Hindawi
International Journal of Mathematics and Mathematical Sciences
Volume 2020, Article ID 5689458, 9 pages
https://doi.org/10.1155/2020/5689458

Research Article

θ-ϕ-Contraction on (α, η)-Complete Rectangular b-Metric Spaces

Abdelkarim Kari,[1] Mohamed Rossafi ,[2] El Miloudi Marhrani,[1] and Mohamed Aamri[1]

[1]*Laboratory of Algebra, Analysis and Applications Faculty of Sciences Ben M'Sik, Hassan II University, Casablanca, Morocco*
[2]*Department of Mathematics, University of Ibn Tofail, B.P. 133 Kenitra, Morocco*

Correspondence should be addressed to Mohamed Rossafi; rossafimohamed@gmail.com

Received 31 May 2020; Revised 9 July 2020; Accepted 14 July 2020; Published 1 August 2020

Academic Editor: Nageswari Shanmugalingam

In this paper, we present some fixed point results for generalized θ-ϕ-contraction in the framework of $(\alpha, \eta)-$compete rectangular b-metric spaces. Further, we establish some fixed point theorems for this type of mappings defined on such spaces. Our results generalize and improve many of the well-known results. Moreover, to support our main results, we give an illustrative example.

1. Introduction

The well-known Banach contraction theory is one of the methods used, which states that if (X, d) is a complete metric space and $T: X \longrightarrow X$ is self-mapping with contraction, then T has a unique fixed point [1].

In 2000, Branciari [2] introduced the notion of generalized metric spaces, for example, the triangle inequality is replaced by the inequality $d(x, y) \leq d(x, u) + d(u, v) + d(v, y)$ for all pairwise distinct points x, y, u, and $v \in X$. Since then, several results have been proposed by many mathematicians on such spaces (see [3–8]).

The concept of metric space, as an ambient space in fixed point theory, has been generalized in several directions, such as, $b-$metric spaces [9] and generalized metric spaces.

Combining conditions are used for definitions of b-metric and generalized metric spaces. Roshan et al. [10] announced the notion of rectangular b-metric space.

Hussain et al. [11] introduced the concept of $\alpha-\eta-$complete rectangular $b-$metric space and proved certain results of fixed point theory on such spaces.

In this paper, we provide some fixed point results for generalized $\theta-\phi-$contraction in the framework of $(\alpha, \eta)-$compete rectangular $b-$metric spaces, and also we give two examples to support our results.

2. Preliminaries

Definition 1 (see[10]). Let X be a nonempty set, $s \geq 1$ be a given real number, and let $d: X \times X \longrightarrow [0, +\infty[$ be a mapping such that for all $x, y \in X$ and all distinct points $u, v \in X$, each distinct from x and y:

(1) $d(x, y) = 0$, if only if $x = y$
(2) $d(x, y) = d(y, x)$
(3) $d(x, y) \leq s[d(x, u) + d(u, v) + d(v, y)]$ ($b-$rectangular inequality)

Then, (X, d) is called a $b-$rectangular metric space.

Lemma 1 (see [10]). *Let (X, d) be a rectangular b-metric space.*

(a) *Suppose that sequences (x_n) and (y_n) in X are such that $x_n \longrightarrow x$ and $y_n \longrightarrow y$ as $n \longrightarrow \infty$, with $x \neq y$, $x_n \neq x$, and $y_n \neq y$ for all $n \in \mathbb{N}$. Then, we have*

$$\frac{1}{s} d(x, y) \leq \lim_{n\longrightarrow\infty} \inf d(x_n, y_n) \leq \lim_{n\longrightarrow\infty} \sup d(x_n, y_n) \leq s d(x, y). \tag{1}$$

(b) *If $y \in X$ and (x_n) is a Cauchy sequence in X with $x_n \neq x_m$ for any $m, n \in \mathbb{N}$, $m \neq n$, converging to $x \neq y$, then*

$$\frac{1}{s} d(x, y) \leq \lim_{n \longrightarrow \infty} \inf d(x_n, y) \leq \lim_{n \longrightarrow \infty} \sup d(x_n, y) \leq s d(x, y), \tag{2}$$

for all $x \in X$.

Zheng et al. [12] introduced a new type of contractions called $\theta - \phi$-contractions in metric spaces and proved a new fixed point theorems for such mapping.

Definition 2 (see [13]). We denote by Θ the set of functions θ: $]0, \infty \longrightarrow 1, \infty[$, satisfying the following conditions:

(θ_1) θ is increasing

(θ_2) for each sequence $(x_n) \in \ ]0, \infty[, \lim_{n \longrightarrow \infty} \theta(x_n) = 1 \Leftrightarrow \lim_{n \longrightarrow \infty} x_n = 0$

(θ_3) θ is continuous on $]0, \infty[$

Definition 3 (see [12]). We denote by Φ the set of functions ϕ: $[1, \infty[\longrightarrow [1, \infty[$ satisfying the following conditions:

(Φ_1) ϕ: $[1, \infty[\longrightarrow [1, \infty[$ is nondecreasing

(Φ_2) for each $t > 1$, $\lim_{n \longrightarrow \infty} \phi^n(t) = 1$

(Φ_3) ϕ is continuous on $[1, \infty[$

Lemma 2 (see [12]). *If $\phi \in \Phi$, then $\phi(1) = 1$ and $\varphi(t) < t$ for each $t > 1$.*

In 2014, Hussain et al. [14] introduced a weaker notion than the concept of completeness and called it α-completeness for metric spaces.

Definition 4 (see [14]). Let T: $X \longrightarrow X$ and α, η :$X \times X \longrightarrow [0, +\infty[$. We say that T is a triangular (α, η)–admissible mapping if

$(T_1) \alpha(x, y) \geq 1 \Longrightarrow \alpha(Tx, Ty) \geq 1,\ x, y \in X$

$(T_2) \eta(x, y) \leq 1 \Longrightarrow \eta(Tx, Ty) \leq 1,\ x, y \in X$

$$(T_3) \begin{cases} \alpha(x, y) \geq 1 \\ \alpha(y, z) \geq 1 \end{cases} \Longrightarrow \alpha(x, z) \geq 1 \text{ for all } x, y, z \in X$$

$$(T_4) \begin{cases} \eta(x, y) \leq 1 \\ \eta(y, z) \leq 1 \end{cases} \Longrightarrow \eta(x, z) \leq 1 \text{ for all } x, y, z \in X$$

Definition 5 (see [14]). Let (X, d) be a b-rectangular metric space and let α, η:$X \times X \longrightarrow [0, +\infty[$ be two mappings. The space is said to be as follows:

(a) T is α–continuous mapping on (X, d), if for given point $x \in X$ and sequence (x_n) in X, $x_n \longrightarrow x$ and $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$ imply that $Tx_n \longrightarrow Tx$.

(b) T is η subcontinuous mapping on (X, d), if for given point $x \in X$ and sequence (x_n) in X, $x_n \longrightarrow x$ and $\eta(x_n, x_{n+1}) \leq 1$ for all $n \in \mathbb{N}$ imply that $Tx_n \longrightarrow Tx$.

(c) T is (α, η)–continuous mapping on (X, d), if for given point $x \in X$ and sequence (x_n) in X, $x_n \longrightarrow x$ and $\alpha(x_n, x_{n+1}) \geq 1$ or $\eta(x_n, x_{n+1}) \leq 1$ for all $n \in \mathbb{N}$ imply that $Tx_n \longrightarrow Tx$.

The following definitions were given by Hussain et al. [11].

Definition 6 (see [11]). Let $d(X, d)$ be a rectangular b-metric space and let α, η:$X \times X \longrightarrow [0, +\infty[$ be two mappings. The space X is said to be

(a) α–complete, if every Cauchy sequence (x_n) in X with $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$ converges in X

(b) η – subcomplete, if every Cauchy sequence (x_n) in X with $\eta(x_n, x_{n+1}) \leq 1$ for all $n \in \mathbb{N}$ converges in X

(c) (α, η)–complete, if every Cauchy sequence (x_n) in X with $\alpha(x_n, x_{n+1}) \geq 1$ or $\eta(x_n, x_{n+1}) \leq 1$ for all $n \in \mathbb{N}$ converges in X

Definition 7 (see [11]). Let (X, d) be a rectangular b-metric space and let α, η:$X \times X \longrightarrow [0, +\infty[$ be two mappings. The space X is said to be

(a) (X, d) is α-regular, if $x_n \longrightarrow x$, where $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$ implies $\alpha(x_n, x) \geq 1$ for all $n \in \mathbb{N}$

(b) (X, d) is η–subregular, if $x_n \longrightarrow x$, where $\eta(x_n, x_{n+1}) \leq 1$ for all $n \in \mathbb{N}$ implies $\eta(x_n, x) \leq 1$ for all $n \in \mathbb{N}$

(c) (X, d) is (α, η)-regular, if $x_n \longrightarrow x$, where $\alpha(x_n, x_{n+1}) \geq 1$ or $\eta(x_n, x_{n+1}) \leq 1$ for all $n \in \mathbb{N}$ implies that $\alpha(x_n, x) \geq 1$ or $\eta(x_n, x) \leq 1$ for all $n \in \mathbb{N}$.

3. Main Results

Definition 8. Let $d(X, d)$ be a (α, η)-rectangular b-metric space with parameter $s > 1$ and let T be a self-mapping on X. Suppose that α, η: $X \times X \longrightarrow [0, +\infty[$ are two functions. We say that T is an $(\alpha, \eta) - \theta - \phi-$ contraction, if for all $x, y \in X$ with $(\alpha(x, y) \geq 1$ or $\eta(x, y) \leq 1)$ and $d(Tx, Ty) > 0$, we have

$$\theta\left(s^2 d(Tx, Ty)\right) \leq \phi\left[\theta\left(\beta_1 d(x, y) + \beta_2 d(Tx, x) + \beta_3 d(Ty, y) + \beta_4 d(y, Tx)\right)\right], \tag{3}$$

where $\theta \in \Theta, \phi \in \Phi$, $\beta_i \geq 0$ for $i \in \{1, 2, 3, 4\}$, $\sum_{i=0}^{i=4} \beta_i \leq 1$, and $\beta_3 < (1/s)$.

Definition 9. Let (X, d) be a (α, η)-complete rectangular b–metric space and T: $X \longrightarrow X$ be a mapping.

(1) T is said to be a $\theta - \phi$–Kannan-type contraction if there exist $\theta \in \Theta$ and $\phi \in \Phi$ with $\alpha(x, y) \geq 1$ or $\eta(x, y) \leq 1$ for any $x, y \in X$, $Tx \neq Ty$, we have

$$\theta\left[s^2 d(Tx, Ty)\right] \leq \phi\left[\theta\left(\frac{(d(Tx, x) + d(Ty, y))}{2s}\right)\right]. \tag{4}$$

(2) T is said to be a $\theta-\phi$–Reich-type contraction if there exist $\theta \in \Theta$ and $\phi \in \Phi$ with $\alpha(x,y) \geq 1$ or $\eta(x,y) \leq 1$ for any $x, y \in X$, $Tx \neq Ty$, we have

$$\theta\left[s^2 d(Tx,Ty)\right] \leq \phi\left[\theta\left(\frac{d(x,y)+d(Tx,x)+d(Ty,y)}{3s}\right)\right]. \quad (5)$$

(3) T is said to be a Kannan-type mapping, that is, if there exists $\alpha \in \,]0,(1/2s)[$ with $\alpha(x,y) \geq 1$ or $\eta(x,y) \leq 1$ for any $x, y \in X$, $Tx \neq Ty$, we have

$$s^2 d(Tx,Ty) \leq \alpha(d(Tx,x)+d(y,Ty)). \quad (6)$$

(4) T is said to be a Reich-type mapping, that is, if there exists $\lambda \in \,]0,(1/3s)[$ with $\alpha(x,y) \geq 1$ or $\eta(x,y) \leq 1$ for any $x, y \in X$, $Tx \neq Ty$, we have

$$s^2 d(Tx,Ty) \leq \lambda[d(x,y)+d(Tx,x)+d(Ty,y)]. \quad (7)$$

Theorem 1. *Let (X,d) be a (α,η)–complete rectangular b-metric and let α,η: $X \times X \longrightarrow [0,+\infty[$ be two functions. Let T: $X \times X \longrightarrow X$ be a self-mapping satisfying the following conditions:*

(i) T is a triangular (α,η)–admissible mapping

(ii) T is an $(\alpha,\eta)-\theta-\phi$–contraction

(iii) There exists $x_0 \in X$ such that $\alpha(x_0,Tx_0) \geq 1$ or $\eta(x_0,Tx_0) \leq 1$

(iv) T is a (α,η)–continuous.

Then T has a fixed point. Moreover, T has a unique fixed point when $\alpha(x,y) \geq 1$ or $\eta(x,y) \leq 1$ for all $x, y \in X$.

Proof. Let $x_0 \in X$ such that $\alpha(x_0,Tx_0) \geq 1$ or $\eta(x_0,Tx_0) \leq 1$.

Define a sequence $\{x_n\}$ by $x_n = T^n x_0 = Tx_{n-1}$. Since T is a triangular (α,η)–admissible mapping, then $\alpha(x_0,x_1) = \alpha(x_0,Tx_0) \geq 1 \Longrightarrow \alpha(Tx_0,Tx_1) \geq 1 = \alpha(x_1,x_2)$ or $\eta(x_0,x_1) = \eta(x_0,Tx_0) \leq 1 \Longrightarrow \alpha(Tx_0,Tx_1) \leq 1 = \alpha(x_1,x_2)$.

Continuing this process, we have $\alpha(x_{n-1},x_n) \geq 1$ or $\eta(x_{n-1},x_n) \leq 1$, for all $n \in \mathbb{N}$. By (T_3) and (T_4), one has

$$\alpha(x_m,x_n) \geq 1 \quad \text{or} \quad \eta(x_m,x_n) \leq 1, \quad \forall \text{m}, \quad \text{n} \in \mathbb{N}, \quad \text{m} \neq \text{n}. \quad (8)$$

Suppose that there exists $n_0 \in \mathbb{N}$ such that $x_{n_0} = Tx_{n_0}$. Then, x_{n_0} is a fixed point of T and the proof is finished. Hence, we assume that $x_n \neq Tx_n$, i.e., $d(x_{n-1},x_n) > 0$ for all $n \in \mathbb{N}$. We have

$$x_n \neq x_m, \quad \forall m, \quad n \in \mathbb{N}, \quad m \neq n. \quad (9)$$

Indeed, suppose that $x_n = x_m$ for some $m = n+k > n$, so we have

$$x_{n+1} = Tx_n = Tx_m = x_{m+1}. \quad (10)$$

Denote $d_m = d(x_m,x_{m+1})$. Then, (3) and Lemma 2 imply that

$$\begin{aligned}\theta(d_n) = \theta(d_m) &\leq \theta\left(s^2 d_m\right) = \theta\left(s^2 d(Tx_{m-1},Tx_m)\right)\\ &\leq \phi(\theta(\beta_1 d_{m-1}+\beta_2 d_{m-1}+\beta_3 d_m))\\ &< \theta(\beta_1 d_{m-1}+\beta_2 d_{m-1}+\beta_3 d_m).\end{aligned} \quad (11)$$

As θ is increasing, so

$$d_n = d_m < \beta_1 d_{m-1}+\beta_2 d_{m-1}+\beta_3 d_m. \quad (12)$$

Hence,

$$d_m < \frac{\beta_1+\beta_2}{1-\beta_3} d_{m-1}. \quad (13)$$

Since

$$\beta_1+\beta_2+\beta_3 \leq 1. \quad (14)$$

Thus

$$d_m < d_{m-1}. \quad (15)$$

Continuing this process, we can prove that $d_n = d_m < d_n$, which is a contradiction. Thus, in the following, we can assume that (8) and (9) hold.

We shall prove that

$$\lim_{n\longrightarrow+\infty} d(x_n,x_{n+1}) = 0 \quad \text{and} \quad \lim_{n\longrightarrow+\infty} d(n_n,x_{n+2}) = 0. \quad (16)$$

Since $T$is $(\alpha,\eta)-\theta-\phi$–contraction, we get

$$\begin{aligned}\theta(d_n) = \theta(d(Tx_{n-1},Tx_n)) &\leq \theta\left(s^2 d(Tx_{n-1},Tx_n)\right)\\ &\leq \phi(\theta(\beta_1 d_{n-1}+\beta_2 d_{n-1}+\beta_3 d_n))\\ &< \theta(\beta_1 d_{n-1}+\beta_2 d_{n-1}+\beta_3 d_n).\end{aligned} \quad (17)$$

Since θ is increasing, we deduce that $d_n < \beta_1 d_{n-1}+\beta_2 d_{n-1}+\beta_3 d_n$, and thus

$$d_n < \frac{\beta_1+\beta_2}{1-\beta_3} d_{n-1}. \quad (18)$$

Since $\beta_1+\beta_2/1-\beta_3 \leq 1$, then

$$d_n < d_{n-1}. \quad (19)$$

Therefore, $d(x_n,x_{n+1})$ is monotone strictly decreasing sequence of nonnegative real numbers. Consequently, there exists $\alpha \geq 0$, such that

$$\lim_{n\longrightarrow+\infty} d(x_n,x_{n+1}) = \alpha, \quad (20)$$

which again by (3) and (19) and property of (θ), we have

$$\begin{aligned}1 < \theta(d_n) &\leq \phi(\theta(\beta_1 d_{n-1}+\beta_2 d_{n-1}+\beta_3 d_n))\\ &\leq \phi(\theta(d_{n-1})) \leq \phi^2(\theta(d_{n-2})) \leq \ldots \leq \phi^n(\theta(d_0))\\ &= \phi^n(\theta(d(x_0,x_1))).\end{aligned} \quad (21)$$

By taking the limit as $n \longrightarrow \infty$ in (21) and using (Φ_2), we have

$$1 \leq \lim_{n \longrightarrow +\infty} \theta(d(x_n, x_{n+1})) \leq \phi^n (\theta(d(x_0, x_1))). \tag{22}$$

Then, $\lim_{n \longrightarrow +\infty} \theta(d(x_n, x_{n+1})) = 1$, by Θ_2, we obtain

$$\lim_{n \longrightarrow +\infty} d(x_n, x_{n+1}) = 0. \tag{23}$$

On the other hand,

$$\begin{aligned} \theta\left(s^2 d(x_n, x_{n+2})\right) &\leq \phi[\theta(\beta_1 d(x_{n-1}, x_{n+1}) + \beta_2 d(x_{n-1}, x_n) + \beta_3 d(x_{n+1}, x_{n+2}) + \beta_4 d(x_{n+1}, x_n))] \\ &\leq \phi[\theta(s\beta_1 d(x_{n-1}, x_{n+2}) + s\beta_1 d(x_{n+2}, x_n) + s\beta_1 d(x_n, x_{n+1}) + \beta_2 d(x_{n-1}, x_n) + \beta_3 d(x_{n+1}, x_{n+2}) + \beta_4 d(x_{n+1}, x_n))], \\ &\leq \phi\left[\theta\left(s^2\beta_1 d(x_{n-1}, x_n) + s^2\beta_1 d(x_n, x_{n+1}) + s^2\beta_1 d(x_{n+1}, x_{n+2}) + s\beta_1 d(x_{n+2}, x_n)\right) + s\beta_1 d(x_n, x_{n+1}) \right. \\ &\quad \left. + \beta_2 d(x_{n-1}, x_n) + \beta_3 d(x_{n+1}, x_{n+2}) + \beta_4 d(x_{n+1}, x_n)\right]. \end{aligned} \tag{24}$$

By θ_1 and Lemma 2, we obtain

$$\begin{aligned} s^2 d(x_n, x_{n+2}) < s^2\beta_1 d(x_{n-1}, x_n) + s^2\beta_1 d(x_n, x_{n+1}) + s^2\beta_1 d(x_{n+1}, x_{n+2}) + s\beta_1 d(x_{n+2}, x_n) \\ + s\beta_1 d(x_n, x_{n+1}) + \beta_2 d(x_{n-1}, x_n) + \beta_3 d(x_{n+1}, x_{n+2}) + \beta_4 d(x_{n+1}, x_n). \end{aligned} \tag{25}$$

Therefore,

$$\begin{aligned} \left(s^2 - s\beta_1\right) d(x_n, x_{n+2}) < s^2\beta_1 d(x_{n-1}, x_n) + s^2\beta_1 d(x_n, x_{n+1}) + s\beta_1 d(x_{n+1}, x_{n+2}) \\ + s\beta_1 d(x_n, x_{n+1}) + \beta_2 d(x_{n-1}, x_n) + \beta_3 d(x_{n+1}, x_{n+2}) + \beta_4 d(x_{n+1}, x_n). \end{aligned} \tag{26}$$

Taking the limit as $n \longrightarrow \infty$ in (28) and using (23), since $s^2 - s\beta_1 > 0$, we have

$$\lim_{n \longrightarrow +\infty} d(x_n, x_{n+2}) = 0. \tag{27}$$

Hence, (16) is proved.

Next, we show that $\{x_n\}$ is an Cauchy sequence in X, if otherwise there exists an $\varepsilon > 0$ for which we can find sequences of positive integers $\{x_{n_{(k)}}\}_k$ and $\{x_{m_{(k)}}\}_k$ of (x_n) such that, for all positive integers k, $n_{(k)} > m_{(k)} > k$,

$$d\left(x_{m_{(k)}}, x_{n_{(k)}}\right) \geq \varepsilon, \tag{28}$$

$$d\left(x_{m_{(k)}}, x_{n_{(k)-1}}\right) < \varepsilon. \tag{29}$$

From (30) and using the rectangular inequality, we get

$$\begin{aligned} \varepsilon \leq d\left(x_{m_{(k)}}, x_{n_{(k)}}\right) &\leq sd\left(x_{m_{(k)}}, x_{m_{(k)+1}}\right) + sd\left(x_{m_{(k)+1}}, x_{n_{(k)+1}}\right) \\ &\quad + sd\left(x_{n_{(k)+1}}, x_{n_{(k)}}\right). \end{aligned} \tag{30}$$

Taking the upper limit as $k \longrightarrow \infty$ in (32) and using (16), we get

$$\frac{\varepsilon}{s} \lim_{n \longrightarrow +\infty} \sup d\left(x_{m_{(k)+1}}, x_{n_{(k)+1}}\right). \tag{31}$$

Moreover,

$$\begin{aligned} d\left(x_{m_{(k)}}, x_{n_{(k)}}\right) &\leq sd\left(x_{m_{(k)}}, x_{n_{(k)-1}}\right) + sd\left(x_{n_{(k)-1}}, x_{n_{(k)+1}}\right) \\ &\quad + sd\left(x_{n_{(k)+1}}, x_{n_{(k)}}\right). \end{aligned} \tag{32}$$

Then, from (23) and (31), we get

$$\lim_{n \longrightarrow +\infty} \sup d\left(x_{m_{(k)}}, x_{n_{(k)}}\right) \leq s\varepsilon. \tag{33}$$

On the other hand, we have

$$\begin{aligned} d\left(x_{n_{(k)}}, x_{m_{(k)+1}}\right) &\leq sd\left(x_{n_{(k)}}, x_{n_{(k)-1}}\right) + sd\left(x_{n_{(k)-1}}, x_{m_{(k)}}\right) \\ &\quad + sd\left(x_{x_{m_{(k)}}}, x_{m_{(k)+1}}\right). \end{aligned} \tag{34}$$

Then, from (23) and 31 we get

$$\lim_{n \longrightarrow +\infty} \sup d\left(x_{n_{(k)}}, x_{m_{(k)+1}}\right) \leq s\varepsilon. \tag{35}$$

Applying (3) with $x = x_{m_{(k)}}$ and $y = x_{n_{(k)}}$, we have

$$\begin{aligned} \theta\left(s^2 d\left(x_{m_{(k)+1}}, x_{n_{(k)+1}}\right)\right) &= \theta\left(s^2 d\left(Tx_{m_{(k)}}, Tx_{n_{(k)}}\right)\right) \\ &\leq \phi\left(\theta\left(\begin{array}{c} \beta_1 d\left(x_{m_{(k)}}, x_{n_{(k)}}\right) + \beta_2 d\left(x_{m_{(k)}}, Tx_{m_{(k)}}\right) \\ +\beta_3 d\left(x_{n_{(k)}}, Tx_{n_{(k)}}\right) + \beta_4 d\left(x_{n_{(k)}}, Tx_{m_{(k)}}\right) \end{array}\right)\right). \end{aligned} \tag{36}$$

Now taking the upper limit as $k \longrightarrow \infty$ in (38) and using (θ_1), (θ_3), (ϕ_3), (23), (33), (35), (37), and Lemma 2, we have

$$\begin{aligned}\theta\left(s^2.\frac{\varepsilon}{s}\right) = \theta(\varepsilon.s) &\leq \theta\left(s^2 \limsup_{k\longrightarrow+\infty} d\left(x_{m_{(k)+1}}, x_{n_{(k)+1}}\right)\right)\\ &\leq \phi\Big(\theta\Big(\beta_1 \limsup_{k\longrightarrow+\infty} d\left(x_{m_{(k)}}, x_{n_{(k)}}\right)\\ &\quad + \beta_2 \limsup_{k\longrightarrow+\infty} d\left(x_{m_{(k)}}, Tx_{m_{(k)}}\right)\\ &\quad + \beta_3 \limsup_{k\longrightarrow+\infty} d\left(x_{n_{(k)}}, Tx_{n_{(k)}}\right)\\ &\quad + \beta_4 \limsup_{k\longrightarrow+\infty} d\left(x_{n_{(k)}}, Tx_{m_{(k)}}\right)\Big)\Big)\\ &\leq \phi(\theta(\beta_1 s\varepsilon + \beta_4 s\varepsilon)) < \theta(s\varepsilon(\beta_1 + \beta_4)).\end{aligned} \tag{37}$$

Therefore, $\varepsilon.s < s\varepsilon(\beta_1 + \beta_4)$ implies $s < \beta_1 + \beta_4$, which is a contradiction.

Consequently, $\{x_n\}$ is a Cauchy sequence in $\alpha - \eta$–complete rectangular b-metric space (X, d). Since $\alpha(x_{n-1}, x_n) \geq 1$ or $\eta(x_{n-1}, x_n) \leq 1$, for all $n \in \mathbb{N}$.

This implies that the sequence $\{x_n\}$ converges to some $z \in X$. Suppose that $z \neq Tz$. Then, we have all the assumption of Lemma 1 and T is (α, η)–continuous, then $Tx_n \longrightarrow Tz$ as $n \longrightarrow \infty$. Therefore,

$$\frac{1}{s} d(z, Tz) \leq \limsup_{n\longrightarrow+\infty} d(x_n, Tx_n) = 0. \tag{38}$$

Hence, we have $d(z, Tz) = 0$ and so $Tz = z$. Thus, z is a fixed point of T. □

3.1. Uniqueness. Let $z, u \in$ Fix (T) where $z \neq u$ and $\alpha(z, u) \geq 1$ or $\eta(z, u) \leq 1$.

Applying (3) with $x = z$ and $y = u$, we have

$$\begin{aligned}\theta(d(z, u)) = \theta(d(Tz, Tu)) &\leq \theta\left(s^2 d(Tz, Tu)\right)\\ &\leq \phi(\theta(\beta_1 d(z, u) + \beta_2 d(z, Tz) + \beta_3 d(u, Tu)\\ &\quad + \beta_4 d(Tz, u)))\\ &\leq \phi(\theta(\beta_1 d(z, u) + \beta_4 d(Tz, u))) \leq \phi(\theta(d(z, u))).\end{aligned} \tag{39}$$

Since θ is increasing, therefore

$$d(z, u) < d(z, u), \tag{40}$$

which is a contradiction. Hence, $z = u$ and T have a unique fixed point.

Recall that a self-mapping T is said to have the property P, if $Fix(T) = Fix(T^n)$ for every $n \in \mathbb{N}$.

Theorem 2. *Let α, η: $X \times X \longrightarrow \mathbb{R}^+$ be two functions and let (X, d) be an (α, η)–complete rectangular b-metric space. Let T: $X \longrightarrow X$ be a mapping satisfying the following conditions:*

(i) T is a triangular (α, η)–admissible mapping

(ii) T is an $(\alpha, \eta) - \theta - \phi$–contraction

(iii) $\alpha(z, Tz) \geq 1$ or $\eta(z, Tz) \leq 1$, for all $z \in$ Fix (T)

Then T has the property P.

Proof. Let $z \in$ Fix (T^n) for some fixed $n > 1$. As $\alpha(z, Tz) \geq 1$ or $\eta(z, Tz) \leq 1$ and T is a triangular (α, η)-admissible mapping, then

$$\alpha\left(Tz, T^2 z\right) \geq 1 \text{ or } \eta\left(T^2 z, Tz\right) \leq 1. \tag{41}$$

Continuing this process, we have

$$\alpha\left(T^n z, T^{n+1} z\right) \geq 1 \text{ or } \eta\left(T^n z, T^{n+1} z\right) \leq 1, \tag{42}$$

for all $n \in \mathbb{N}$. By (T_3) and (T_4), we get

$$\alpha(T^m z, T^n z) \geq 1 \quad \text{or} \quad \eta(T^m z, T^n z) \leq 1, \quad \forall m, \quad n \in \mathbb{N}, \quad n \neq m. \tag{43}$$

Assume that $z \notin$ Fix (T), i.e., $d(z, Tz) > 0$.

Applying (3) with $x = T^{n-1} z$ and $y = z$, we get

$$d(z, Tz) = d(T^n z, Tz) = d\left(TT^{n-1} z, Tz\right) \leq s^2 d\left(TT^{n-1} z, Tz\right), \tag{44}$$

which implies that

$$\begin{aligned}\theta\left(d\left(TT^{n-1} z, Tz\right)\right) &\leq \phi\left[\theta\left(\beta_1 d\left(T^{n-1} z, z\right) + \beta_2 d\left(T^{n-1} z, T^n z\right) + \beta_3 d(z, Tz) + \beta_4 d(z, T^n z)\right)\right]\\ &< \theta\left[\beta_1 d\left(T^{n-1} z, z\right) + \beta_2 d\left(T^{n-1} z, T^n z\right) + \beta_3 d(z, Tz) + \beta_4 d(z, T^n z)\right]\\ &= \theta\left[\beta_1 d\left(T^{n-1} z, T^n z\right) + \beta_2 d\left(T^{n-1} z, T^n z\right) + \beta_3 d(z, Tz)\right].\end{aligned} \tag{45}$$

Since θ is increasing, therefore,

$$d(z, Tz) < \frac{\beta_1 + \beta_2}{1 - \beta_3} d\left(T^{n-1} z, T^n z\right) \leq d\left(T^{n-1} z, T^n z\right), \tag{46}$$

which is a contradiction as $d(T^{n-1} z, T^n z) \longrightarrow 0$ and $d(z, Tz) > 0$.

Assuming the following conditions, we prove that Theorem 2 still holds for T not necessarily continuous. □

Theorem 3. *Let $\alpha, \eta: X \times X \longrightarrow \mathbb{R}^+$ be two functions and let $d(X, d)$ be an (α, η)–complete rectangular b-metric space.*

Let $T: X \longrightarrow X$ be a mapping satisfying the following assertions:

(i) T is triangular (α, η)–admissible

(ii) T is $(\alpha, \eta) - \theta - \phi$–contraction

(iii) There exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$ or $\eta(x_0, Tx_0) \leq 1$

(iv) (X, d) is an (α, η)-regular rectangular b-metric space

Then T has a fixed point. Moreover, T has a unique fixed point whenever $\alpha(z, u) \geq 1$ or $\eta(z, u) \leq 1$ for all $z, u \in Fix(T)$.

Proof. Let $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$ or $\eta(x_0, Tx_0) \leq 1$. Similar to the proof of Theorem 3, we can conclude that

$$(\alpha(x_n, x_{n+1}) \geq 1 \quad \text{or} \quad \eta(x_n, x_{n+1}) \leq 1), \quad \text{and} \quad x_n \longrightarrow z \quad \text{as} \quad n \longrightarrow \infty, \tag{47}$$

where $x_{n+1} = Tx_n$.

From (iv), $\alpha(x_{n+1}, z) \geq 1$ or $\eta(x_{n+1}, z) \leq 1$ holds for $n \in \mathbb{N}$.

Suppose that $Tz = x_{n_{0+1}} = Tx_{n_0}$ for some $n_0 \in \mathbb{N}$. From Theorem 3, we know that the members of the sequence $\{x_n\}$ are distinct. Hence, we have $Tz \neq Tx_n$, i.e., $d(Tz, Tx_n) > 0$ for all $n > n_0$. Thus, we can apply (3), to x_n and z for all $n > n_0$ to get

$$\theta(d(Tx_n, Tz)) \leq \theta\left(s^2 d(Tx_n, Tz)\right) \leq \phi(\theta(\beta_1 d(x_n, z) + \beta_2 d(x_n, Tx_n) + \beta_3 d(z, Tz) + \beta_4 d(z, Tx_n))). \tag{48}$$

By Lemma 2 and (θ_1), we obtain

$$d(Tx_n, Tz) < (\beta_1 d(x_n, z) + \beta_2 d(x_n, Tx_n) + \beta_3 d(z, Tz) + \beta_4 d(z, Tx_n)). \tag{49}$$

By taking the limit as $n \longrightarrow \infty$ in (51), we have

$$\lim_{n\longrightarrow\infty} \sup d(Tx_n, Tz) \leq \beta_3 d(z, Tz). \tag{50}$$

Assume that $z \neq Tz$. Then, from Lemma 1,

$$\frac{1}{s} d(z, Tz) \leq \lim_{n\longrightarrow\infty} \sup d(Tx_n, Tz) \leq \beta_3 d(z, Tz). \tag{51}$$

By assumption $\beta_3 < 1/s$, we have $d(z, Tz) = 0$ and so $z = Tz$. Thus, z is a fixed point of T.

The proof of the uniqueness is similarly to that of Theorem 3.

above theorems, if we take $\phi(t) = t^k$, for some fixed $k \in \,]0, 1[$, where $\beta_1 = 1 and \beta_2 = \beta_3 = \beta_4 = 0$. We obtain the following extension of Jamshaid et al. result (Theorem 1) [13] of (α, η)–complete rectangular b-metric space. ☐

Corollary 1. *Let $\alpha, \eta: X \times X \longrightarrow [0, +\infty[$ be two functions and $d(X, d)$ be an (α, η)–complete rectangular b-metric space and let $T: X \longrightarrow X$ be self-mapping. Suppose for all $x, y \in X$ with $\alpha(x, y) \geq 1$ or $\eta(x, y) \leq 1$ and $d(Tx, Ty) > 0$, we have*

$$\theta\left[s^2 d(Tx, Ty) \leq [\theta(d(x, y))]\right]^k, \tag{52}$$

where $\theta \in \Theta$ and $k \in \,]0, 1[$. If the mapping T satisfies the following assertions: point, if

(i) T is a triangular (α, η)–admissible mapping

(ii) There exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$ or $\eta(x_0, Tx_0) \leq 1$

(iii) T is (α, η)-continuous or

(iv) is an (α, η)–regular rectangular b–metric space

Then T has a fixed point. Moreover, T has a unique fixed point whenever $\alpha(z, u) \geq 1$ or $\eta(z, u) \leq 1$ for all $z, u \in Fix(T)$.

Proof. Let $\phi(t) = t^k$, we prove that T is an $(\alpha, \eta) - \theta - \phi$–contraction, Hence, T satisfies in assumption of Theorem 3 or 2 and is the unique fixed point of T.

It follows from Theorem 3; we obtain the following fixed point theorems for $\theta - \phi$-Kannan-type contraction and $\theta - \phi$-Reich-type contraction. ☐

Theorem 4. *Let (X, d) be a (α, η)–complete rectangular b-metric space and let $\alpha, \eta: X \times X \longrightarrow [0, +\infty[$ be two functions. Let $T: X \times X \longrightarrow X$ be a self-mapping satisfying the following conditions:*

(i) T is a triangular (α, η)–admissible mapping

(ii) T is a $(\alpha, \eta) - \theta - \phi$-Kannan-type contraction

(iii) There exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$ or $\eta(x_0, Tx_0) \leq 1$

(iv) $T$is a (α, η)–continuous

Then T has a fixed point. Moreover, T has a unique fixed point when $\alpha(x, y) \geq 1$ or $\eta(x, y) \leq 1$ for all $x, y \in X$.

Proof. If T is a $(\alpha, \eta) - \theta - \phi$-Kannan-type contraction, thus there exist $\theta \in \Theta$ and $\phi \in \Phi$ with $\alpha(x, y) \geq 1$ or $\eta(x, y) \leq 1$ for any $x, y \in X$, $Tx \neq Ty$, we have

$$\theta\left[s^2 d(Tx, Ty)\right] \leq \phi\left[\theta\left(\frac{(d(Tx, x) + d(Ty, y))}{2s}\right)\right]. \tag{53}$$

Therefore,

$$\theta\left[s^2 d(Tx, Ty)\right] \leq \phi[\theta(\beta_1 d(x, y) + \beta_2 d(Tx, x) + \beta_3 d(Ty, y) + \beta_4 d(Tx, y)], \tag{54}$$

where $\beta_1 = \beta_4 = 0$, $\beta_2 = \beta_3 = 1/2s$, which implies that T is a $(\alpha, \eta) - \theta - \phi$ contraction Therefore, from Theorem 2, T has a unique fixed point. ☐

Theorem 5. *Let (X, d) be a (α, η)–complete rectangular b-metric space and let $\alpha, \eta: X \times X \longrightarrow [0, +\infty[$ be two functions. Let $T: X \times X \longrightarrow X$ be a self-mapping satisfying the following conditions:*

(i) T is a triangular (α,η)–admissible mapping

(ii) T is a $(\alpha,\eta)-\theta-\phi$–Reich-type contraction

(iii) There exists $x_0\in X$ such that $\alpha(x_0,Tx_0)\geq 1$ or $\eta(x_0,Tx_0)\leq 1$

(iv) T is a (α,η)–continuous

Then T has a fixed point. Moreover, T has a unique fixed point when $\alpha(x,y)\geq 1$ or $\eta(x,y)\leq 1$ for all $x,y\in X$.

Proof. If T is a $(\alpha,\eta)-\theta-\phi$–Reich-type contraction, thus there exist $\theta\in\Theta$ and $\phi\in\Phi$ with $\alpha(x,y)\geq 1$ or $\eta(x,y)\leq 1$ for any $x,y\in X$, $Tx\neq Ty$, we have

$$\theta\left[s^2 d(Tx,Ty)\right]\leq\phi\left[\theta\left(\frac{(d(x,y)+d(Tx,x)+d(Ty,y))}{3s}\right)\right]. \tag{55}$$

Therefore,

$$\theta\left[s^2 d(Tx,Ty)\right]\leq\phi\left[\theta\left(\beta_1 d(x,y)+\beta_2 d(Tx,x)+\beta_3 d(Ty,y)+\beta_4 d(Tx,y)\right)\right], \tag{56}$$

where $\beta_1=\beta_2=\beta_3=(1/3s)$ and $\beta_4=0$, which implies that T is a $(\alpha,\eta)-\theta-\phi$ contraction. Therefore, from Theorem 3, T has a unique fixed point. □

Corollary 2. *Let (X,d) be a (α,η)–complete rectangular b-metric space and let $\alpha,\eta\colon X\times X\longrightarrow[0,+\infty[$ be two functions. Let $T\colon X\times X\longrightarrow X$ be a self-mapping satisfying the following conditions:*

(i) T is a triangular (α,η)–admissible mapping

(ii) T is a (α,η)–Kannan-type mapping

(iii) There exists $x_0\in X$ such that $\alpha(x_0,Tx_0)\geq 1$ or $\eta(x_0,Tx_0)\leq 1$

(iv) $T$is a (α,η)–continuous

Then T has a fixed point. Moreover, T has a unique fixed point when $\alpha(x,y)\geq 1$ or $\eta(x,y)\leq 1$ for all $x,y\in X$. Then T has a unique fixed point $x\in X$.

Proof. Let $\theta(t)=e^t$ for all $t\in\ ]0,+\infty[$, and $\phi(t)=t^{2s\alpha}$ for all $t\in[1,+\infty[$.

It is obvious that $\theta\in\Theta$ and $\phi\in\Phi$. We prove that T is a $\theta-\phi$-Kannan-type contraction:

$$\begin{aligned}\theta\left(s^2 d(Tx,Ty)\right)&=e^{s^2 d(Tx,Ty)}\leq e^{\alpha(d(Tx,x)+d(y,Ty))}\\&=e^{2s\alpha\left(\frac{d(Tx,x)+d(y,Ty)}{2s}\right)}\\&=\left[e^{\left(\frac{d(Tx,x)+d(y,Ty)}{2s}\right)}\right]^{2s\alpha}\\&=\phi\left[\theta\left(\frac{d(Tx,x)+d(y,Ty)}{2s}\right)\right].\end{aligned} \tag{57}$$

Therefore, from Theorem 3, T has a unique fixed point $x\in X$. □

Corollary 3. *Let (X,d) be a (α,η)–complete rectangular b-metric space and let $\alpha,\eta\colon X\times X\longrightarrow[0,+\infty[$ be two functions. Let $T\colon X\times X\longrightarrow X$ be a self-mapping satisfying the following conditions:*

(i) T is a triangular (α,η)–admissible mapping

(ii) T is a (α,η)–Reich-type mapping

(iii) There exists $x_0\in X$ such that $\alpha(x_0,Tx_0)\geq 1$ or $\eta(x_0,Tx_0)\leq 1$

(iv) $T$is a (α,η)–continuous

Then T has a fixed point. Moreover, T has a unique fixed point when $\alpha(x,y)\geq 1$ or $\eta(x,y)\leq 1$ for all $x,y\in X$.

Proof. Let $\theta(t)=e^t$ for all $t\in\ ]0,+\infty[$, and $\phi(t)=t^{3s\lambda}$ for all t $t\in[1,+\infty[$.

We prove that T is a $\theta-\phi$-Reich-type contraction:

$$\begin{aligned}\theta\left(s^2 d(Tx,Ty)\right)&=e^{s^2 d(Tx,Ty)}\leq e^{3\lambda s((d(x,y)+d(Tx,x)+d(y,Ty))/3s)}\\&=\left[e^{\frac{(d(x,y)+d(Tx,x)+d(y,Ty))}{3s}}\right]^{3\lambda s}\\&=\phi\left[\theta\left(\left(\frac{(d(x,y)+d(Tx,x)+d(y,Ty))}{3}\right)\right)\right].\end{aligned} \tag{58}$$

Therefore, from Theorem 3, T has a unique fixed point $x\in X$. □

Example 1. Consider the set $X=\{1,2,3,4\}$. It is easy to check that the mapping $d\colon X\times X\longrightarrow[0,+\infty[$ given by

(i) $d(x,y)=d(y,x)$, $d(x,x)=0$ for all $x,y\in X$

(ii) $d(1,2)=1/24$, $d(1,3)=3$, $d(1,4)=4$

(iii) $d(2,3)=5$, $d(2,4)=6$, and $d(3,4)=18$

Clearly (X,d) is a rectangular b–metric space with parameter $s=2$.

Define mapping $T\colon X\longrightarrow X$ and $\alpha,\eta\colon X\times X\longrightarrow[0,+\infty[$ by

$$\begin{cases}T(1)=1,\\T(2)=1,\\T(3)=1,\\T(4)=2.\end{cases} \tag{59}$$

$$\alpha(x,y)=\frac{x+y}{\max\{x,y\}}, \tag{60}$$

$$\eta(x,y)=\frac{|x-y|}{\max\{x,y\}}. \tag{61}$$

Then, T is an (α,η)–continuous triangular (α,η)–admissible mapping.

Let $\theta(t)=\sqrt{t}+1$, $\phi(t)=2t+1/3$, and $\beta_1=4/10$, $\beta_2=1/10$, $\beta_3=3/10$, and $\beta_4=2/10$. It is obvious that $\theta\in\Theta$ and $\phi\in\Phi$. Evidently, $(\alpha(x,y)\geq 1$ or $(x,y)\leq 1)$ and

$d(Tx, Ty) > 0$ are when, or $\{x, y\} = \{3, 4\}$. Consider the following four possibilities:

For $x = 1$ and $y = 4$, then

$$\theta\left(s^2 d(T1, T4)\right) = \sqrt{\frac{1}{6}} + 1 = 1.4, \tag{62}$$

$$\begin{aligned}&\phi\left(\theta\left(\beta_1 d(1,4) + \beta_2 d(1, T1)\right) + \beta_3 d(4, T4) + \beta_4 d(4, T1)\right)\\&\quad = \phi\left(\theta\left(\frac{21}{5}\right)\right) = 2.36.\end{aligned} \tag{63}$$

Then

$$\begin{aligned}\theta\left(s^2 d(T1, T4)\right) \le \phi\left(\theta\left(\beta_1 d(1,4) + \beta_2 d(1, T1)\right)\right.\\ \left. + \beta_3 d(4, T4) + \beta_4 d(4, T1)\right).\end{aligned} \tag{64}$$

For $x = 2$ and $y = 4$, then

$$\theta\left(s^2 d(T2, T4)\right) = \sqrt{\frac{1}{6}} + 1 = 1.4, \tag{65}$$

$$\begin{aligned}&\phi\left(\theta\left(\beta_1 d(2,4) + \beta_2 d(2, T2)\right) + \beta_3 d(4, T4) + \beta_4 d(4, T2)\right)\\&\quad = \phi(\theta(3.65)) = 2.27.\end{aligned} \tag{66}$$

Then

$$\begin{aligned}\theta\left(s^2 d(T2, T4)\right) \le \phi\left(\theta\left(\beta_1 d(2,4) + \beta_2 d(2, T2)\right)\right.\\ \left. + \beta_3 d(4, T4) + \beta_4 d(4, T2)\right).\end{aligned} \tag{67}$$

For $x = 3$ and $y = 4$, then

$$\theta\left(s^2 d(T3, T4)\right) = \sqrt{\frac{1}{6}} + 1 = 1.4, \tag{68}$$

$$\begin{aligned}&\phi\left(\theta\left(\beta_1 d(3,4) + \beta_2 d(3, T3)\right) + \beta_3 d(4, T4) + \beta_4 d(4, T3)\right)\\&\quad = \phi(\theta(10.1)) = 3.13.\end{aligned} \tag{69}$$

Then

$$\begin{aligned}\theta\left(s^2 d(T3, T4)\right) \le \phi\left(\theta\left(\beta_1 d(3,4) + \beta_2 d(3, T3)\right) + \beta_3 d(4, T4)\right.\\ \left. + \beta_4 d(4, T3)\right).\end{aligned} \tag{70}$$

Hence, T satisfying the assumption of Theorems 3 and 1 is the unique fixed point of T.

Example 2. Let $X = A \cup B$, where $A = \{(1/n): n \in \{2, 3, 4, 5, 6, 7\}\}$ and $B = [1, 2]$. Define $d: X \times X \longrightarrow [0, +\infty[$ as follows:

$$\begin{cases} d(x, y) = d(y, x) \quad \text{for all} \quad x, y \in X, \\ d(x, y) = 0 \Leftrightarrow y = x, \end{cases} \tag{71}$$

$$\begin{cases} d\left(\frac{1}{2}, \frac{1}{3}\right) = d\left(\frac{1}{4}, \frac{1}{5}\right) = d\left(\frac{1}{6}, \frac{1}{7}\right) = 0.05, \\ d\left(\frac{1}{2}, \frac{1}{4}\right) = d\left(\frac{1}{3}, \frac{1}{7}\right) = d\left(\frac{1}{5}, \frac{1}{6}\right) = 0.08, \\ d\left(\frac{1}{2}, \frac{1}{6}\right) = d\left(\frac{1}{3}, \frac{1}{4}\right) = d\left(\frac{1}{5}, \frac{1}{7}\right) = 0.4, \\ d\left(\frac{1}{2}, \frac{1}{5}\right) = d\left(\frac{1}{3}, \frac{1}{6}\right) = d\left(\frac{1}{4}, \frac{1}{7}\right) = 0.24, \\ d\left(\frac{1}{2}, \frac{1}{7}\right) = d\left(\frac{1}{3}, \frac{1}{5}\right) = d\left(\frac{1}{4}, \frac{1}{6}\right) = 0.15, \\ d(x, y) = (|x - y|)^2 \quad \text{otherwise}. \end{cases} \tag{72}$$

Then, (X, d) is a rectangular b-metric space with coefficient $s = 3$. However, we have the following:

(1) (X, d) is not a metric space, as $d((1/5), (1/7)) = 0.4 > 0.29 = d((1/5), (1/4)) + d((1/4), (1/7))$

(2) (X, d) is not a b-metric space for $s = 3$, as $d((1/3), (1/4)) = 0.4 > 0.39 = 3[d((1/3), (1/2)) + d((1/2), (1/4))]$

(3) (X, d) is not a rectangular metric space, as $d((1/5), (1/7)) = 0.4 > 0.28 = d((1/5), (1/4)) + d((1/4), (1/2)) + d((1/2), (1/7))$

Define mapping $T: X \longrightarrow X$ and $\alpha, \eta: X \times X \longrightarrow [0, +\infty[$ by

$$T(x) = \begin{cases} \sqrt[6]{x}, & \text{if} \quad x \in [1, 2], \\ 1, & \text{if} \quad x \in A, \end{cases} \tag{73}$$

$$\alpha(x, y) = \begin{cases} \sinh(x + y), & \text{if} \quad x, y \in [1, 2], \\ \dfrac{1}{e^{x+y}}, & \text{otherwise}, \end{cases} \tag{74}$$

$$\eta(x, y) = \begin{cases} \dfrac{x + y}{4}, & \text{if} \quad x, y \in [1, 2], \\ 1 + e^{-(x+y)}, & \text{otherwise}. \end{cases} \tag{75}$$

Then, T is an (α, η)−continuous triangular (α, η)−admissible mapping.

Let $\theta(t) = \sqrt{t} + 1$, $\phi(t) = t + 1/2$ and taking $\beta_1 = 1$ and $\beta_2 = \beta_3 = \beta_3 = 0$. It is obvious that $\theta \in \Theta$ and $\phi \in \Phi$.

Evidently, $(\alpha(x,y) \geq 1 \text{ or } (x,y) \leq 1)$ and $d(Tx,Ty) > 0$ are when $\{x,y\} \in [1,2]$ with $x \neq y$.

Consider two cases:

Case 1: $x > y$:

$$\theta s^2 d(Tx,Ty) = 3\left(\sqrt[6]{x} - \sqrt[6]{y}\right) + 1, \tag{76}$$

$$\phi[\theta d(x,y)] = \frac{x-y}{2} + 1. \tag{77}$$

On the other hand

$$\begin{aligned}\theta\left(s^2 d(Tx,Ty)\right) - \phi[\theta(d(x,y))] &= \frac{6\left(\sqrt[6]{x} - \sqrt[6]{y}\right) - (x-y)}{2} \\ &= \frac{1}{2}\left(\left(\sqrt[6]{x} - \sqrt[6]{y}\right)\right)\left[6 - \left(\sqrt[6]{x^5} + \sqrt[6]{x^4 y} + \sqrt[6]{x^3 y^2} + \sqrt[6]{x^2 y^3}\right.\right. \\ &\quad \left.\left. + \sqrt{6xy^4} + \sqrt[6]{y^5}\right)\right].\end{aligned} \tag{78}$$

Since $x, y \in [1,2]$, then

$$\left[6 - \left(\sqrt[6]{x^5} + \sqrt[6]{x^4 y} + \sqrt[6]{x^3 y^2} + \sqrt[6]{x^2 y^3} + \sqrt[6]{xy^4} + \sqrt[6]{y^5}\right)\right] \leq 0, \tag{79}$$

which implies that

$$\begin{aligned}\theta s^2 d(Tx,Ty) \leq \phi[\theta(d(x,y))] &= \phi\left[\theta\left(\beta_1 d(x,y)\right) + \beta_2 d(x,Tx) + \beta_3 d(y,Ty)\right. \\ &\quad \left. + \beta_4 d(y,Tx)\right].\end{aligned} \tag{80}$$

Case 2: $y > x$:

$$\theta s^2 d(Tx,Ty) = 3\left(\sqrt[6]{y} - \sqrt[6]{x}\right) + 1, \tag{81}$$

$$\phi[\theta(d(x,y))] = \frac{y-x}{2} + 1. \tag{82}$$

Similarly for Case 2, we conclude that

$$\begin{aligned}\theta s^2 d(Tx,Ty) \leq \phi\left[\theta\left(\beta_1 d(x,y)\right) + \beta_2 d(x,Tx) + \beta_3 d(y,Ty)\right. \\ \left. + \beta_4 d(y,Tx)\right].\end{aligned} \tag{83}$$

Hence, condition (3) is satisfied. Therefore, T has a unique fixed point $z = 1$.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

References

[1] S. Banach, "Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales," *Funding Mathematics*, vol. 3, no. 1, pp. 133–181, 1922.

[2] A. Branciari, "Fixed point theorem of Banach-Caccioppoli type on a class of generalized metric spaces," *Publicationes Mathematics*, vol. 57, pp. 31–37, 2000.

[3] C. D. Bari and P. Vetro, "Common fixed points in generalized metric spaces," *Applied Mathematics and Computation*, vol. 218, no. 13, pp. 7322–7325, 2012.

[4] C.-M. Chen and W. Y. Sun, "Periodic points and fixed points for the weaker (ψ, φ)-contractive mappings in complete generalized metric spaces," *Journal of Applied Mathematics*, vol. 2012, Article ID 856974, 7 pages, 2012.

[5] P. Das, "A fixed point theorem on a class of generalized metric spaces," *Korean Journal of Mathematics*, vol. 9, pp. 29–33, 2002.

[6] I. R. Sarma, J. M. Rao, and S. S. Rao, "Contractions over generalized metric spaces," *Journal of Nonlinear Sciences and Applications*, vol. 2, no. 3, pp. 180–182, 2009.

[7] M. Jleli and B. Samet, "A new generalization of the Banach contraction principale," *Journal of Inequalities and Applications*, vol. 2014, no. 1, p. 8, 2014.

[8] W. Kirk and N. Shahzad, "Generalized metrics and Caristi's theorem," *Fixed Point Theory and Applications*, vol. 2013, p. 129, 2013.

[9] S. Czerxik, "Contraction mapping in b-metric spaces," *Acta Mathematica et Informatica Universitatis Ostraviensis*, vol. 1, pp. 5–11, 1993.

[10] J. R. Roshan, V. Parvaneh, V. Parvaneh, Z. Kadelburg, and N. Hussain, "New fixed point results in b-rectangular metric spaces," *Nonlinear Analysis: Modelling and Control*, vol. 21, no. 5, pp. 614–634, 2016.

[11] N. Hussain, V. Parvaneh, B. A. S. Alamri, and Z. Kadelburg, "F-HR-type contractions on (\alpha,\eta)-complete rectangular b-metric spaces," *The Journal of Nonlinear Sciences and Applications*, vol. 10, no. 3, pp. 1030–1043, 2017.

[12] D. Zheng, Z. Cai, and P. Wang, "New fixed point theorems for θ-ϕ contraction in complete metric spaces," *The Journal of Nonlinear Sciences and Applications*, vol. 10, no. 5, pp. 2662–2670, 2017.

[13] A. Jamshaid, E. Al-Mazrooei, Y. J. Cho, and Y.-O. Yan, "Fixed point result for generalized θ-contraction," *Journal of Nonlinear Sciences and Applications*, vol. 10, pp. 2350–2358, 2017.

[14] N. Hussain, M. A. Kutbi, and P. Salimi, "Fixed point theory in α-complete metric spaces with applications," *Abstract and Applied Analysis*, vol. 2014, Article ID 280817, 11 pages, 2014.