C([0,T];L^2(\Omega))$.

**Theorem 3.1.** *Under the above hypothesis, the problem* (10) *admits a weak positive solution* $u \in \mathcal{W}(H^1)$.

We consider the application:

$$\begin{array}{cccc} \mathfrak{L}_n: & L^2(Q_T) & \rightarrow & L^2(Q_T) \\ & v & \rightarrow & u_n \end{array}$$

where $u_n$ satisfies the following problem

$$\begin{cases} \int\limits_\Omega \frac{\partial u_n}{\partial t}\varphi + \int\limits_\Omega D(J_\rho(\nabla u_{n\sigma}))\nabla u_n \nabla\varphi = \int\limits_\Omega f_n(t,x,v)\varphi \qquad \forall\varphi \in H^1(\Omega) \\ u_n(0) = u_0^n \end{cases} \tag{11}$$

To prove the existence of solution of (11), we need to prove that $\mathcal{L}_n$ admits a fixed point. To this end, we prove through the following lemma, that the hypothesis for the Schauder fixed point theorem are satisfied.

**Lemma 3.2.** *1.* $\mathcal{L}_n$ *is a continous operator on* $L^2(Q_T)$.
*2.* $\mathcal{L}_n(B) \subset B$ *with*

$$B = \{U \in L^2(Q_T) \ \ \textit{such that} \ ||U||_{L^2(Q_T)} \leq \sqrt{T(\frac{2C_{n,T}^2}{\alpha_0} + ||u_0||^2_{L^2(\Omega)})} \ \ \}$$

*3.* $\mathcal{L}_n$ *is a compact operator.*

*Proof.* 1. Let's consider $v$ and $\overline{v}$ in $L^2(Q_T)$ such that:

$$\mathcal{L}_n(v) = u_n \ \text{ and } \mathcal{L}_n(\overline{v}) = \overline{u}_n$$

we have for every $\varphi \in H^1(\Omega)$:

$$\int\limits_\Omega \frac{\partial(u_n - \overline{u}_n)}{\partial t}\varphi + \int\limits_\Omega D(J_\rho(\nabla u_{n\sigma}))\nabla u_n \nabla\varphi - \int\limits_\Omega D(J_\rho(\nabla \overline{u}_{n\sigma}))\nabla \overline{u}_n \nabla\varphi =$$

$$= \int\limits_\Omega (f_n(t,x,v) - f_n(t,x,\overline{v}))\varphi$$

We choose $\varphi = u_n - \overline{u}_n$, we have:

$$\frac{1}{2}\frac{\partial}{\partial t}||u_n - \overline{u}_n||^2_{L^2(\Omega)} + \int_\Omega D(J_\rho(\nabla u_{n\sigma}))\nabla u_n \nabla(u_n - \overline{u}_n) - \int_\Omega D(J_\rho(\nabla \overline{u}_{n\sigma}))\nabla \overline{u}_n \nabla(u_n - \overline{u}_n)$$
$$= \int_\Omega (f_n(v) - f_n(\overline{v}))(u_n - \overline{u}_n).$$

which implies that

$$\frac{1}{2}\frac{\partial}{\partial t}||u_n - \overline{u}_n||^2_{L^2(\Omega)} + \int_\Omega D(J_\rho(\nabla u_{n\sigma}))|\nabla u_n - \nabla \overline{u}_n|^2$$
$$- \int_\Omega (D(J_\rho(\nabla \overline{u}_{n\sigma})) - D(J_\rho(\nabla u_{n\sigma})))\nabla \overline{u}_n \nabla(u_n - \overline{u}_n)$$
$$= \int_\Omega (f_n(v) - f_n(\overline{v}))(u_n - \overline{u}_n).$$

Since the diffusivities $D$ is smooth and uniformly positive definite, then

$$\frac{1}{2}\frac{\partial}{\partial t}||u_n - \overline{u}_n||^2_{L^2(\Omega)} + \nu||\nabla(u_n - \overline{u}_n)||^2_{L^2(\Omega)} \leq \frac{1}{\alpha}||f_n(v) - f_n(\overline{v})||^2_{L^2(\Omega)}$$
$$+ \alpha||u_n - \overline{u}_n||^2_{L^2(\Omega)} + \frac{2c}{\nu}||\overline{u}_n||^2_{H^1(\Omega)}||u_n - \overline{u}_n||^2_{L^2(\Omega)} + \frac{\nu}{2}||\nabla u_n - \nabla \overline{u}_n||^2_{L^2(\Omega)}$$

and that implies

$$\frac{1}{2}\frac{\partial}{\partial t}||u_n - \overline{u}_n||^2_{L^2(\Omega)} + \frac{\nu}{2}||\nabla(u_n - \overline{u}_n)||^2_{L^2(\Omega)} \leq \frac{1}{\alpha}||f_n(v) - f_n(\overline{v})||^2_{L^2(\Omega)}$$
$$+ (\alpha + \frac{2c}{\nu}||\overline{u}_n||^2_{H^1(\Omega)})||u_n - \overline{u}_n||^2_{L^2(\Omega)}.$$

Choosing $\nu$ such that $\frac{\nu}{2} = \alpha + \frac{2c}{\nu}||\overline{u}_n||^2_{H^1(\Omega)}$ and setting $\alpha_0 = \alpha > 0$, we obtain by integration over 0 and $t$

$$||u_n - \overline{u}_n||^2_{L^2(\Omega)} \leq \frac{2}{\alpha_0}\int_0^t ||f_n(v) - f_n(\overline{v})||^2_{L^2(\Omega)}.$$

Since the nonlinearity $f_n$ is locally Lipschitz and it's bounded then $f_n$ is globally Lipschitz which enable us to deduce the existence of a constant $C_{n,T}$ depending only on $n$ and $T$ such that:

$$||f_n(v) - f_n(\overline{v})||_{L^2(Q_T)} \leq C_{n,T}||v - \overline{v}||_{L^2(Q_T)}.$$

Integrating over 0 and $T$, we obtain that

$$||u_n - \overline{u}_n||^2_{L^2(Q_T)} \leq \frac{2TC^2_{n,T}}{\alpha_0}||v - \overline{v}||^2_{L^2(Q_T)}.$$

Finally:

$$||\mathcal{L}_n(v) - \mathcal{L}_n(\overline{v})||_{L^2(Q_T)} \leq \sqrt{\frac{2TC^2_{n,T}}{\alpha_0}}||v - \overline{v}||_{L^2(Q_T)}.$$

2. We set $\mathcal{B} = \{U \in\ L^2(Q_T)/||U||_{L^2(Q_T)} \leq \sqrt{T(\frac{2C_{n,T}^2}{\alpha_0} + ||u_0||^2_{L^2(\Omega)})}\}$. Let's consider $v \in B$ such that: $u_n = \mathcal{L}_n(v)$. We have:

$$||u_n||^2_{L^2(\Omega)} + 2\nu||\nabla u||^2_{L^2(Q_T)} \leq \frac{2}{\alpha}||f_n(v)||^2_{L^2(Q_T)} + 2\alpha||\nabla u||^2_{L^2(Q_T)} + ||u_0||^2_{L^2(\Omega)}.$$

Choosing $\alpha = \nu$, we obtain

$$||u_n||^2_{L^2(\Omega)} \leq \frac{2C_{n,T}^2}{\nu} + ||u_0||^2_{L^2(\Omega)}$$

which implies that:

$$||u_n||_{L^2(Q_T)} \leq \sqrt{T(\frac{2C_{n,T}^2}{\alpha_0} + ||u_0||^2_{L^2(\Omega)})}.$$

Consequently, we conclude that: $\mathcal{L}_n(\mathcal{B}) \subset \mathcal{B}$.

3. Now, we prove that $\mathcal{L}_n$ is a compact operator. Let $v_k \in L^2(Q_T)$ such that: $\mathcal{L}_n(v_k) = u_n^k$. We have:

$$\frac{1}{2}||u_n^k||^2_{L^2(\Omega)} + \nu||\nabla u_n^k||^2_{L^2(Q_T)} \leq \frac{1}{\alpha}||f_n(v_k)||^2_{L^2(Q_T)} + \alpha||u_n^k||^2_{L^2(Q_T)} + \frac{1}{2}||u_0||^2_{L^2(\Omega)}$$

for $\alpha = \frac{\nu}{2}$, we obtain that

$$||\nabla u_n^k||_{L^2(Q_T)} \leq \sqrt{T(\frac{2C_{n,T}^2}{\alpha_0} + ||u_0||^2_{L^2(\Omega)})}.$$

Consequently,

$$||u_n^k||_{L^2(0,T;H^1(\Omega))} \leq C.$$

Otherwise $u_n^k$ satisfies for all $\varphi \in H^1(\Omega)$

$$\int_\Omega \frac{\partial u_n^k}{\partial t}\varphi + \int_\Omega D(J_\rho(\nabla u_{n\sigma}^k))\nabla u_n^k \nabla\varphi = \int_\Omega f_n(v_k)\varphi.$$

Then

$$|\int_\Omega \frac{\partial u_n^k}{\partial t}\varphi| \leq ||f_n(v_k)||_{L^2(\Omega)}||\varphi||_{L^2(\Omega)} + \alpha||u_n^k||_{L^2(\Omega)}||\varphi||_{L^2(\Omega)}.$$

Therefore

$$||\frac{\partial u_n^k}{\partial t}||_{L^2(0,T;H^{-1}(\Omega))} \leq C.$$

So, the sequences $(u_n^k)_{k\geq 0}$ and $(\frac{\partial u_n^k}{\partial t})_{k\geq 0}$ are respectively bounded in $L^2(0,T;H^1(\Omega))$ and $\mathcal{L}^2(0,T;H^{-1}(\Omega))$. Since the embedding $L^2(0,T;H^1(\Omega)) \subset L^2(\Omega_T)$ and $L^2(0,T;H^{-1}(\Omega)) \subset L^2(\Omega_T)$ are compact, then the operator $\mathcal{L}_n$ is compact. Finally, we conclude that the operator $\mathcal{L}_n$ admits a fixed point. □

**Lemma 3.3.** *Let $u_n$ be a weak solution of* (10) *and suppose that $u_0 \geq 0$ in $\Omega$. Then $u_n \geq 0$ in $Q_T$.*

Let us introduce the following function defined on $\mathbb{R}$ by

$$sign^- r = \begin{cases} -1 & r < 0 \quad (12) \\ 0 & r \geq 0 \quad (13) \end{cases}$$

as $sign^-$ is an increasing function, we consider the convex function $j_\epsilon \in C^2(\mathbb{R})$ such that

$$j_\epsilon^{'}(r) \to sign^- r \text{ when } \epsilon \to 0.$$

In the first time, we will prove that the solution $u$ is positive. We consider the following problem:

$$\begin{cases} \frac{\partial u_n}{\partial t} - \operatorname{div}(D(J_\rho(\nabla u_{n\sigma}))\nabla u_n) = f_n(t,x,u_n) & \text{in } ]0,T[\times\Omega \\ u_n(0,x) = u_0^n(x) & \text{in } \Omega \\ < D(J_\rho(\nabla u_{n\sigma}))\nabla u_n, \xi >= 0 & \text{on } ]0,T[\times\partial\Omega \end{cases} \tag{14}$$

We multiply both sides of the first equation by $j_\epsilon^{'}(u_n)$ in (14) and integrating on $Q_T$ we obtain

$$\int\limits_{Q_T} \frac{\partial u_n}{\partial t} j_\epsilon^{'}(u_n) = -\int\limits_{Q_T} D(J_\rho(\nabla u_{n\sigma}))\nabla u_n \nabla(j_\epsilon^{'}(u_n)) + \int\limits_{Q_T} f(u_n) j_\epsilon^{'}(u_n). \tag{15}$$

Let's note by $I_1$ and $I_2$, respectively the two members in the right side of the equality (15). Using the convexity of $j_\epsilon$ and the properties of the diffusion tensor $D$, we deduce for the first integral

$$I_1 = -\int\limits_{Q_T} D(J_\rho(\nabla u_{n\sigma}))|\nabla u_n|^2 j_\epsilon^{''}(u_n)dxdt \leq 0.$$

Concerning the second member $I_2$, we have

$$\lim_{\epsilon\to 0} I_2 = \lim_{\epsilon\to 0} \int\limits_{[u_n\geq 0]} f_n(u_n) j_\epsilon^{'}(u_n) + \lim_{\epsilon\to 0} \int\limits_{[u_n\leq 0]} f_n(u_n) j_\epsilon^{'}(u_n).$$

According to (5) we have

$$\lim_{\epsilon\to 0} I_2 = -\int\limits_{[u\leq 0]} f_n(t,x,u_n) \leq 0.$$

Then

$$\lim_{\epsilon\to 0} \int\limits_{Q_T} \frac{\partial u_n}{\partial t} j_\epsilon^{'}(u_n)dxdt \leq 0$$

consequently

$$\int\limits_0^T \int\limits_\Omega \frac{\partial (u_n)^-}{\partial t} \leq 0$$

which implies

$$\int\limits_\Omega (u_n)^-(t,x) \leq \int\limits_\Omega (u_n)^-(0,x)$$

as $u_0 \geq 0$ almost for every $x \in \Omega$, we deduce that

$$\int\limits_\Omega (u_n)^-(t,x) \leq 0.$$

Since $(u_n)^-(t,x) \geq 0$ we obtain that $(u_n)^-(t,x) = 0$; therefore $u_n \geq 0$.

## 4. A priori estimates

**Lemma 4.1.** *There exists a constant $M$ depending on $||u_0||_{L^1(\Omega)}$ such that:*

$$\int_\Omega u_n(t) \leq M \text{ for all } t \in [0,T]. \tag{16}$$

*Proof.* We have for all $\varphi \in H^1(\Omega)$

$$\int_\Omega \frac{\partial u_n}{\partial t}\varphi - \int_\Omega \operatorname{div}(D(J_\rho(\nabla u_{n\sigma}))\nabla u_n)\varphi = \int_\Omega f_n(t,x,u_n)\varphi$$

taking $\varphi = 1$, we obtain

$$\int_\Omega u_n(t) = \int_\Omega f_n(t,x,u_n) + \int_\Omega u_0$$

and by hypothesis (5), we have

$$\int_\Omega u_n(t) \leq \int_\Omega u_0.$$

□

**Lemma 4.2.** *There exists a constant $R_1$ depending on $T$ and $||u_0||_{L^1(\Omega)}$ such that:*

$$\int_{Q_T} |f_n(t,x,u_n)| \leq R_1 \text{ for all } t \in [0,T] \tag{17}$$

*Proof.* We have

$$\int_\Omega u_n(t) - \int_\Omega f_n(t,x,u_n) = \int_\Omega u_0$$

According to (5) we obtain

$$\int_{Q_T} |f_n(t,x,u_n)| \leq T\int_\Omega |u_0|.$$

□

**Lemma 4.3.** *There exists a constant $R_3$ depending on the $T$ and $||u_0||_{L^1(\Omega)}$ such that:*

$$\int_{Q_T} |u_n f_n(t,x,u_n)| \leq R_3. \tag{18}$$

*Proof.* We have

$$\frac{1}{2}||u_n||^2_{L^2(\Omega)} + \nu||\nabla u_n||^2_{L^2(Q_T)} - \int_{Q_T} u_n f_n(t,x,u_n) \leq \frac{1}{2}||u_0||^2_{L^2(\Omega)}$$

then

$$\int_{Q_T} |u_n f_n(t,x,u_n)| \leq \frac{1}{2}||u_0||^2_{L^2(\Omega)}.$$

□

## 5. Convergence

Our objective is to show that: $u_n$ converges to some $u$ solution of the problem (3). We have by Lemma 1, the existence of a subsequence noted also $u_n \in L^2(0,T;H^1(\Omega))$ and $u \in L^2(0,T;H^1(\Omega))$ such that:

$$\begin{gathered} u_n \rightharpoonup u \ \text{ in } L^2(0,T;H^1(\Omega)) \\ u_n \to u \ \text{ in } L^2(\Omega_T) \\ \frac{\partial u_n}{\partial t} \rightharpoonup \frac{\partial u}{\partial t} \ \text{ in } L^2(0,T;H^{-1}(\Omega)) \\ u_n * \nabla K_\sigma \to u * \nabla K_\sigma \ \text{ in } L^2(\Omega_T) \\ \nabla u_{n\sigma} \otimes \nabla u_{n\sigma} \to \nabla u_\sigma \otimes \nabla u_\sigma \ \text{ in } L^2(\Omega_T) \\ J_\rho(\nabla u_{n\sigma}) \to J_\rho(\nabla u_\sigma) \ \text{ in } L^2(\Omega_T) \\ D(J_\rho(\nabla u_{n\sigma})) \to D(J_\rho(\nabla u_\sigma)) \ \text{ in } L^2(\Omega_T). \end{gathered}$$

We have by the hypothesis (6) that

$$f_n(t,x,u_n) \to f(t,x,u) \ \text{ a.e in } Q_T.$$

It remains to show that $(f_n)_n$ are equi-integrable in $L^1(Q_T)$. For this we show that: for each $\epsilon > 0$, there exists $\delta > 0$ such that for all $A \subset Q_T$ measurable with $|A| < \delta$, we have

$$\int_A |f_n(t,x,u_n)| dx dt \le \epsilon.$$

Let $A$ be a measurable subset of $Q_T$, $\epsilon > 0$ and $k > 0$. We have:

$$\int_A |f_n(t,x,u_n)| = \int_{A\cap[u_n \le k]} |f_n(t,x,u_n)| + \int_{A\cap[u_n \ge k]} |f_n(t,x,u_n)|.$$

For the first term on the right-hand side, we have

$$\int_{A\cap[u_n \le k]} |f_n(t,x,u_n)| \le \int_A \sup_{|r|\le k} |f(t,x,r)| dx$$

We have $\sup_{|r|\le k} |f(t,x,r)| \in L^1(Q_T)$ is uniformly integrable in $L^1(Q_T)$, therefore for each $\epsilon > 0$ there exist $\delta > 0$ such that if $|E| < \delta$ then

$$\int_A \sup_{|u|\le k} |f(t,x,u)| dx \le \frac{\epsilon}{2}$$

Which implies that

$$\int_{A\cap[u_n \le k]} |f_n(t,x,u_n)| \le \frac{\epsilon}{2}$$

On the other hand, we have

$$\int_{A\cap[u_n \ge k]} |f_n(t,x,u_n)| \le \frac{1}{k} \int_{Q_T} |u_n f_n(t,x,u_n)|$$

Using Lemma 4.3, we obtain

$$\int_{A\cap[u_n \ge k]} |f_n(t,x,u_n)| \le \frac{1}{2k} ||u_0||^2_{L^2(\Omega)}$$

If we choose $k \ge \frac{2||u_0||^2_{L^2(\Omega)}}{\epsilon}$, we have

$$\int_{A\cap[u_n \ge k]} |f_n(t,x,u_n)| \le \frac{\epsilon}{2}.$$

Finally, $\int_A |f_n(t,x,u_n)| \leq \epsilon$. This completes the proof.

## 6. Numerical Simulation

Equation can be solved numerically using finite differences. Spatial derivatives are usually replaced by central differences, while the easiest way to discretize $\frac{\partial u}{\partial t}$ consists of using a forward difference approximation. The resulting so-called explicit scheme allows to calculate all values at a new time level directly from the ones in the previous level without solving linear or non linear systems of equations. An explicit scheme has the basic structure

$$\frac{u_{i,j}^{k+1} - u_{i,j}^{k}}{\tau} = A_{i,j}^{k} * u_{i,j}^{k} + f(u_{i,j}^{k})$$

where $\tau$ is the time step size and $u_{i,j}^k$ denotes the approximation of $u(x,t)$ in the pixel $(i,j)$ at time $k\tau$. The expression $A_{i,j}^k * u_{i,j}^k$ is a discretization of $\nabla(D\nabla u)$. The stencil notation of the nonnegative discretization for $A_{i,j}^k$ are shown in Figure 1.

| | | |
|---|---|---|
| $\frac{\lvert b_{i-1,j+1}\rvert - b_{i-1,j+1}}{4h_1h_2} + \frac{\lvert b_{i,j}\rvert - b_{i,j}}{4h_1h_2}$ | $\frac{c_{i,j+1}+c_{i,j}}{2h_2^2} - \frac{\lvert b_{i,j+1}\rvert + \lvert b_{i,j}\rvert}{2h_1h_2}$ | $\frac{\lvert b_{i+1,j+1}\rvert + b_{i+1,j+1}}{4h_1h_2} + \frac{\lvert b_{i,j}\rvert + b_{i,j}}{4h_1h_2}$ |
| $\frac{a_{i-1,j}+a_{i,j}}{2h_1^2} - \frac{\lvert b_{i-1,j}\rvert + \lvert b_{i,j}\rvert}{2h_1h_2}$ | $-\frac{a_{i-1,j}+2a_{i,j}+a_{i+1,j}}{2h_1^2} - \frac{\lvert b_{i-1,j+1}\rvert - b_{i-1,j+1} + \lvert b_{i+1,j+1}\rvert + b_{i+1,j+1}}{4h_1h_2} - \frac{\lvert b_{i-1,j-1}\rvert + b_{i-1,j-1} + \lvert b_{i+1,j-1}\rvert - b_{i+1,j-1}}{4h_1h_2} + \frac{\lvert b_{i-1,j}\rvert + \lvert b_{i+1,j}\rvert + \lvert b_{i,j-1}\rvert + \lvert b_{i,j+1}\rvert + 2\lvert b_{i,j}\rvert}{2h_1h_2} - \frac{c_{i,j-1}+2c_{i,j}+c_{i,j+1}}{2h_2^2}$ | $\frac{a_{i+1,j}+a_{i,j}}{2h_1^2} - \frac{\lvert b_{i+1,j}\rvert + \lvert b_{i,j}\rvert}{2h_1h_2}$ |
| $\frac{\lvert b_{i-1,j-1}\rvert + b_{i-1,j-1}}{4h_1h_2} + \frac{\lvert b_{i,j}\rvert + b_{i,j}}{4h_1h_2}$ | $\frac{c_{i,j-1}+c_{i,j}}{2h_2^2} - \frac{\lvert b_{i,j-1}\rvert + \lvert b_{i,j}\rvert}{2h_1h_2}$ | $\frac{\lvert b_{i+1,j-1}\rvert - b_{i+1,j-1}}{4h_1h_2} + \frac{\lvert b_{i,j}\rvert - b_{i,j}}{4h_1h_2}$ |

FIGURE 1. Nonnegative discretization.

We use an objective criterion giving an idea of the quality of the image filtered and enhanced compared with that reference image. In general, the PSNR is used in the image restoration to validate the filtering model used. But in the enhancement therefore, this criterion is adapted as follows

$$MSE = \frac{1}{MN}\sum_{i=1}^{M}\sum_{j=1}^{N}[U(i,j) - U_0(i,j)]^2$$

$$PSNR = 10Log_{10}\left(\frac{255^2}{MSE}\right)$$

where $U$ is the filtered image with enhancement and $U_0$ is the reference image.

The evaluation results provided by the methods of enhancement are difficult to compare and that only the naked eye can detect the difference between the methods of image enhancement. The choice of one image enhancement technique over another is completely subjective and depends used in comparing the image enhancement techniques is called measure of enhancement $EME$ or measure of improvement. Let an image $I(N, M)$ be split into $k_1k_2$ blocks $w_{k,l}(i, j)$ of sizes $l_1l_2$ then we define

$$EME = \frac{1}{k_1k_2}\sum_{l=1}^{k_1}\sum_{k=1}^{k_2} 20log(\frac{I^w_{max;k,l}}{I^w_{min;k,l}})$$

where $I^w_{max;k,l}$ and $I^w_{min;k,l}$ are respectively maximum and minimum values of the image $I(N, M)$ inside the block $w_{k,l}$. The higher the value of $EME$, the higher the image contrast and information clarity in the image.

**Experiments Results:**

We consider a blurred image by a Gaussian white noise with variance $\sigma = 0.01$ and the initial PSNR=28.75.

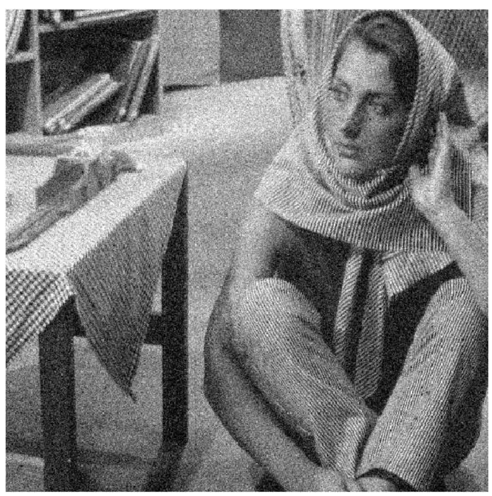

Figure 2. Barbara image.

In Figures 3 and 4, we show the results obtained using Weickert and the proposed models on Barbara image where there is a high presence of textures combined with nontextured parts.

In order to show the robustness of the proposed method, we tested a color image as shown in Figure 5.

We note that in Figure 7 the new model separates better the textured details from the larger regions: the small textured details are in the texture component, while the larger regions are kept in the cartoon component. Using Weickert model (Fig. 6), small textured details are still kept in the $u$ component, while contours of larger

FIGURE 3. Cartoon and Texture part by Weickert model with PSNR=31.16 and EME=2.77.

FIGURE 4. Cartoon and Texture part by Our model with PSNR=34.46 and EME=5.36

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regions can be showed in the $u_0 - u$ component. Therefore, using this model, we could not separate texture and non texture parts very well. Indeed, if we look to the $u_0 - u$ components from Fig. 3, we still see the hands and the hair in the result produced by the Weickert model. These are not seen in the $u_0 - u$ component produced by Our model (Fig. 4).

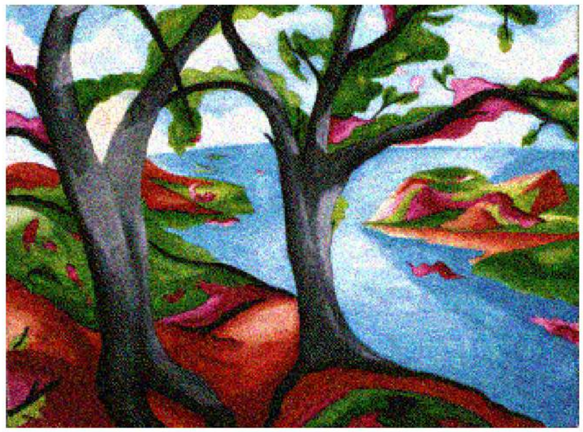

FIGURE 5. Tree image blurred by a Gaussian White noise of variance 0.01.

FIGURE 6. Cartoon and Texture part by Weickert model with PSNR=67.25 and EME=5.44.

FIGURE 7. Cartoon and Texture part by Our model with PSNR=64.91 and EME=6.73.

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