The Annals of Mathematical Statistics
1972, Vol. 43, No. 3, 701-732

THE 1971 RIETZ LECTURE
SUMS OF INDEPENDENT RANDOM VARIABLES—WITHOUT MOMENT CONDITIONS[1]

BY HARRY KESTEN

Cornell University

Analogues of classical limit laws for sums of independent random variables (central limit theorem, laws of large numbers and law of the iterated logarithm) are discussed. We stress results which go through without moment or smoothness assumptions on the underlying distributions. These include (i) estimates for the spread of the distribution of $S_n = \sum_1^n X_i$ in terms of concentration functions (Lévy-Rogozin inequality), (ii) comparison of the distribution of S_n on different intervals (ratio limit theorems and Spitzer's theorem for the existence of the potential kernel for recurrent random walk), (iii) study of the set of accumulation points of $S_n/\gamma(n)$ for suitable $\gamma(n) \uparrow \infty$. Only the following parallel to the law of the iterated logarithm is new: If $X_1, X_2, \cdots$ are independent random variables all with distribution F, $S_n = \sum_1^n X_1$, $m_n = \operatorname{med}(S_n)$, then there exists a sequence $\{\gamma(n)\}$ such that $\gamma(n) \to \infty$ and $-\infty < \liminf (S_n - m_n)/\gamma(n) < \limsup (S_n - m_n)/\gamma(n) < \infty$ w.p. 1, if and only if F belongs to the domain of partial attraction of the normal law.

1. Introduction. The classical limit laws for sums of independent random variables are the laws of large numbers, the central limit theorem and the more refined law of the iterated logarithm. Typically the hypotheses of these theorems include a moment assumption, such as the existence of the mean for the strong law of large numbers or the existence of the variance for the central limit theorem and the law of the iterated logarithm. Analogues of the central limit theorem which imply convergence to a stable law other than the normal law require regularity of the tail of the underlying distributions instead of the existence of the 2nd moment. Our purpose here is to survey some reasonable generalizations or analogues of the classical limit theorems which do not make a priori moment or smoothness assumptions on the underlying distributions, e.g. results which hold for *any* random walk. We shall, however, always assume that the random variables which we add are independent, and in most cases we shall even assume them identically distributed. We shall also restrict ourselves to one dimensional real random variables, although most problems and some results carry over to vector (or even group) valued random variables. To put it in nonprobabilistic terms, we investigate general inequalities and smoothness properties of convolutions of probability distributions on the real line $\mathbb{R}$. The abstract above states more specifically what kind of results are discussed. All

Received September 9, 1971.

[1] Invited Rietz Lecture given at the annual meeting of the Institute of Mathematical Statistics, Fort Collins, September 20-23, 1971. This work was supported by the N.S.F. under grants GP 19658 and GP 28109.

Institute of Mathematical Statistics is collaborating with JSTOR to digitize, preserve, and extend access to *The Annals of Mathematical Statistics.* www.jstor.org

results are already in the literature, with the exception of Theorem 2c and the necessary and sufficient condition for the existence of $\{\gamma(n)\}$ satisfying

$$-\infty < \liminf \frac{S_n - m_n}{\gamma(n)} < \limsup \frac{S_n - m_n}{\gamma(n)} < \infty \tag{1.1}$$

mentioned in the abstract. The latter condition appears in Theorem 6 and is proved in the Appendix.

Of course the choice of topics here was largely dictated by personal tastes and, what is perhaps the same thing, a desire to draw attention to several open problems and conjectures which are stated in Section 3 and 4.

Throughout $X_1, X_2, \cdots$ will denote a sequence of independent real random variables and $S_n = \sum_1^n X_i$. The distribution function of X_i will be denoted by F_i, and just by F if the X_i are identically distributed, $\tilde{X}_1, \tilde{X}_2, \cdots$ will be another sequence of independent random variables, independent of $\{X_i\}_{i\geq 1}$, and with the same distribution as $\{X_i\}_{i\geq 1}$. We put $X_i^s = X_i - \tilde{X}_i =$ "the symmetrization" of X_i, and of course

$$\tilde{S}_n = \sum_1^n \tilde{X}_i\,, \qquad S_n^s = S_n - \tilde{S}_n\,.$$

The distribution function F_i^s of X_i^s is given by

$$F_i^s(x) = \int F(y + x)\,dF(y)\,. \tag{1.2}$$

For any distribution function F, F^k denotes $F * F * \cdots * F$, its k-fold convolution, $\operatorname{med}(S_n) =$ median of S_n, i.e., any number m for which $P\{S_n \geq m\} \geq \frac{1}{2}$ and $P\{S_n \leq m\} \geq \frac{1}{2}$. $I(A)$ is the indicator function of the event A.

2. The spread of a convolution. One of the most elegant chapters of early probability theory deals with the convergence of series of independent random variables. Kolmogorov [38] gave necessary and sufficient conditions for S_n to converge w.p. 1 in his celebrated three series criterion. Lévy, [41] Theorem 44, proved that S_n converges w.p. 1, if and only if it converges in probability and Doob [11], Corollary 2 to Theorem III. 2.7, showed that convergence w.p. 1 of S_n is even equivalent to convergence in distribution (for a good account of these results see [11], Chapter III. 2 with references in the appendix, page 626; different proofs can be found in [41], Section 42–46). In these investigations it proved useful to ask whether $S_n - c_n$ converges w.p. 1 for suitable c_n, even when S_n itself diverges w.p. 1. Such "centering constants" c_n exist if and only if S_n^s converges w.p. 1 (see [11], middle of page 120). If $S_n - c_n$ diverges w.p. 1 for every choice of c_n, or equivalently, if S_n^s diverges w.p. 1 then the sequence of partial sums is called essentially divergent. It turns out that this occurs exactly when the distribution of S_n gets "spread out" more and more over the whole real line. More precisely, S_n is essentially divergent if and only if

$$Q(S_n; L) = \sup_a P\{a \leq S_n \leq a + L\} \to 0 \qquad (n \to \infty)$$

for each $L > 0$. (See [41], Section 42, 43, [11], Theorem III. 2.9.) The proof of these results as well as the final result itself raises the question to give a quantitative estimate of the spread of the distribution of S_n. A common measure for the spread is the variance, but this may of course be infinite. Lévy circumvented this difficulty by introducing the concentration function $Q(Y; L)$ and its inverse $D(Y; q)$ for an arbitrary random variable Y. These functions are defined by[2]

$$Q(Y; L) = \sup_a P\{a \leqq Y \leqq a + L\}, \tag{2.1}$$

$$\begin{aligned} D(Y; q) &= \inf\{L : Q(Y; L) \geqq q\} \\ &= \inf\{L : \exists a \text{ such that } P\{a \leqq Y \leqq a + L\} \geqq q\}. \end{aligned} \tag{2.2}$$

In (2.2) $0 \leqq q \leqq 1$ and the inf over the empty set is taken as ∞. Doeblin and Lévy [10], (see also [9] and [41], Section 48) were the first to give a quantitative estimate for the spread of the distribution of S_n in terms of a lower bound for $D(S_n; q)$. Their result shows more or less that the spread is smallest in the cases where the central limit theorem applies; under some uniformity conditions $D(S_n; q)$ grows at least like $n^{\frac{1}{2}}$. Specifically they prove the following result: If $0 < q_1$, $q_2 < 1$ then there exist constants $0 < C = C(q_1, q_2)$, $N = N(q_1, q_2) < \infty$ such that for any $L > 0$ and independent random variables $X_1, \cdots, X_n$ which satisfy $n \geqq N$ and

$$D(X_i; q_1) > 2L, \quad \text{or equivalently,} \quad Q(X_i; 2L) < q_1, \tag{2.3}$$

one has

$$D(S_n; q_2) > CLn^{\frac{1}{2}}. \tag{2.4}$$

This result is a rather crude analogue of the *global* central limit theorem; it tells us something about the probability mass of intervals with length of the order $n^{\frac{1}{2}}$. The Doeblin-Lévy inequality (2.4) has been modified and improved by Kolmogorov [39], Rogozin [50], [51], LeCam [40], Esséen [13], [14] and Kesten [32]. The best version at present is an analogue of the *local* central limit theorem in that it deals with the probability mass of intervals of fixed length. It looks as follows:

THEOREM 1 ([32]). *There exists a universal constant $C < \infty$ such that for any n and arbitrary, $0 < \lambda_1, \cdots, \lambda_n < 2L$ and independent random variables $X_1, \cdots, X_n$*

$$\begin{aligned} Q(S_n; L) &\leqq CL \frac{\max_{k \leqq n} Q(X_k; L)}{[\sum_{i=1}^n \int_{|y| \leqq \lambda_i} y^2 \, dF_i^s(y) + \lambda_i^2 P\{|X_i^s| > \lambda_i\}]^{\frac{1}{2}}} \\ &\leqq C \frac{L}{[\sum_{i=1}^n \lambda_i^2\{1 - Q(X_i; 2\lambda_i)\}]^{\frac{1}{2}}}. \end{aligned} \tag{2.5}$$

To give a feeling for the inequality (2.5) we write out two special cases:

(i) $X_1, X_2, \cdots, X_n$ identically distributed and $0 < \lambda_i = \lambda \leqq 2L$, $1 \leqq i \leqq n$.

---

[2] The notation in (2.1) and (2.2) is convenient, but not very logical because Q and D depend only on the distribution function, G say, of Y rather than on Y itself. Sometimes we shall therefore write $Q(G; L)$ and $D(G; q)$ instead of $Q(Y; L)$, respectively $D(Y; q)$.

(2.5) now gives

$$Q(S_n; L) = \max_a P\{a \leqq S_n \leqq a + L\}$$
$$(2.6) \qquad \leqq \frac{CLQ(X_1; L)}{\lambda n^{\frac{1}{2}}[\lambda^{-2} \int_{|y| \leqq \lambda} y^2 \, dF_1^s(y) + P\{|X_1^s| > \lambda\}]^{\frac{1}{2}}}$$
$$\leqq \frac{CL}{\lambda n^{\frac{1}{2}}[1 - Q(X_1; 2\lambda)]^{\frac{1}{2}}} .$$

In analogy with the local central limit theorem, the probability mass of any interval of fixed length is at most $O(n^{-\frac{1}{2}})$. (In the identically distributed case with infinite variance, $Q(S_n; L)$ is actually $o(n^{-\frac{1}{2}})$; see [14], Theorem 4.1.)

(ii) $X_1, X_2, \cdots, X_n$ identically distributed and integer valued, $\lambda_i = L = \frac{1}{4}$, $1 \leqq i \leqq n$. Let

$$p = \max_k P\{X_1 = k\} .$$

(2.5) now gives

$$(2.7) \qquad Q(S_n; \tfrac{1}{4}) = \max_k P\{S_n = k\} \leqq \frac{Cp}{n^{\frac{1}{2}}(1 - p)^{\frac{1}{2}}} .$$

The maximal probability of any integer for S_n is at most $C[n(1 - p)]^{-\frac{1}{2}}$ times the maximal probability for any integer for X_1.

REMARK 1. Even though several analogues of Theorem 1 in higher dimensions have been given (see Sazonov [53] and Esséen [13], [14]) they are not quite satisfactory. The constants in [53] are not universal, but depend on the underlying distributions, whereas Esséen [13] has to make symmetry assumptions or (Theorem 6.1 in [14]) has to be content with a weak estimate. The difficulty is due, in part, to the existence of distributions in $\mathbb{R}^d$, which are concentrated on subspaces of dimension less than d. For instance if $X_1, \cdots, X_n$ are identically distributed in $\mathbb{R}^2$ with a distribution function F concentrated on a line and with finite second moment, then

$$(2.8) \qquad \sup_a P\{S_n \in a + U\} \geqq \frac{C(F, U)}{n^{\frac{1}{2}}}$$

for any rectangle U, where $C(F, U) > 0$. On the other hand, if F is not concentrated on a line, then the left-hand side of (2.8) is $O(n^{-1})$ (see Theorem 6.2 of [14]) so that it is hard to see what should replace the denominator in the second or third member of (2.5). Compare also Section 3 of [13] and Section 6 of [14].

REMARK 2. It seems worth pointing out that we do not have a purely analytic proof of (2.5)—(2.7). Esséen [13] and [14] has a very neat analytic proof for

$$(2.9) \qquad Q(S_n; L) \leqq CL[\textstyle\sum_{i=1}^n \int_{|y| \leqq \lambda_i} y^2 \, dF_i^s(y) + \lambda_i^2 P\{|X_i^s| > \lambda_i\}]^{-\frac{1}{2}} ,$$

but one needs an additional combinatorial argument to obtain (2.5). In the special case (ii) (2.9) gives only

$$\max_k P\{S_n = k\} \leqq \frac{C}{n^{\frac{1}{2}}(1 - p)^{\frac{1}{2}}} .$$

One would expect (2.7) to be provable by methods of Fourier series only, and even (2.5) should be provable analytically.

REMARK 3. Even though it is only marginally related to our present subject, we point out the usefulness of the dispersion function in the formulation of certain limit theorems. The function $D(S_n; q)$ often provides us automatically with the right norming constants. For example, if G_n is any sequence of distribution functions and $\gamma_n > 0$, δ_n constants such that $G_n(\gamma_n x + \delta_n)$ converges weakly to a function $G(x)$, then[2] $G_n(D(G_n; q)x + \delta_n)$ also converges weakly to $G(D(G; q)x)$, whenever $q < 1$ and q is a continuity point of $D(G; \cdot)$ and $D(G; q) > 0$. Thus in many cases we can take norming constants to be of the form $D(G_n; q)$. Also, if $\{G_\alpha\}$ is a family of distribution functions, then there exist constants $\gamma(\alpha) > 0$, $\delta(\alpha)$, such that any sequence $G_{\alpha_n}(\gamma(\alpha_n)x + \delta(\alpha_n))$ has a subsequence which converges weakly to a nondegenerate limit, if and only if there exists a $q_0 < 1$ such that $D(G_\alpha; q_0) > 0$ for all α and

$$\sup_\alpha \frac{D(G_\alpha; q_1)}{D(G_\alpha; q_2)} < \infty \qquad \text{for all} \quad q_0 \leqq q_1, q_2 < 1 .$$

In that case we can take $\gamma(\alpha) = D(G_\alpha; q_0)$ and $\delta(\alpha) =$ median of G_α.

3. Smoothness of convolutions powers. In this section we shall always *assume that* $X_1, X_2, \cdots$ *are independent identically distributed* random variables. Their common distribution function and characteristic function will be denoted by F, respectively φ. We use the following terminology: F is called *arithmetic* if its support is contained in an arithmetic sequence $\{nh: n = 0, \pm 1, \pm 2, \cdots\}$ for some $h > 0$. In all other cases we call it *non-arithmetic*. For simplicity we formulate results for the arithmetic case only for the *integer valued* case, i.e. when $h = 1$.

In the last section we discussed upper bounds for

$$P\{S_n \in a + I\} \tag{3.1}$$

for intervals I. Without further assumptions on F one cannot hope to find the exact asymptotic behavior of F. Nevertheless one can prove remarkable smoothness properties of (3.1) as a function of a, with practically no assumptions on F. The first result in this direction is probably Doeblin's ratio theorem for a Markov chain $\{Z_n\}_{n\geq 0}$ with a discrete state space (see [8] or [5], Corollary 2 to Theorem I.9.4):

$$\lim_{N\to\infty} \frac{\sum_{k=1}^N P\{Z_k = j \mid Z_0 = i\}}{\sum_{k=1}^N P\{Z_k = m \mid Z_0 = l\}} \tag{3.2}$$

exists and is positive and finite if i, j, l, m all lie in one class (see [5], Section I. 3 for definition of "class"). For an integer valued random walk S_n, this becomes

$$\lim_{n\to\infty} \frac{\sum_{k=1}^n P\{S_k = i\}}{\sum_{k=1}^n P\{S_k = j\}} \tag{3.3}$$

exists and is positive and finite, whenever

$$P\{S_n = a \ \text{ for some } \ n\} > 0 \quad \text{for} \quad a = i, j, j - i, i - j . \tag{3.4}$$

For a persistent random walk the limit in (3.3) equals one (see [5], Theorem I. 5.3 and 6, [55], Proposition 1.5). A considerably stronger result was proved by Chung and Erdös ([6], Theorem 3.1 and 4). Their result was improved by Kemeny [26], Doob, Snell and Williamson [12] and Neveu [42] to the following

THEOREM 2a. *If S_n is an integer valued random walk, then*

$$\lim_{n\to\infty} \frac{P\{S_{n+m} = i_n + j\}}{P\{S_n = i_n\}} = 1 , \tag{3.5}$$

whenever

$$\lim_{n\to\infty} [P\{S_n = i_n\}]^{1/n} = 1 , \tag{3.6}$$

and for some integers $k \geqq 0$, a,

$$P\{S_k = a\}P\{S_{k+m} = a + j\} > 0 . \tag{3.7}$$

Notice that $m < 0$ is permitted in (3.5) and (3.6) as long as $k + m > 0$ in (3.6). Actually none of the papers [6], [26], [12] or [42] states the theorem in this generality, but their proofs easily yield the general result; indeed it is apparent from those works as well as those of Ornstein [45] and Stone [56] that the heart of the Chung-Erdös proof is an inequality of the following form: If (3.7) holds, then for every $\varepsilon > 0$ there exists a $\delta > 0$ such that sufficiently large n

$$P\{S_n = i_n\} \leqq (1 + \varepsilon)P\{S_{n+m} = i_n + j\} + e^{-\delta n} \tag{3.8}$$

and

$$P\{S_{n+m} = i_n + j\} \leqq (1 + \varepsilon)P\{S_n = i_n\} + e^{-\delta n} . \tag{3.9}$$

These inequalities are implicit in the proofs of Theorems 3.1 and 4 of [6], and of course they imply (3.5) when (3.6) holds. It proved quite difficult to generalize the "individual ratio limit theorem" to the non-arithmetic case. This was done by Ornstein [45] and Stone [56], [57], essentially by generalizing (3.8) and (3.9). [56] even proves such inequalities for a random walk on a locally compact Abelian group. For real valued X_i with a distribution F, which is not a lattice distribution, i.e., which satisfies

(3.10) Support of F does not lie in a set of the form

$$\{k + nh \colon n = 0, \pm 1, \cdots\} ,$$

Ornstein [45] and Stone [56], [57] derive

THEOREM 2b. *For any finite intervals I, J with lengths $|I|$, $|J| > 0$ and satisfying*

$$\lim_{n\to\infty} [P\{S_n \in x_n + I\}]^{1/n} = 1 , \tag{3.11}$$

and any integer m, one has

$$\lim_{n\to\infty} \frac{P\{S_{n+m} \in x_n + y + J\}}{P\{S_n \in x_n + I\}} = \frac{|J|}{|I|} , \tag{3.12}$$

uniformly for y in a compact set.

We observe that for symmetric random walk with infinite variance one can even compare intervals whose distance increases like $n^{\frac{1}{2}}$. Specifically, it is possible to prove the following theorem (however by methods somewhat different from those used for Theorems 2a and 2b):

THEOREM 2c. *Let F be symmetric, with*

$$\int x^2\, dF(x) = \infty \; . \tag{3.13}$$

If F is concentrated on the integers and for each j there exists an n(j) such that

$$P\{S_n = j\} > 0 \qquad \text{for} \quad n \geqq n(j) \, , \tag{3.14}$$

then for all $A < \infty$ *and* $|i_n|, |j_n| \leqq An^{\frac{1}{2}}$

$$\lim_{n\to\infty} \frac{P\{S_n = i_n\}}{P\{S_n = j_n\}} = 1 \; . \tag{3.15}$$

If F is symmetric, satisfies (3.13), *but F is non-arithmetic, then for all* $A < \infty$ *and* $|x_n|, |y_n| \leqq An^{\frac{1}{2}}$ *and fixed intervals I, J,*

$$\lim_{n\to\infty} \frac{P\{S_n \in x_n + I\}}{P\{S_n \in y_n + J\}} = \frac{|I|}{|J|} \; . \tag{3.16}$$

REMARK 4. We have not written down a ratio theorem for lattice distributions F, which are non-arithmetic, i.e., distributions F whose support lies in $\{k + nh: n = 0, \pm 1, \cdots\}$ for some k, h which are independent over the rationals. This case is, however, covered by Theorem 4 in [57]. Note also that the definitive paper [56] of Stone goes considerably further than Theorem 2b. It proves a ratio theorem for more general sets than intervals I and J and even has a ratio theorem for densities. In [26], [12], [56], [57] the authors have also evaluated limits of the left-hand sides of (3.5) and (3.12) in cases where the limits in (3.6) resp. (3.11) are less than one. Orey [43] and Kingman and Orey [36], have given simple and rather weak conditions for the validity of the individual ratio limit theorem for general countable Markov chains.

REMARK 5. We do not discuss here almost-everywhere convergence analogues of (3.3), or the more general problem of convergence with probability one of the random variables

$$\frac{\sum_{k=1}^{n} f(S_k)}{\sum_{k=1}^{n} g(S_k)} \tag{3.17}$$

for suitable functions f and g. Several authors investigated special cases of this problem, but a general attack on it for general Markov chains was made by Harris and Robbins [19] (via Hopf's ergodic theorem) and for countable Markov chains by Chung [4] (using "Doeblin's trick"; see also [5], Chapter I. 15 for an account of these results for countable Markov chains). Both [19] (Theorem 4 and Corollary) and [4] (Corollary 2 to Theorem I. 15.1) explicitly specialize their result regarding (3.17) to a random walk S_n. The closest correspondence with Theorem 2 is found when f and g are indicator functions. If we write

$$N_n(A) = \sum_{k=1}^{N} I_A(S_k) = \# \text{ of } k \in [1, n] \quad \text{with} \quad S_k \in A\,, \tag{3.18}$$

then for a persistent random walk one obtains

$$\frac{N_n(i)}{N_n(j)} \to 1 \quad \text{w.p. } 1 \tag{3.19}$$

if S_n is integer valued and $P\{S_{k_1} = i\} > 0$, $P\{S_{k_2} = j\} > 0$ for some k_1, k_2; if F is non-arithmetic (3.19) has to be replaced by

$$\frac{N_n(I)}{N_n(J)} \to \frac{|I|}{|J|} \quad \text{w.p. } 1 \tag{3.20}$$

for fixed intervals I and J.

We also remark that [19] and [4] require that the functions f and g are integrable w.r.t. a certain measure, which in the case of a random walk on the integers is the counting measure, and in the case of a non-arithmetic random walk is Lebesgue measure on $\mathbb{R}$. This adds interest to a paper of Robbins [49] which proves convergence w.p. 1 of

$$\frac{1}{n} \sum_{k=1}^{n} f(S_k)$$

for a random walk S_n and an almost periodic function f. Such functions are *not* integrable w.r.t. counting measure on the integers or Lebesgue measure on $\mathbb{R}$.

Theorem 2 shows that under mild assumptions convolutions are quite smooth as long as we do not go out "too far in the tails." (3.6), respectively (3.11), usually holds unless $\mu = EX$ exists and $|i_n - n\mu|$, respectively $|x_n - n\mu|$, grows too fast with n; ((3.7), (3.10) and (3.14) are merely aperiodicity assumptions). However, Theorem 2 only tells us something about the probabilities of intervals of fixed length. Can we compare probabilities of intervals whose length grows with n? Specifically we have in mind the following

PROBLEM 1. Find sufficient conditions for

$$\limsup_{n\to\infty} \frac{\sum_{k=1}^{n} P\{S_k \in [ak^{\frac{1}{2}}, bk^{\frac{1}{2}}]\}}{\sum_{k=1}^{n} P\{S_k \in [(a+c)k^{\frac{1}{2}}, (b+c)k^{\frac{1}{2}}]\}} < \infty \tag{3.21}$$

for all $a < b$ and c for which the denominator in (3.21) is eventually positive.

For any F for which (3.21) holds Conjecture 3 in Section 4 is valid, and this fact is our main motivation for the problem. There do exist distribution functions F for which (3.21) fails for suitable a, b, c. So far these examples do not disprove Conjecture 3, though, since what we need for Conjecture 3 is less than (3.21).[3]

[3] From Theorem 3 in [34] it follows that in order to establish Conjecture 3 we should probe

$$\sum_{n=1}^{\infty} P\{S_n \in [an^{\frac{1}{2}}, bn^{\frac{1}{2}}\} [\sum_{l=0}^{n-1} P\{|S_l| < n^{\frac{1}{2}}\}]^{-1} = \infty$$

if and only if

$$\sum_{n=1}^{\infty} P\{S_n \in [(a+c)n^{\frac{1}{2}}, (b+c)n^{\frac{1}{2}}]\} [\sum_{l=0}^{n-1} P\{|S_l| < n^{\frac{1}{2}}\}^{-1} = \infty\,.$$

Partial summation shows that this equivalence is implied by (3.21).

On the other hand, if

$$\int x\,dF(x) = 0\,, \qquad \int x^2\,dF(x) < \infty\,,$$

then (3.21) follows from the central limit theorem. If

$$\int x\,dF(x) \neq 0\,, \qquad \int x^2\,dF(x) < \infty\,,$$

then both numerator and denominator of (3.21) converge by virtue of Theorem 1 in [25]. Finally if $\int x^2\,dF(x) = \infty$, then we know from the inequality (2.6) or Theorem 4.1 in [14] that for each $k > 0$

$$Q(S_n; kn^{\frac12}) = \sup_a P\{a \leqq S_n \leqq a + kn^{\frac12}\} \to 0 \qquad (n \to \infty)\,.$$

Thus the distribution of S_n will be spread out over regions large w.r.t. $n^{\frac12}$ and (3.21) asks for a certain smoothness of the distributions of S_n over distances which are small w.r.t. their spread. If $\int x^2\,dF(x) = \infty$ and F is symmetric, then Theorem 2c implies immediately

$$\lim_{n\to\infty} \frac{P\{S_n \in [an^{\frac12}, bn^{\frac12}]\}}{P\{S_n \in [(a+c)n^{\frac12}, (b+c)n^{\frac12}]\}} = 1 \tag{3.22}$$

for all $a < b$ and c, which is much stronger than (3.21).

So far we have concentrated on ratios of probabilities. Another measure for the smoothness of the distribution of S_n is the behavior of

$$W_n(x, B) = P\{S_n \in B\} - P\{S_n \in x + B\} \tag{3.23}$$

and

$$V_n(x) = \sup_{B \text{ Borel set}} |W_n(x, B)|\,. \tag{3.24}$$

It is comparatively easy to prove the following theorem (see [21], [29], [46], Theorem 7, [44]):

THEOREM 3. *If F is concentrated on the sequence $\{k + nh : n = 0, \pm 1, \cdots\}$ then for each fixed integer i with $P\{S_l - lk = ih\} > 0$ for some l*

$$\begin{aligned} V_n(i) &= \tfrac12 \textstyle\sum_j |P\{S_n - nk = jh\} - P\{S_n - nk = (i+j)h\}| \\ &= O(1/n^{\frac12}) \qquad (n \to \infty)\,. \end{aligned} \tag{3.25}$$

If F^k has an absolutely continuous component for some k, then for all x

$$V_n(x) = O(1/n^{\frac12}) \qquad (n \to \infty)\,. \tag{3.26}$$

If F is not a lattice distribution, i.e., satisfies (3.10), *then*

$$\begin{aligned} &\lim_{n\to\infty} \int |W_n(x, y + B)|\,dy \\ &\quad = \lim_{n\to\infty} \int |P\{S_n \in y + B\} - P\{S_n \in x + y + B\}|\,dy = 0 \end{aligned} \tag{3.27}$$

for any x and bounded Borel set B; moreover

$$\lim_{n\to\infty} \left|\int f(y)P\{S_n \in dy\} - \int f(x+y)P\{S_n \in dy\}\right| = 0\,, \tag{3.28}$$

for any x and bounded uniformly continuous function f. For fixed x and F the rate *of convergence in* (3.28) *depends only on the upper bound of* $|f|$ *and the modules of continuity of* f.

Orey [44] even proves a generalization of (3.25) for countable persistent Markov chains which improves Kolmogorov's ergodic theorem for countable Markov chains.

Much deeper than Theorem 3 are the results of Spitzer and several other authors which state that for suitable choices of x and B

$$\sum_n W_n(x, B) \quad \text{converges.}$$

The precise statement is as follows:

THEOREM 4. *If* S_n *is an integer valued random walk and* $P\{S_k = i\} > 0$ *for some* k, *then for any* j

$$U(i, j) = \sum_n W_n(i, \{j\}) = \sum_n [P\{S_n = j\} - P\{S_n = i + j\}] \tag{3.29}$$

converges. If F *is non-arithmetic, then*

$$Uf(y) = \sum_n \int f(x + y) P\{S_n \in dx\} \tag{3.30}$$

converges for any x *and any continuous function* f *which satisfies*

$$\int x^2 |f(x)|\, dx < \infty \quad \text{and} \quad \hat{f}(\theta) = \int e^{i\theta x} f(x)\, dx \quad \text{has compact support} \tag{3.31}$$

and

$$\int f(x)\, dx = 0\,. \tag{3.32}$$

If F^k *has an absolutely continuous component for some* k, *then the series in* (3.30) *converges for every bounded Borel function* f *with compact support which satisfies* (3.32). *In particular, in the last case*

$$U(x, I) = \sum_n W_n(x, I) = \sum_n [P\{S_n \in I\} - P\{S_n \in x + I\}] \tag{3.33}$$

converges for any x *and interval* I.

Theorem 4 is trivial for a transient random walk. The special case where S_n is integer valued and F satisfies

$$\int x\, dF(x) = 0\,, \qquad \int x^2\, dF(x) < \infty \tag{3.34}$$

or F is symmetric is still rather simple. (3.29) for this case was proved by Hoeffding [24] (Theorem 4.5) and independently also by Kemeny and Snell [28] (Theorem 1 there implies (3.29) by Theorem III. 14 in [27] or Proposition 28.2 and the comment at the bottom of page 351 in [55]). The general case of (3.29) is much more delicate and was proved by Spitzer [54] (see also Section 28 of [55]). To go from the arithmetic case to the non-arithmetic case is highly non-trivial. Special cases were proven by Herz [22] and Bretagnolle and Dacunha-Castelle [3]. The convergence of (3.30) and (3.33) if f satisfies (3.32) and F^k has an absolutely continuous component is due to Ornstein [46]. A rather different proof for the convergence of (3.30) if f satisfies (3.31) and (3.32) was given by Port and Stone [48] and Stone [58] (especially Theorem 5 in [58]). The above papers go on to study the asymptotic behavior of $U(i, j)$ as $j \to \pm\infty$, respectively of $Uf(y)$ and $U(x, I + y)$ as $y \to \pm\infty$. For transient random walk this asymptotic behavior of U is precisely the content of the classical renewal theorem and these delicate asymptotic studies therefore form analogues of the renewal theorem for persistent random walk.

Even though none of the proofs of Theorem 4 go so far, it is tempting to conjecture that the series for U actually converge absolutely, under some obviously necessary aperiodicity assumption. For simplicity we only state this in the integer valued case.

CONJECTURE 1. *If S_n is an integer valued random walk such that for each integer i*

$$P\{S_k = i\} > 0\,, \qquad \text{for} \quad k \geqq k(i)\,,\ \text{say}, \tag{3.35}$$

then

$$\textstyle\sum_n |W_n(i, \{j\})| = \sum_n |P\{S_n = j\} - P\{S_n = i + j\}| < \infty\,. \tag{3.36}$$

In support of this conjecture we point out that it holds if F is symmetric (see (5.3)), or under not too stringent regularity assumptions as spelled out in Proposition 1 below. A similar propositon is indicated in part IV-c of [3].

PROPOSITION 1. *Let*[4]

$$C^s(m) = \textstyle\sum_{|k| \leqq m} k^2 P\{X_i^s = k\}\,.$$

Then (3.36) *holds whenever* (3.35) *holds for all i as well as one of the two conditions* (i) *or* (ii)*:*

(i) $\sum_{m=m_0}^{\infty} (mC^s(m))^{-1} < \infty$ *for some* m_0 *(e.g. if* $C^s(m) \geqq (\log m)^{1+\varepsilon}$ *for some* $\varepsilon > 0$ *and large* m*).*

(ii) $C^s(2m) \leqq (2^{\frac{1}{2}} - \varepsilon)C^s(m)$ *for some* $\varepsilon > 0$ *and large* m.

In particular, (3.36) holds whenever F is in the domain of (not necessarily normal) attraction of a stable law L (if L is the normal law then $C^s(\cdot)$ is slowly varying so that (ii) holds; if L is not normal then (i) holds (see [16], Theorem 17. 5.1)).

We postpone the proof of this proposition till the appendix, but it relies on rather coarse estimates only.

REMARK 6. The random variable analogue of the ratio theorems was (3.19). Clearly

$$\textstyle\sum_{k=1}^{n} W_k(i, \{j\}) = E\{N_n(j) - N_n(i + j)\}$$

so that random variable analogue of the study of W would be the study of the distribution of $N_n(j) - N_n(i + j)$. Without assumptions on F not much can be said about possible limits of this distribution. It is, however, an easy consequence of Theorem 2 in [31] that if S_n is persistent and $P\{S_{k_1} = i\} > 0$, $P\{S_{k_2} = j\} > 0$ for some k_1, k_2 then the distribution of

$$\frac{N_n(j) - N_n(i + j)}{(N_n(j))^{\frac{1}{2}}}$$

[4] The proof shows that $C^s(m)$ may be replaced by $C(m) = \sum_{|k| \leqq m} k^2 P\{X_1 = k\}$.

converges to a normal distribution with mean zero and some finite, strictly positive variance.

We close this section with a slightly more esoteric, but fascinating conjecture. The fascination lies in its generality and the very explicit and simple answer it proposes for the limit of a certain ratio. Again we restrict ourselves to the integer valued case; [47] and Section 5 of [48] form indications of what one should consider in the non-arithmetic case. In [35] and [30] (3.29) and further results of Spitzer [54] were used to prove ratio theorems for the taboo probabilities

$$Q_\Omega{}^n(i, j) = P\{i + S_n = j,\, i + S_k \notin \Omega \text{ for } 1 \leqq k \leqq n - 1\}\,.$$

It was shown that if S_n persistent and satifies (3.35) for all i and $EX_1{}^2 = \infty$, then there exist for every finite set Ω constants $c_n = c_n(\Omega)$ such that for all m, i, j

$$\frac{Q_\Omega{}^{n+m}(i, j)}{c_n} = \tilde{g}_\Omega(i)\tilde{g}_{-\Omega}(-j)\,.$$

The functions $\tilde{g}$ are certain potential theoretical functions defined in [30]. Dr. B. Belkin (private communication) showed that the assumption $EX_1{}^2 = \infty$ can be dropped. We were, however, unable in general to compare the $Q_\Omega(\cdot, \cdot)$ for different Ω. That is, we do not know if for $\Omega \subset \Omega'$, $Q_\Omega{}^n(i, j)$ and $Q^n_{\Omega'}(i, j)$ are of the same order, or if their ratio has a limit. Such comparisons all hinge on the following

CONJECTURE 2. *Let S_n be an integer valued random walk, and put*

$$f_n{}^{(r)} = P\{r\text{th } \textit{return to the origin occurs at time } n\}$$
$$= P\{S_n = 0 \text{ and } S_k = 0 \text{ for exactly } (r-1) \text{ values of } k \in [1, n-1]\}\,.$$

If S_n is persistent and

$$P\{S_n = 0\} > 0 \quad \textit{eventually}, \tag{3.37}$$

then

$$\lim_{n\to\infty} \frac{f_n{}^{(r)}}{f_n{}^{(1)}} = r\,. \tag{3.38}$$

We proved in [30] that the conjecture holds if F belongs to the domain of normal attraction of a symmetric stable law, and later, in [33], for every symmetric F. More generally (3.38) holds when

$$\operatorname{Im} \varphi(\theta) = O(|\theta|^3)\,, \qquad \theta \to 0\,. \tag{3.39}$$

With no conditions but persistence and (3.37) (see 11.20 and following lines of [30], for $r = 2$)

$$\liminf_{n\to\infty} \frac{f_n{}^{(r)}}{f_n{}^{(1)}} = r\,,$$

and (3.38) holds in the following "Abelian sense"

$$\lim_{t\uparrow 1} \frac{\sum_{n=1}^\infty nt^n f_n{}^{(r)}}{\sum_{n=1}^\infty nt^n f_n{}^{(1)}} = r\,, \qquad \sum_1^\infty nf_n{}^{(1)} = \infty\,.$$

REMARK 7. We draw the reader's attention to the fact that Problem 1 as well as Conjectures 1, 2 and 3 (to be formulated in Section 4) have been settled if F is symmetric. The great simplification in that case is that then $\mathrm{Im}\,\varphi(\theta) = 0$. The major stumbling block in the general case is that we do not know very well how $\mathrm{Im}\,\varphi(\theta)$ can behave as $\theta \to 0$. Many estimates would probably become easy if we could find explicit information about the relation between $\mathrm{Im}\,\varphi(\theta)$ and $1 - \mathrm{Re}\,\varphi(\theta)$ as $\theta \to 0$ for a general characteristic function φ.

4. Limit points of normalized random walk. With the exception of Remarks 5 and 6 the comments so far have dealt with the distribution of S_n, rather than the actual random variables S_n themselves. Here we want to discuss strong laws for $S_n/\gamma(n)$ for suitable sequences $\gamma(n)$. We are interested here in the set of accumulation points of $S_n/\gamma(n)$, i.e. in the set

$$A(S_n, \{\gamma(n)\}) = \bigcap_{m=1}^{\infty} \overline{\left\{\frac{S_n}{\gamma(n)} : n \geqq m\right\}} . \tag{4.1}$$

The bar in the right-hand side of (4.1) denotes closure in the extended real line $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$ with its usual topology. Thus we allow $+\infty$ or $-\infty$ as accumulation points. In this section $X_1, X_2, \cdots$ *will again always be identically distributed* with distribution function F and characteristic function φ, and the topology and closure will always be that of $\overline{\mathbb{R}}$.

As the notation indicates $A(S_n, \{\gamma_n\})$ is a random set which depends on the sample sequence $\{S_n\}_{n\geqq 1}$. It is not hard to show though, that A is w.p. 1 independent of the sample sequence. More precisely (Theorem 1 of [34]), there exists a closed set $B = B(F; \{\gamma_n\}) \subset \overline{\mathbb{R}}$ such that

$$A(S_n; \{\gamma_n\}) = B(F; \{\gamma_n\}) \quad \text{w.p. } 1 .\text{[5]} \tag{4.2}$$

Several classical results can be reformulated in terms of B. The oldest one perhaps is Kolomogorov's strong law of large numbers in which one takes $\gamma(n) = n$: If

$$\mu = \int x \, dF(x) \tag{4.3}$$

is well defined ($+\infty$ or $-\infty$ allowed as possible values), then

$$B(F; \{n\}) = \{\mu\} , \tag{4.4}$$

i.e., in view of (4.2),

$$\lim_{n\to\infty} \frac{S_n}{n} = \mu \quad \text{w.p. } 1 .$$

(See for instance Theorem 3.30 in [2]). Next, one has the law of the iterated logarithm, which takes $\gamma(n) = (2n \log \log n)^{\frac{1}{2}}$. The Hartman and Wintner [20] result, when specialized to identically distributed variables, can be formulated

[5] In Theorem 1 of [34] we made the assumption $\gamma(n) \to \infty$, but this is not used at all in the proof and therefore superfluous.

as follows: If

$$\int x\,dF(x) = 0 \qquad \text{and} \qquad \sigma^2 = \int x^2\,dF(x) < \infty$$

then

$$\{-\sigma, +\sigma\} \subset B(F; \{(2n \log\log n)^{\frac{1}{2}}\}) \subset [-\sigma, +\sigma] \tag{4.5}$$

(the first member in (4.5) denotes the set consisting of the two points $+\sigma$ and $-\sigma$: the last member the closed interval from $-\sigma$ to $+\sigma$). Strassen [60] and [61] proved a more detailed theorem, which in particular implies

$$B(F; \{(2n \log\log n)^{\frac{1}{2}}\}) = [-\sigma, +\sigma] \tag{4.6}$$

and if $\sigma^2 = \infty$ then at least one of the points $+\infty$ and $-\infty$ belongs to $B(F; \{(2n \log\log n)^{\frac{1}{2}}\})$. Another statement about B, perhaps the first which makes no a priori moment assumptions, is the celebrated Chung-Fuchs criterion for persistence or transience of a random walk (Theorem 3 in [7]); we give here the improved version of Spitzer [55], Theorem 8.2, Ornstein [46], Section 4 and Stone [58], Theorem 2. Here one takes $\gamma(n) \equiv 1$:

$$B(F; \{1\}) \subset \{-\infty, +\infty\} \tag{4.7}$$

if and only if

$$\int_{-a}^{+a} \mathrm{Re}\, \frac{1}{1 - \varphi(\theta)}\, d\theta < \infty \quad \text{for some} \quad a > 0\,.$$

(By (4.2) (4.7) is equivalent to $\lim_{n\to\infty} |S_n| = \infty$ w.p. 1, i.e., transience of S_n.)

We largely owe our interest in $B(F; \{\gamma(n)\})$ to questions and conjectures of K. C. Binmore and M. Katz (private communication). They were interested in the structure of $B(F; \{n\})$. In [1] they gave a necessary and sufficient condition for $B(F; \{n\}) = \{+\infty\}$; i.e., for $S_n \to \infty$ w.p. 1, and in a letter to the author they gave a sufficient condition for a point b to belong to $B(F; \{n\})$. The set $B(F; \{n\})$ can be viewed as a sort of generalized mean of F. For, if the mean (4.3) is well defined, then by (4.4) B consists only of one point, the mean of F. For the same reason one can view the determination of the structure of $B(F; \{n\})$ as a generalization of the strong law of large numbers without moment conditions. What are the possibilities for B, other than one point sets which can always be obtained by virtue of (4.4)? Simple examples (see [34], Section 5) show that $B = \{-\infty, +\infty\}$ is possible. But what if B contains a finite point b plus at least one other point? The answer to this question came as a surprise to us (see [34], Section 4 and 5):

THEOREM 5. *If $B(F; \{n\})$ contains at least two points, then $B(F; \{n\})$ has to contain $+\infty$ and $-\infty$. Any closed set B of $\overline{\mathbb{R}}$ which contains $+\infty$ and $-\infty$ is the $B(F; \{n\})$ for some F.*

Thus for closed sets B of $\overline{\mathbb{R}}$ of more than one point, the only restriction is that $+\infty \in B$ and $-\infty \in B$; it is for instance possible to construct a random walk such that w.p. 1 the accumulation points of S_n/n are exactly $-\infty$, 0 and

$+\infty$ or $-\infty$, $+\infty$ and all integers (but nothing else). It is, however, not possible to find a random walk with limit points 0 and $+\infty$ only for S_n/n.

The case $\gamma(n) = n$ may well be special because we can translate $B(F; \{n\})$ by a constant a by replacing $F(x)$ by $F(x - a)$. Probably another special case is $\gamma(n) = n^{\frac{1}{2}}$, because the concentration function inequalities of Theorem 1 tell us that the distribution of S_n is spread out at least over a distance of $n^{\frac{1}{2}}$. An indication of this effect is provided by Stone's result [59]:

$$\text{If } P\left\{\limsup_{n\to\infty} \frac{S_n}{n^{\frac{1}{2}}} = -\infty\right\} < 1\,, \qquad \text{then } P\left\{\limsup_{n\to\infty} \frac{S_n}{n^{\frac{1}{2}}} = +\infty\right\} = 1\,.$$

We believe that a stronger result holds which we state as

CONJECTURE 3. *If $B(F; \{n^{\frac{1}{2}}\})$ contains any finite point, then $B(F; \{n^{\frac{1}{2}}\}) = \overline{\mathbb{R}}$.*

Good indications for the truth of this conjecture are given in [34]: If $\alpha < \frac{1}{2}$ and $B(F; \{n^\alpha\})$ contains any finite point, then $B(F; \{n^\alpha\}) = \overline{\mathbb{R}}$. If $B(F; \{n^{\frac{1}{2}}\})$ contains the finite point b, then it contains $[b, \infty]$ or $[-\infty, b]$. The conjecture is true for any persistent random walk S_n, as well as when F is symmetric, or more generally

$$|\mathrm{Im}\, \varphi(\theta)| = O(1 - \mathrm{Re}\, \varphi(\theta))\,, \qquad \theta \to 0$$

(see Section 3 of [34]). Lastly, the conjecture is true for any F for which (3.21) holds (see the comments following Problem 1).

Apart from the stated results we do not know much about $B(F; \{n^\alpha\})$. Thus there is the natural

PROBLEM 2. Find the possible structures of $B(F; \{n^\alpha\})$ for $\alpha > \frac{1}{2}$, $\alpha \neq 1$.

Several related problems have been stated in [34]. We restricted ourselves there to sequences $\gamma(n) = n^\alpha$, but one may of course consider more general sequences, and, as we shall see, there are good reasons to allow for a sequence of translation parameters $\delta(n)$. This leads to the more general, but vaguer

PROBLEM 3. Find the structure of the set of accumulation points of

$$\frac{S_n - \delta(n)}{\gamma(n)}$$

for suitable sequences $\{\gamma(n)\}$, $\{\delta(n)\}$.

This is not just an idle generalization, but it is closely related to the law of the iterated logarithm. Recently Feller [17], Rogozin [52], Heyde [23] (also a question of P. Révész in a conversation with the author) dealt with the following version of Problem 3: When is the set of accumulation points of $(S_n - \delta(n))/\gamma(n)$ bounded, and if so, what are lim sup and lim inf of $(S_n - \delta(n))/\gamma(n)$? Actually, this is not quite their formulation and more precision is clearly needed to make the problem interesting. For one can always take $\gamma(n)$ so large that

$$\frac{S_n - \delta(n)}{\gamma(n)} \to 0 \quad \text{w.p. } 1\,.$$

Thus it is more reasonable to require

$$(4.8) \qquad -\infty < \liminf_{n\to\infty} \frac{S_n - \delta(n)}{\gamma(n)} < \limsup_{n\to\infty} \frac{S_n - \delta(n)}{\gamma(n)} < +\infty .$$

Even then the question is unreasonable, for if $\gamma(n)$ is so large that $S_n/\gamma(n) \to 0$ w.p, 1, then the accumulation points of $(S_n - \delta(n))/\gamma(n)$ are precisely those of $-\delta(n)/\gamma(n)$. If $\delta(n)$ is unrestricted we can therefore still obtain anything for the accumulation points of $(S_n - \delta(n))/\gamma(n)$. However, we are interested in the fluctuations of S_n, not of $\delta(n)$, and in order to catch these we take $\delta(n)$ "well in the support" of the distribution of S_n, i.e., we add the requirement

$$(4.9) \qquad P\{S_n \geqq \delta(n)\} \geqq \pi \quad \text{and} \quad P\{S_n \leqq \delta(n)\} \geqq \pi$$

for some fixed $0 < \pi < 1$. The simplest choice would be

$$\delta(n) = \text{median } S_n ,$$

corresponding to $\pi = \frac{1}{2}$. This choice is strongly suggested by [52] and [23]. The problem now becomes: When do there exist sequences $\{\gamma(n)\}$ and $\{\delta(n)\}$ satisfying (4.8) and (4.9) for some fixed $0 < \pi < 1$? This is completely answered by

THEOREM 6. *If $X_1, X_2, \cdots$ are independent random variables with common distribution function F and if $\delta(n)$ is a sequence of numbers satisfying* (4.9), *then there exists a strictly positive sequence $\{\gamma(n)\}$ for which* (4.8) *holds if and only if F belongs to the domain of partial attraction of the normal law. If such a $\{\gamma(n)\}$ exists, it can be chosen such that $n^{-\frac{1}{2}+\varepsilon}\gamma(n)$ is increasing, for any fixed $\varepsilon > 0$; also if $\{\delta'(n)\}$ is another sequence satisfying* (4.9) *(possibly with a $0 < \pi' < \pi$) then $\gamma(n)^{-1}|\delta(n) - \delta'(n)| \to 0$ so that* (4.8) *also holds with δ replaced by δ' and*

$$(4.10) \qquad \lim {\inf \atop \sup} \frac{S_n - \delta(n)}{\gamma(n)} = \lim {\inf \atop \sup} \frac{S_n - \delta'(n)}{\gamma(n)} .$$

The necessity of the condition was proved by Heyde [23] and Rogozin [52] and the sufficiency is proved in the Appendix.

As we pointed out, there is no point in admitting arbitrary $\delta(n)$, but it still seems worthwhile to single out the case $\delta(n) = 0$. Thus we have

PROBLEM 4. Find necessary and sufficient conditions for the existence of $\{\gamma(n)\}$ satisfying

$$(4.11) \qquad -\infty < \liminf_{n\to\infty} \frac{S_n}{\gamma(n)} < \limsup_{n\to\infty} \frac{S_n}{\gamma(n)} < \infty \quad \text{w.p. } 1 .$$

There may well be unexpected aspects in this problem, for one can construct an F *with mean zero* and $\gamma(n)$ such that

$$-\infty < \liminf \frac{S_n - \text{med}\,(S_n)}{\gamma(n)} < \limsup \frac{S_n - \text{med}\,(S_n)}{\gamma(n)} < \infty$$

but

$$\limsup \frac{S_n}{\gamma(n)} = \infty \quad \text{w.p. } 1 .$$

Thus even though the mean may be zero, zero is not necessarily a good centering constant. We list two more problems which suggest themselves immediately.

PROBLEM 5. If (4.8) and (4.9) hold, find the set of all accumulation points of $(S_n - \delta(n))/\gamma(n)$, or more generally the set of accumulation points in a suitable function space of the random functions η_n defined on [0, 1] by

$$\eta_n(t) = \gamma(n)^{-1}\{(k + 1 - nt)(S_k - \delta(k)) + (nt - k)(S_{k+1} - \delta(k + 1))\}$$

on $[k/n, (k + 1)/n]$, $0 \leqq k < n$.

The problem is, of course, suggested by Strassen's work [60]. The results of [60] (see also (4.6)) suggest that the set of accumulation points of $(S_n - \delta(n))/\gamma(n)$ is exactly the closed interval from $\liminf (S_n - \delta(n))/\gamma(n)$ to $\limsup (S_n - \delta(n))/\gamma(n)$.

Much of the work on generalizations of the law of the iterated logarithm deals with conditions under which (4.8) holds for specific sequences $\gamma(n)$ defined in terms of F (see for instance Feller [17], Tomkins [62] and several references cited there). Perhaps one should first discuss

PROBLEM 6. What sequences $\{\gamma(n)\}$ can satisfy (4.8) for any F and $\delta(n)$ satisfying (4.9)?

As stated in Theorem 6, we may assume $n^{-\frac{1}{2}+\varepsilon}\gamma(n) \uparrow$ and if F is not concentrated on one point, then we prove in Section 5, Lemma 4 (proof of (4.16)) that necessarily

$$n^{-\frac{1}{2}}\gamma(n) \to \infty \ . \tag{4.12}$$

On the other hand Rogozin [52], Theorem 4, proved that if $\gamma(n)$ satisfies

$$\gamma(n) \uparrow \quad \text{and} \quad \sum_{k=n}^{\infty} (\gamma(k))^{-2} \leqq Cn(\gamma(n))^{-2} \tag{4.13}$$

for some $C < \infty$, then $\gamma(n)$ cannot satisfy (4.8). Thus $\gamma(n) = n^{\frac{1}{2}+\varepsilon}L(n)$ with $\varepsilon > 0$ and L slowly varying is not a possible choice in (4.8) for any F. We see from (4.12) and (4.13) that if we restrict ourselves to "nice" sequences $\{\gamma(n)\}$, then they must be large w.r.t. $n^{\frac{1}{2}}$ but still "close to $n^{\frac{1}{2}}$." The most explicit information about nice sequences is summarized in the following theorem which is essentially due to Feller [17]. It is a slight modification of Theorems 1 and 3 of [17] and essentially proved in the same way. We shall therefore be content with only a remark on the proof in the Appendix. Fristedt [18] has proved an analogous theorem for processes with stationary independent increments. Notice that the restriction to symmetric F is for convenience only, for as we shall show in the proof of Theorem 6, (4.8) is under condition (4.9) equivalent to

$$-\infty < \liminf_{n\to\infty} \frac{S_n^s}{\gamma(n)} < 0 < \limsup_{n\to\infty} \frac{S_n^s}{\gamma(n)} < +\infty \quad \text{w.p. } 1 \ . \tag{4.14}$$

THEOREM 7. *Let $X_1, X_2, \cdots$ be independent random variables, all with the symmetric distribution function F, not concentrated on $\{0\}$. Put*

$$H(x) = \int_{|y| \leqq x} y^2 \, dF(y) \ .$$

If $\gamma(n)$ is any sequence of positive numbers such that

$$\limsup \frac{S_n}{\gamma(n)} < \infty$$

then

$$\lim_{n\to\infty} nP\{|X_1| > \eta\gamma(n)\} = 0 \quad \textit{for all} \quad \eta > 0\,, \tag{4.15}$$

and

$$\lim_{n\to\infty} \frac{\gamma^2(n)}{nH(\gamma(n))} = \infty\,. \tag{4.16}$$

If $\gamma(n)$ satisfies

$$\gamma(n)\uparrow \quad \textit{and} \quad \inf_{1\leqq k\leqq n} \frac{\gamma(n)}{n^\xi}\left(\frac{\gamma(k)}{k^\xi}\right)^{-1} > 0 \quad \textit{for some} \quad \xi > 0 \tag{4.17}$$

and

$$\sum P\{|X_1| > \gamma(n)\} = \infty\,, \tag{4.18}$$

then

$$\limsup \frac{S_n}{\gamma(n)} = \infty \quad \text{w.p. } 1\,. \tag{4.19}$$

Let

$$\chi_k = 1 \quad \textit{if there is some} \quad \gamma(n) \in (2^{k-1}, 2^k]$$
$$= 0 \quad \textit{otherwise}\,,$$

and[6]

$$\rho = \inf\{\lambda \colon \textstyle\sum_{k=1}^\infty \chi_k k^{-\lambda} < \infty\}\,.$$

If $\gamma(n)$ satisfies (4.17) *and*

$$\sum P\{|X_1| > \gamma(n)\} < \infty \tag{4.20}$$

and $\rho > 0$ and

$$nH(\gamma(n))[\log\log nH(\gamma(n))]^\tau \leqq (\gamma(n))^2 = o\{nH(\gamma(n))[\log\log nH(\gamma(n))]\} \tag{4.21}$$

for some $0 < \tau < 1$ then (4.19) *holds again. If $\gamma(n)\uparrow$ and*

$$0 < \lim_{n\to\infty} \frac{(\gamma(n))^2}{nH(\gamma(n))[\log\log nH(\gamma(n))]} = A < \infty \tag{4.22}$$

and (4.20) *holds, then*

$$\limsup_{n\to\infty} \frac{S_n}{\gamma(n)} = \left(\frac{2\rho}{A}\right)^{\frac{1}{2}} \quad \text{w.p. } 1\,.$$

Finally, if $\gamma(n)$ satisfies (4.17) *and* (4.20) *and*

$$\lim_{n\to\infty} \frac{(\gamma(n))^2}{nH(\gamma(n))[\log\log nH(\gamma(n))]} = \infty \tag{4.23}$$

then

$$\limsup_{n\to\infty} \frac{S_n}{\gamma(n)} = 0 \quad \text{w.p. } 1\,. \tag{4.24}$$

---

[6] If $\gamma(n)$ does not have too large gaps $\rho = 1$.

If any reader wants more work we only need to point out that almost all the results and problems of this section are meaningful for non-identically distributed X_i and that very little has been done in this generality (even though [23] and [52] allow the X_i to have different distributions).

5. Appendix. We give here the proofs of Proposition 1 and Theorem 6.

PROOF OF PROPOSITION 1.

$$
\begin{aligned}
W_n(k, \{j\}) &= P\{S_n = j\} - P\{S_n = k + j\} \\
&= \frac{1}{2\pi} \int_{-\pi}^{+\pi} (e^{-ij\theta} - e^{-i(k+j)\theta})\varphi^n(\theta)\, d\theta \qquad (5.1)\\
&= \frac{-k}{2\pi} \int_{-\pi}^{+\pi} \theta \operatorname{Im} \varphi^n(\theta)\, d\theta + \frac{1}{2\pi} \int_{-\pi}^{+\pi} O(\theta)^2 \varphi^n(\theta)\, d\theta ,
\end{aligned}
$$

where $\theta^{-2}O(\theta^2)$ remains bounded as $\theta \to 0$. Now $|\varphi(\theta)|^2$ is the characteristic function of X_i^s and

$$
\begin{aligned}
1 - |\varphi(\theta)|^2 &= \sum_{-\infty}^{+\infty} (1 - \cos k\theta) P\{X_i^s = k\} \qquad (5.2)\\
&\geqq \sum_{k \leqq \pi/|\theta|} (1 - \cos k\theta) P\{X_i^s = k\} \geqq \frac{2}{\pi^2} \theta^2 C^s\left(\left[\frac{\pi}{|\theta|}\right]\right) .
\end{aligned}
$$

It is well known that (3.35) for all i implies $|\varphi(\theta)| \neq 1$ for $0 < |\theta| \leqq \pi$ ([55], Proposition 7.8) so that it follows from (5.2) that

$$\sum_n \int_{-\pi}^{+\pi} \theta^2 |\varphi(\theta)|^n \, d\theta < \infty$$

and (3.36) will therefore follow from

$$\sum_n \int_{-\pi}^{+\pi} |\theta| \, |\operatorname{Im} (\varphi(\theta))^n| \, d\theta < \infty . \qquad (5.3)$$

(Of course this shows that (3.36) holds whenever F is symmetric.) We can estimate trivially the left-hand side of (5.3) by

$$
\begin{aligned}
&\sum_n \int_{-\pi}^{+\pi} |\theta| \, |\varphi(\theta)|^n \, d\theta \\
&\quad = \int_{-\pi}^{+\pi} \frac{|\theta|}{1 - |\varphi(\theta)|} d\theta \leqq 2 \int_{-\pi}^{+\pi} \frac{|\theta|}{1 - |\varphi(\theta)|^2} d\theta \leqq 2\pi^2 \int_0^{\pi m_0^{-1}} \frac{d\theta}{|\theta| C^s(\pi/|\theta|)} + O(1) \\
&\quad \leqq 2\pi^2 \sum_{m=m_0}^{\infty} \int_{\pi(m+1)^{-1}}^{\pi m^{-1}} \frac{d\theta}{\theta C^s(m)} + O(1) \leqq 2\pi^2 \sum_{m=m_0}^{\infty} \frac{1}{m C^s(m)} + O(1)
\end{aligned}
$$

so that (3.36) indeed holds, if (i) in Proposition 1 is fulfilled. Slightly more tricky is the proof under condition (ii). First observe that (ii) implies

$$C^s(m) = O(m^{\frac{1}{2}}) , \qquad m \to \infty . \qquad (5.4)$$

Thus, a simple summation by parts shows that

$$E\{|X_1^s|\} = \sum_{m=1}^{\infty} m^{-1}[C^s(m) - C^s(m-1)] < \infty .$$

Since, when A is so large that $P\{|\tilde{X}_1| \leqq A\} \geqq \frac{1}{2}$,

$$\tfrac{1}{2} P\{|X_1| \geqq m + A\} \leqq P\{|X_1| \geqq m + A\} P\{|\tilde{X}_1| \leqq A\} \leqq P\{|X_1^s| \geqq m\} \qquad (5.5)$$

it follows that also $E|X_1| < \infty$.

Also, if we put

$$C(m) = \sum_{|k|\leq m} k^2 P\{X_1 = k\} = E\{X_1^2 I[|X_1| \leqq m]\}$$

then

$$\begin{aligned} C^s(m) &= E\{(X_1 - \tilde{X}_1)^2 I[|X_1 - \tilde{X}_1| \leqq m\} \\ &\geqq E\{(X_1^2 + \tilde{X}_1^2 - 2EX_1\tilde{X}_1) I[|X_1| \leqq \tfrac{1}{2}m, |\tilde{X}_1| \leqq \tfrac{1}{2}m]\} \\ &= 2E\{X_1^2 I[|X_1| \leqq \tfrac{1}{2}m]\} P\{|\tilde{X}_1| \leqq \tfrac{1}{2}m\} - 2E\{X_1 I[|X_1| \leqq \tfrac{1}{2}m]\}^2 . \end{aligned} \tag{5.6}$$

If $\int x^2 dF(x) < \infty$, then $C(m)$ is bounded and for suitable m_1 and $D < \infty$ (use Assumption (ii) again)

$$C([\tfrac{1}{2}m]) \leqq DC^s(m) \leqq 4DC^s([\tfrac{1}{2}m]) , \qquad m \geqq m_1 ,$$

and consequently

$$C(m) \leqq 4DC^s(m) , \qquad m \leqq m_1 . \tag{5.7}$$

If, on the other hand, $\int x^2 dF(x) = \infty$, then for any B

$$\begin{aligned} E\{|X_1| I[|X_1| \leqq \tfrac{1}{2}m]\} &\leqq B + \textstyle\int_{B \leq |x| \leq [\frac{1}{2}m]} |x|\, dF(x) \\ &\leqq B + [\textstyle\int_{|x| \leq \frac{1}{2}m} |x|^2 dF(x) \int_{|x| \geq B} dF(x)]^{\frac{1}{2}} . \end{aligned}$$

Thus, if $\int x^2 dF(x) = \infty$

$$E\{X_1 I[|X_1| \leqq \tfrac{1}{2}m]\}^2 = o(\textstyle\int_{|x| \leq \frac{1}{2}m} |x|^2 dF(x)) = o(C(\tfrac{1}{2}m)) \tag{5.8}$$

as $m \to \infty$, and it follows easily from (5.6) and (5.8) that (5.7) again holds.

We now use the trick of Lemma 3.2 of [24].

$$\operatorname{Im} \varphi(\theta)^n = \frac{1}{2i}[\varphi^n(\theta) - \varphi^n(-\theta)] = \operatorname{Im} \varphi(\theta) \textstyle\sum_{j=0}^{n-1} \varphi^j(\theta)\varphi^{n-1-j}(-\theta)$$

so that the left-hand side of (5.3) is bounded by

$$\begin{aligned} &\textstyle\sum_n \int_{-\pi}^{+\pi} |\theta|\, |\operatorname{Im} \varphi(\theta)| n |\varphi(\theta)|^{n-1} d\theta \\ &= \int_{-\pi}^{+\pi} \frac{|\theta|\, |\operatorname{Im} \varphi(\theta)|}{\{1 - |\varphi(\theta)|\}^2} d\theta \leqq 2\pi^4 \int_0^{\pi m_0^{-1}} \frac{|\operatorname{Im} \varphi(\theta)|}{\theta^3 (C^s[\pi/\theta])^2} d\theta + O(1) \\ &\leqq 2\pi^4 \textstyle\sum_{m=m_0}^{\infty} (C^s(m))^{-2} \int_{\pi(m+1)^{-1}}^{\pi m^{-1}} \theta^{-3} |\operatorname{Im} \varphi(\theta)|\, d\theta + O(1) . \end{aligned}$$

We proceed to estimate $|\operatorname{Im} \varphi(\theta)|$ on $[\pi(m+1)^{-1}, \pi m^{-1}]$. We already know $E|X_1| < \infty$. If $EX_1 \neq 0$, then S_n is transient and (3.36) is trivial, so that we may assume $EX_1 = 0$. Then, if we write $p_k = P\{X_1 = k\}$ we have for $\theta \in [\pi(m+1)^{-1}, \pi m^{-1}]$

$$\begin{aligned} |\operatorname{Im} \varphi(\theta)| &\leqq \textstyle\sum_{k=1}^{\infty} |\sin k\theta - k\theta|\, |p_k - p_{-k}| \\ &\leqq \textstyle\sum_{k=1}^{m} k^3\theta^3 (p_k + p_{-k}) + 2 \sum_{k=m+1}^{\infty} k\theta (p_k + p_{-k}) . \end{aligned}$$

In view of (5.7) it therefore suffices to prove for m_0 so large that $C(m_0) > 0$:

$$\textstyle\sum_{m=m_0}^{\infty} \{mC(m)\}^{-2} \sum_{k=1}^{m} k^3 (p_k + p_{-k}) < \infty , \tag{5.9}$$

and

$$\textstyle\sum_{m=m_0}^{\infty} \{C^s(m)\}^{-2} \sum_{k=m+1}^{\infty} k(p_k + p_{-k}) < \infty . \tag{5.10}$$

But,

$$\begin{aligned}
&\sum_{m=m_0}^{\infty} \{mC(m)\}^{-2} \sum_{k=1}^{m} k^3(p_k + p_{-k}) \\
&\quad = \sum_{k=1}^{\infty} k^3(p_k + p_{-k}) \sum_{m \geqq \max(m_0,k)} \{mC(m)\}^{-2} \\
&\quad \leqq O(1) + \sum_{k=m_0}^{\infty} \{C(k)\}^{-2} k^3(p_k + p_{-k}) \sum_{m \geqq k} m^{-2} \\
&\quad \leqq O(1) + \sum_{k=m_0}^{\infty} \frac{k^2(p_k + p_{-k})}{\{C(k)\}^2}
\end{aligned}$$

which is finite, because

$$C(k) = \sum_{j=1}^{k} j^2(p_j + p_{-j})$$

(see Abel-Dini's theorem, [37], Section 39). Thus (5.9) always holds. As for (5.10), after interchange of the order of summation we obtain at most

$$\begin{aligned}
&\sum_{k=m_0}^{\infty} k(p_k + p_{-k}) \sum_{r=0}^{\infty} \sum_{2^{-r-1}k < m \leqq 2^{-r}k, m \geqq m_0} \{C^s(m)\}^{-2} \\
&\quad \leqq \sum_{k=m_0}^{\infty} k(p_k + p_{-k})\{C^s(k)\}^{-2} \sum_{r=0}^{\infty} 2^{-r}k(2^{\frac{1}{2}} - \varepsilon)^{2r+2} \quad \text{(condition (ii))} \\
&\quad = O\left(\sum_{k=m_0}^{\infty} \frac{k^2(p_k + p_{-k})}{\{C(k)\}^2}\right) < \infty ,
\end{aligned}$$

again by the Abel-Dini theorem.

REMARK 8. There exist integer valued random walks with mean zero for which (5.3) fails. One therefore needs more delicate procedures to establish (3.36) than the ones used here.

PROOF OF THEOREM 6. This proof is broken down into short lemmas.

LEMMA 1. *If* (4.9) *holds, then* (4.8) *is equivalent to*

$$-\infty < \liminf_{n\to\infty} \frac{S_n^s}{\gamma(n)} < 0 < \limsup_{n\to\infty} \frac{S_n^s}{\gamma(n)} < +\infty \quad \text{w.p. } 1 . \tag{5.11}$$

PROOF. By the Hewitt-Savage 0–1 law ([2], Section 3.9) lim inf and lim sup of $(S_n - \delta(n))/\gamma(n)$ are constants w.p. 1, and hence equal to lim inf respectively lim sup of $(\tilde{S}_n - \delta(n))/\gamma(n)$ w.p. 1. Therefore, if we write

$$\lim \begin{matrix}\inf \\ \sup\end{matrix} \frac{S_n - \delta(n)}{\gamma(n)} = \begin{matrix}\rho_- \\ \rho_+\end{matrix} ,$$

then

$$\begin{aligned}
\limsup \frac{S_n^s}{\gamma(n)} &= \limsup \frac{S_n - \tilde{S}_n}{\gamma(n)} \\
&\leqq \limsup \frac{S_n - \delta(n)}{\gamma(n)} - \liminf \frac{\tilde{S}_n - \delta(n)}{\gamma(n)} \\
&\leqq \rho_+ - \rho_- \quad \text{w.p. } 1
\end{aligned} \tag{5.12}$$

and similarly

$$\liminf \frac{S_n^s}{\gamma(n)} \geqq \rho_- - \rho_+ \quad \text{w.p. } 1 . \tag{5.13}$$

We now prove that on the other hand

$$\text{(5.14)} \qquad \liminf \frac{S_n^s}{\gamma(n)} \leqq \rho_- \qquad \text{and} \qquad \limsup \frac{S_n^s}{\gamma(n)} \geqq \rho_+ \quad \text{w.p. } 1 .$$

Indeed, let $\varepsilon > 0$, $T_0 = 0$ and

$$T_{k+1} = \inf\left\{n > T_k : \frac{S_n - \delta(n)}{\gamma(n)} \geqq \rho_+ - \varepsilon\right\} .$$

By definition of ρ_+, $T_k < \infty$ for all k w.p. 1, and[7]

$$\begin{aligned} & P\left\{\limsup \frac{S_n^s}{\gamma(n)} = \limsup \frac{(S_n - \delta(n)) - (\tilde{S}_n - \delta(n))}{\gamma(n)} \geqq \rho_+ - \varepsilon\right\} \\ \text{(5.15)} \qquad & \geqq P\left\{T_k < \infty \quad \text{for all } k \text{ and } \frac{\tilde{S}_{T_k} - \delta(T_k)}{\gamma(T_k)} \leqq 0 \text{ i.o.}\right\} \\ & \geqq \limsup_{k\to\infty} P\left\{\frac{\tilde{S}_{T_k} - \delta(T_k)}{\gamma(T_k)} \leqq 0\right\} \geqq \pi > 0 \end{aligned}$$

since T_k is defined in terms of $\{S_n\}_{n\geqq 1}$, which is independent of $\{\tilde{S}_n\}_{n\geqq 1}$. Again by the Hewitt-Savage 0–1 law, (5.15) implies

$$\text{(5.16)} \qquad P\left\{\limsup \frac{S_n^s}{\gamma(n)} \geqq \rho_+ - \varepsilon\right\} = 1 .$$

(5.16) holds for any $\varepsilon > 0$ so that the second statement in (5.14) follows and the first statement is proved in the same way. The equivalence of (4.8) and (5.11) is immediate from the inequalities (5.12)—(5.14) and the simple consequence of symmetry of S_n^s and the Hewitt-Savage 0–1 law

$$\text{(5.17)} \qquad \liminf \frac{S_n^s}{\gamma(n)} = - \limsup \frac{S_n^s}{\gamma(n)} \quad \text{w.p. } 1 .$$

Next we derive a necessary and sufficient condition for (5.11) in terms of convergence of a certain series when $\gamma(n) \uparrow$.

LEMMA 2. *If for all $n \geqq 1$ and some $\alpha > 0$*

$$\text{(5.18)} \qquad P\{S_n \geqq 0\} \geqq \alpha \qquad \text{and} \qquad P\{S_n \leqq 0\} \geqq \alpha ,$$

and $0 < \gamma(n) \uparrow \infty$, then

$$\text{(5.19)} \qquad \limsup \frac{S_n}{\gamma(n)} \geqq \rho \quad \text{w.p. } 1$$

implies

$$\text{(5.20)} \qquad \sum_{r=1}^{\infty} P\{S_{2^r} \geqq (\tfrac{1}{2}\rho - \varepsilon)\gamma(2^r)\} = \infty \quad \text{for all} \quad \varepsilon > 0 ,$$

and, in turn, (5.20) implies

$$\text{(5.21)} \qquad \limsup \frac{S_n}{\gamma(n)} \geqq \tfrac{1}{4}\rho \quad \text{w.p. } 1 .$$

[7] "A_k i.o." stands for "the event A_k occurs infinitely often, i.e., for infinitely many k."

PROOF. First observe that w.p. 1

$$(5.22)\quad \limsup \frac{S_n}{\gamma(n)} = \limsup \frac{X_1 + \sum_2^n X_i}{\gamma(n)} = \limsup \frac{\sum_2^n X_i}{\gamma(n)} = \limsup \frac{S_{n-1}}{\gamma(n)},$$

and (5.22) remains valid if n is restricted to an arbitrary increasing subsequence of the positive integers. Now, let

$$(5.23)\quad \limsup \frac{S_n}{\gamma(n)} = L \text{ w.p. } 1 .$$

Clearly

$$(5.24)\quad \limsup \frac{S_{2n}}{\gamma(2n)} \leqq L \text{ w.p. } 1 ,$$

But we actually must have equality in (5.24). Firstly we must have $\limsup S_{n_i} \geqq 0$ for any subsequence $n_i \to \infty$ by the Hewitt-Savage 0–1 law and

$$P\{S_{n_i} \geqq 0 \text{ i.o.}\} \geqq \limsup P\{S_{n_i} \geqq 0\} \geqq \alpha > 0 .$$

Thus $L \geqq 0$. Secondly, if

$$(5.25)\quad \limsup \frac{S_{2n}}{\gamma(2n)} < L$$

then by $0 < \gamma(n) \uparrow$ and (5.22) also

$$(5.26)\quad \limsup \frac{S_{2n-1}}{\gamma(2n-1)} = \limsup \frac{S_{2n-2}}{\gamma(2n-1)} \leqq \limsup \frac{S_{2n-2}}{\gamma(2n-2)} < L ,$$

and of course (5.25) and (5.26) together contradict (5.23). Thus

$$(5.27)\quad \begin{aligned} L = \limsup \frac{S_{2n}}{\gamma(2n)} &= \limsup \left(\frac{\sum_1^n X_{2i}}{\gamma(2n)} + \frac{\sum_1^n X_{2i-1}}{\gamma(2n)} \right) \\ &\leqq 2 \limsup \frac{S_n}{\gamma(2n)} \quad \text{w.p. } 1 , \end{aligned}$$

so that for all $\varepsilon > 0$ w.p. 1 infinitely many of the events

$$A_r = \{S_n \geqq (\tfrac{1}{2}L - \varepsilon)\gamma(2^r) \text{ for some } 2^{r-1} \leqq n < 2^r\}$$

occur. But

$$\begin{aligned} P\{S_{2^r} &\geqq (\tfrac{1}{2}L - \varepsilon)\gamma(2^r) \text{ and } A_r\} \\ &\geqq \textstyle\sum_{2^{r-1} \leqq n < 2^r} P\{S_n \geqq (\tfrac{1}{2}L - \varepsilon)\gamma(2^r),\ S_j < (\tfrac{1}{2}L - \varepsilon)\gamma(2^r) \\ &\qquad \text{for } 2^{r-1} \leqq j < n\} P\{S_{2^r} - S_n \geqq 0\} \\ &\geqq \alpha \textstyle\sum_{2^{r-1} \leqq n < 2^r} P\{S_n \geqq (\tfrac{1}{2}L - \varepsilon)\gamma(2^r),\ S_j < (\tfrac{1}{2}L - \varepsilon)\gamma(2^r) \\ &\qquad \text{for } 2^{r-1} \leqq j < n\} \\ &= \alpha P\{A_r\} \end{aligned}$$

so that by the extended Borel-Cantelli lemma (see Problem 5.6.9 in [2]) also

$$(5.28)\quad \{S_{2^r} \geqq (\tfrac{1}{2}L - \varepsilon)\gamma(2^r) \text{ i.o.}\} \quad \text{a.e. on} \quad \{A_r \text{ i.o.}\} .$$

Since we already proved that (5.23) implies

$$P\{A_r \text{ i.o.}\} = 1 ,$$

it follows from (5.28) and the Borel-Cantelli lemma that for all $\varepsilon > 0$

$$\textstyle\sum_r P\{S_{2^r} \geqq (\frac{1}{2}L - \varepsilon)\gamma(2^r)\} = \infty .$$

This proves (5.20). Now assume that (5.20) holds. Then it follows from the fact that the $S_{2^{r+1}} - S_{2^r}$ is independent of and has the same distribution as S_{2^r} that also

$$\textstyle\sum_r P\left\{S_{2^r} \geqq 0 \text{ and } \dfrac{S_{2^{r+1}} - S_{2^r}}{\gamma(2^r)} \geqq \frac{1}{2}\rho - \varepsilon\right\} \geqq \alpha \sum_r P\left\{\dfrac{S_{2^{r+1}} - S_{2^r}}{\gamma(2^r)} \geqq \frac{1}{2}\rho - \varepsilon\right\} = \infty . \tag{5.29}$$

Denote by B_r^1 the event between braces in the first member of (5.29) and by B_r^0 the complement of B_r^1. Then for $s > r$ and any choice of $\varepsilon_1, \cdots, \varepsilon_r$, $\varepsilon_i = 0$ or 1,

$$P\{B_s^1 \mid B_1^{\varepsilon_1}, \cdots, B_r^{\varepsilon_r}\} \leqq P\{S_{2^{s+1}} - S_{2^s} \geqq (\tfrac{1}{2}\rho - \varepsilon)\gamma(2^s)\} \leqq \frac{1}{\alpha} P\{B_s^1\} .$$

Now

$$\begin{aligned} \textstyle\sum_N^M P\{B_r^1\} &= E\{\# \text{ of } N \leqq r \leqq M \text{ with } B_r^1\} \\ &= \textstyle\sum_N^M P\{B_r^1 \text{ is the first } B_j^1 \text{ to occur on or after } N\} \\ &\quad (1 + E\{\# \text{ of } r < s \leqq M \text{ with } B_s^1 \mid B_N^0, B_{N+1}^0, \cdots, B_{r-1}^0, B_r^1\}) \\ &\leqq \left(1 + \frac{1}{\alpha} \textstyle\sum_N^M P\{B_s^1\}\right) \cdot P\{\text{some } B_r^1 \text{ with } N \leqq r \leqq M \text{ occurs}\} . \end{aligned} \tag{5.30}$$

Since, by (5.29)

$$\textstyle\sum_N^\infty P\{B_r^1\} = \infty$$

we find from (5.30) by letting $M \to \infty$

$$P\{\text{some } B_r^1 \text{ with } r \geqq N \text{ occurs}\} \geqq \alpha > 0 . \tag{5.31}$$

(5.31) holds for all N and thus

$$P\{B_r^1 \text{ i.o.}\} \geqq \alpha > 0 . \tag{5.32}$$

Again by the Hewitt-Savage 0–1 law the left-hand side of (5.32) must equal 1 and thus

$$P\{S_{2^{r+1}} \geqq (\tfrac{1}{2}\rho - \varepsilon)\gamma(2^r) \text{ i.o.}\} \geqq P\{B_r^1 \text{ i.o.}\} = 1 .$$

It follows that

$$\limsup \frac{S_{2n}}{\gamma(n)} \geqq \tfrac{1}{2}\rho \quad \text{w.p. } 1$$

and (5.21) follows from this and (5.27) with $\gamma(2n)$ replaced by $\gamma(n)$.

LEMMA 3. *If* (5.11) *holds, then it also holds with* $\{\gamma(n)\}$ *replaced by the increasing sequence* $\{\gamma^*(n)\}$ *defined by*

$$\gamma^*(n) = \inf_{k \geqq n} \gamma(k) . \tag{5.33}$$

PROOF. An easy concentration function argument shows that (5.11) implies $\gamma(n) \to \infty$, unless F^s is concentrated on $\{0\}$ (see Lemma 4 for a sharper statement with proof). We ignore this trivial case so that we may assume $\gamma(n) \to \infty$, $\gamma^*(n) \to \infty$ and that for each n there is an $f(n) \geqq n$ with $\gamma^*(n) = \gamma(f(n))$. Now $\gamma^*(n) \leqq \gamma(n)$ so that (5.11) implies

$$(5.34) \qquad 0 < \limsup \frac{S_n^s}{\gamma(n)} \leqq \limsup \frac{S_n^s}{\gamma^*(n)} \quad \text{w.p. } 1 .$$

Assume that the second inequality in (5.34) is strict, and write again L for the almost certain value of $\limsup (\gamma(n))^{-1}S_n^s$. Then for some $\varepsilon > 0$

$$(5.35) \quad 1 = P\{S_n^s \geqq (L+\varepsilon)\gamma^*(n) \text{ i.o.}\} = P\{S_n^s \geqq (L+\varepsilon)\gamma(f(n)) \text{ i.o.}\} .$$

But, again by symmetry,

$$P\{S_{f(n)}^s \geqq (L+\varepsilon)\gamma(f(n)) \mid S_n^s \geqq (L+\varepsilon)\gamma(f(n))\} \geqq P\{S_{f(n)}^s - S_n^s \geqq 0\} \geqq \tfrac{1}{2} ,$$

and by the extended Borel-Cantelli lemma (Problem 5.6.9 in [2]) (5.35) implies

$$P\left\{\limsup \frac{S_n^s}{\gamma(n)} \geqq L+\varepsilon\right\} \geqq P\{S_{f(n)}^s \geqq (L+\varepsilon)\gamma(f(n)) \text{ i.o.}\} = 1 .$$

This contradicts the definition of L so that

$$(5.36) \qquad 0 < \limsup \frac{S_n^s}{\gamma(n)} = \limsup \frac{S_n^s}{\gamma^*(n)} < \infty \quad \text{w.p. } 1$$

and similarly for the lim inf. This proves the lemma.

We can now prove Theorem 6. Firstly, by Lemmas 1 and 3, if (4.8) holds, (5.11) holds with $\{\gamma(n)\}$ replaced by the increasing sequence $\{\gamma^*(n)\}$. As proved by Rogozin [52], Theorem 2, and Heyde [23], Theorem 1, this implies

$$(5.37) \qquad F \in \text{domain of partial attraction of the normal law.}$$

We therefore only have to prove that (5.37) allows us to find $\{\gamma(n)\}$ such that for a given $\varepsilon > 0$, $0 < n^{-\frac{1}{2}+\varepsilon}\gamma(n) \uparrow$ and such that (5.11) holds. In view of Lemma 2, applied to S_n^s (which satisfies (5.18) with $\alpha = \frac{1}{2}$), it suffices to construct a sequence $\{\gamma(n)\}$, $0 < n^{-\frac{1}{2}+\varepsilon}\gamma(n) \uparrow$, for which there exists a $0 < \tau < \infty$ with

$$(5.38) \qquad \sum_r P\{S_{2^r}^s > 2\tau\gamma(2^r)\} < \infty$$

and

$$(5.39) \qquad \sum_r P\{S_{2^r}^s > \tau\gamma(2^r)\} = \infty .$$

For, by Lemma 2, (5.38) and (5.39) entail

$$0 < \tfrac{1}{2}\tau \leqq \limsup \frac{S_n^s}{\gamma(n)} \leqq 4\tau < \infty ,$$

and

$$\limsup \frac{S_n^s}{\gamma(n)} = -\liminf \frac{S_n^s}{\gamma(n)}$$

are constants w.p. 1. Now fix $\varepsilon > 0$, and without loss of generality even $0 < \varepsilon < \frac{1}{2}$, and take

$$\lambda_k = (\tfrac{3}{2} \log k)^{\frac{1}{2}}$$

and let

$$\nu_k = \frac{1}{(2\pi)^{\frac{1}{2}}} \int_{\lambda_k}^{\infty} e^{-\frac{1}{2}t^2} dt , \qquad \nu_k^* = \frac{1}{(2\pi)^{\frac{1}{2}}} \int_{2\lambda_k}^{\infty} e^{-\frac{1}{2}t^2} dt .$$

It is well known ([15], Lemma 7.1.2) that as $k \to \infty$

$$(5.40) \qquad \nu_k \sim \frac{1}{(2\pi)^{\frac{1}{2}}\lambda_k} e^{-\lambda_k^2/2} = \frac{C_1}{(\log k)^{\frac{1}{2}}k^{\frac{3}{4}}} , \qquad C_1 = \frac{1}{(3\pi)^{\frac{1}{2}}} ,$$

$$\nu_k^* \sim \frac{1}{(2\pi)^{\frac{1}{2}}2\lambda_k} e^{-2\lambda_k^2} = \frac{C_1}{2(\log k)^{\frac{1}{2}}k^3} .$$

Now observe that if (5.37) holds, then also F^s lies in the domain of partial attraction of the normal law. By Section 36 of [41] one can now find a sequence $0 < x_k \uparrow \infty$ such that

$$\zeta_k =_{\text{def}} \frac{x_k^2 P\{|X_1^s| > x_k\}}{H(x_k)} \downarrow 0$$

(where $H(x)$ is as in Theorem 7). Without loss of generality we may assume

$$(5.41) \qquad \zeta_k \leqq k^{-2/\varepsilon} ,$$

and the sequence of integers r_k determined by

$$(5.42) \qquad 2^{r_k} \leqq \zeta_k^{-\frac{3}{4}} \frac{x_k^2}{H(x_k)} < 2^{r_k+1}$$

increasing. It is easy to see that for any r satisfying

$$(5.43) \qquad r_k - \varepsilon^{-1} \log_2 k < r \leqq r_k$$

one has (by virtue of (5.41), (5.42))

$$(5.44) \quad 2^r P\{|X_1^s| > x_k\} \leqq 2^{r_k}\zeta_k H(x_k)x_k^{-2} \leqq \zeta_k^{\frac{1}{4}} \leqq k^{-1/2\varepsilon}(\to 0 \text{ as } k \to \infty) ,$$

and

$$(5.45) \quad \frac{\int_{|y| \leq x_k} |y|^3 dF^s}{2^{r/2}\{H(x_k)\}^{\frac{3}{2}}} \leqq \left\{\frac{x_k^2}{2^r H(x_k)}\right\}^{\frac{1}{2}} = 2^{(r_k - r)/2}\left\{\frac{x_k^2}{2^{r_k}H(x_k)}\right\}^{\frac{1}{2}} \leqq \{2^{r_k - r+1}\zeta_k^{\frac{3}{4}}\}^{\frac{1}{2}}$$

$$\leqq \{2k^{1/\varepsilon}\zeta_k^{\frac{3}{4}}\}^{\frac{1}{2}} \leqq 2^{\frac{1}{2}}k^{-1/4\varepsilon}(\to 0 \text{ as } k \to \infty) .$$

Thus, by the Berry-Esséen theorem (Chapter XVI. 5 in [16]), the distribution of

$$[2^r H(x_k)]^{-\frac{1}{2}} S_{2^r}^s$$

converges to a standard normal distribution as $r \to \infty$ and $k \to \infty$ in such a way that (5.43) remains satisfied; moreover, this convergence is uniform in r under the restriction (5.43). In particular we can thin out the sequence x_k (if necessary) such that for all k and all r satisfying (5.43)

$$(5.46) \qquad \tfrac{1}{2}\nu_k \leqq P\{[2^r H(x_k)]^{-\frac{1}{2}} S_{2^r}^s > \lambda_k\} \leqq 2\nu_k$$

and

$$P\{[2^r H(x_k)]^{-\frac{1}{2}} S^s_{2^r} > 2\lambda_k\} \leqq 2\nu_k{}^* . \tag{5.47}$$

Finally, define

$$\gamma(n) = n^{\frac{1}{2}-\varepsilon} 2^{\varepsilon r_k} \lambda_k \{H(x_k)\}^{\frac{1}{2}} \qquad \text{for} \quad 2^{r_k - 1} < n \leqq 2^{r_k}$$

Since $r_k \uparrow$, $\lambda_k \uparrow$ and $H(x_k) \uparrow$ we see that

$$n^{-\frac{1}{2}+\varepsilon} \gamma(n) \uparrow .$$

Also

$$\begin{aligned} \textstyle\sum_r P\{S^s_{2^r} > \gamma(2^r)\} &\geqq \textstyle\sum_k P\{S^s_{2^{r_k}} > \gamma(2^{r_k})\} \\ &= \textstyle\sum_k P\{[2^{r_k} H(x_k)]^{-\frac{1}{2}} S^s_{2^{r_k}} > \lambda_k\} \geqq \frac{1}{2} \sum \nu_k = \infty , \end{aligned}$$

so that (5.39) holds with $\tau = 1$. On the other hand,

$$\begin{aligned} &\textstyle\sum_r P\{S^s_{2^r} > 2\gamma(2^r)\} \\ &\quad \leqq \textstyle\sum_k \{\sum_{r_{k-1} < r \leqq r_k - \varepsilon^{-1} \log_2 k} + \sum_{r_k - \varepsilon^{-1} \log_2 k < r \leqq r_k}\} P\{S^s_{2^r} > 2\gamma(2^r)\} . \end{aligned} \tag{5.48}$$

For $r_{k-1} < r \leqq r_k - \varepsilon^{-1} \log_2 k$ we estimate $P\{S^s_{2^r} > 2\gamma(2^r)\}$ by means of (2.6) with

$$\lambda = 2\gamma(2^r) , \qquad L = 2\gamma(2^{r_k}) = 2[2^{r_k} H(x_k)]^{\frac{1}{2}} \lambda_k \qquad \text{and} \qquad n = 2^{r_k - r + 1} .$$

We merely have to observe that $S^s_{2^{r_k}}$ is the sum of the $2^{r_k - r + 1}$ independent random variables

$$S^s_{j2^{r-1}} - S^s_{(j-1)2^{r-1}} , \qquad j = 1, 2, \cdots, 2^{r_k - r + 1} ,$$

each of which has the same distribution as $S^s_{2^{r-1}}$ and that $(S^s_{2^{r-1}})^s$ has the same distribution as $S^s_{2^r}$. Thus, by (2.6)

$$\begin{aligned} \tfrac{1}{2} &\leqq P\{|S^s_{2^{r_k}}| \leqq [2^{r_k} H(x_k)]^{\frac{1}{2}} \lambda_k\} \quad \text{(see (5.46))} \\ &\leqq Q(S^s_{2^{r_k}}; 2\gamma(2^{r_k})) \\ &\leqq C\gamma(2^{r_k})(\gamma(2^r))^{-1} [2^{r_k - r + 1} P\{|S^s_{2^r}| > 2\gamma(2^r)\}]^{-\frac{1}{2}} \\ &= C 2^{-\varepsilon(r_k - r)} [2P\{|S^s_{2^r}| > 2\gamma(2^r)\}]^{-\frac{1}{2}} . \end{aligned}$$

Consequently

$$P\{|S^s_{2^r}| > 2\gamma(2^r)\} \leqq 2C^2 2^{-2\varepsilon(r_k - r)}$$

and the first double sum in the right-hand side of (5.48) is bounded by

$$\textstyle\sum_k \sum_{r \leqq r_k - \varepsilon^{-1} \log_2 k} 2C^2 2^{-2\varepsilon(r_k - r)} \leqq 4C^2 \sum_k k^{-2} < \infty .$$

The second double sum in the right-hand side of (5.48) is by virtue of (5.47) and (5.40) bounded by

$$\begin{aligned} \textstyle\sum_k \sum_{r_k - \varepsilon^{-1} \log_2 k < r \leqq r_k} &P\{|S^s_{2^r}| > 2\gamma(2^r) \geqq 2[2^r H(x_k)]^{\frac{1}{2}} \lambda_k\} \\ &\leqq \textstyle\sum_k 4\varepsilon^{-1} (\log_2 k) \nu_k{}^* < \infty . \end{aligned}$$

This proves that also (5.38) holds with $\tau = 1$ and by our previous remarks this proves (5.11) and (4.8).

This proves the main portion of Theorem 6. By means of Lemmas 1 and 3 we have already seen that $\gamma(n)$ can be chosen increasing and independent of $\delta(n)$

as long as (4.9) holds, but we still have to prove the more specific statement

$$(\gamma(n))^{-1}|\delta(n) - \delta'(n)| \to 0 \tag{5.49}$$

(see Theorem 6 for notation). We shall actually prove that (5.11) implies

$$P\{|S_n^s| > \varepsilon\gamma(n)\} \to 0 \qquad \text{for all} \quad \varepsilon > 0 . \tag{5.50}$$

Before proving (5.50), we remark that it implies

$$Q(S_n^s; 2\varepsilon\gamma(n)) \to 1 ,$$

and since (see [41], Section 29)

$$Q(S_n^s; 2\varepsilon\gamma(n)) = Q(S_n - \tilde{S}_n; 2\varepsilon\gamma(n)) \leqq Q(S_n; 2\varepsilon\gamma_n) ,$$

also $Q(S_n; 2\varepsilon\gamma_n) \to 1$. Thus, given any $0 < \eta < \min(\pi, \pi')$, for sufficiently large n there exists an interval $[a_n, a_n + 2\varepsilon\gamma_n]$ with

$$P\{a_n \leqq S_n \leqq a_n + 2\varepsilon\gamma_n\} \geqq 1 - \eta .$$

Since $P\{S_n \leqq \delta(n)\} \geqq \pi > \eta$ it follows that $a_n \leqq \delta(n)$ eventually. Similarly $\delta(n) \leqq a_n + 2\varepsilon\gamma_n$ eventually and the same has to be true for $\delta'(n)$, i.e.,

$$a_n \leqq \delta(n), \delta'(n) \leqq a_n + 2\varepsilon\gamma(n)$$

for large enough n. Since $\varepsilon > 0$ was arbitrary (5.50) implies (5.49) and it suffices to prove (5.50). We prove this together with (4.15) and (4.16).

LEMMA 4. *If F is symmetric, not concentrated on* 0, *and $\gamma(n) > 0$ such that*

$$\limsup \left|\frac{S_n}{\gamma(n)}\right| < \infty , \quad \text{w.p. } 1 \tag{5.51}$$

then (4.15) *and* (4.16) *hold and*

$$P\{|S_n| > \varepsilon\gamma(n)\} \to 0 \qquad \text{for all} \quad \varepsilon > 0 . \tag{5.52}$$

PROOF. For any $0 < \eta < 1$ and n, let $\{Y_i\}_{i\geqq 1} = \{Y_i(n, \eta)\}_{i\geqq 1}$ be a sequence of independent random variables, each with the conditional distribution of X_i given that $|X_i| > \eta\gamma(n)$, i.e.,

$$\begin{aligned} (5.53) \qquad & P\{Y_i \in U\} \\ & = P\{|X_i| > \eta\gamma(n)\}^{-1} P\{X_i \in U \cap \{(-\infty, -\eta\gamma(n)) \cup (+\eta\gamma(n), +\infty)\}\} \end{aligned}$$

(we are only interested in the case where $P\{|X_i| > \eta\gamma(n)\} > 0$ as will become apparent in a few lines). In particular, since F is symmetric, $P\{Y_i > \eta\gamma(n)\} = P\{Y_i < -\eta\gamma(n)\} = \frac{1}{2}$ and

$$Q(Y_i; \eta\gamma(n)) = \sup_a P\{a \leqq Y_i \leqq a + \eta\gamma(n)\} \leqq \tfrac{1}{2} . \tag{5.54}$$

Thus, by (2.6) (with $L = 2A\gamma(n)$, $\lambda = \frac{1}{2}\eta\gamma(n)$, X_i replaced by Y_i) for any $A \geqq 1$ and

$$l \geqq l_0 = 128\left(\frac{CA}{\eta}\right)^2 ,$$

$$Q(\textstyle\sum_1^l Y_i; 2A\gamma(n)) \leqq \dfrac{C4A\gamma(n)2^{\frac{1}{2}}}{\eta\gamma(n)l^{\frac{1}{2}}} \leqq \tfrac{1}{2} . \tag{5.55}$$

(5.55) of course implies

$$P\{|\textstyle\sum_1^l Y_i| > A\gamma(n)\} \geqq 1 - Q(\sum_1^l Y_i; 2A\gamma(n)) \geqq \frac{1}{2}\,. \tag{5.56}$$

Assume now that (4.15) fails and that for some $\delta > 0$ and increasing sequence $\{n_k\}$

$$n_k P\{|X_i| > \eta\gamma(n_k)\} \geqq \delta\,.$$

Then, by the Poisson approximation to the binomial approximation

$$P\{|X_i| > \eta\gamma(n_k) \text{ for at least } l_0 \text{ values of } i \leqq n_k\} \geqq d(\delta, l_0) > 0\,, \tag{5.57}$$

for a suitable strictly positive d, which depends on δ and l_0 only. But then, again using symmetry,

$$P\{|S_{n_k}| > A\gamma(n_k)\} \geqq \textstyle\sum_{l \geqq l_0} \sum_{1 \leqq i_1 < \cdots < i_l \leqq n_k} P\{|X_i| > \eta\gamma(n_k)$$

exactly for $i \in \{i_1, \cdots, i_l\}$, $|\sum_{i \in \{i_1, \cdots, i_l\}} X_i| > A\gamma(n_k)$ and

$$\begin{aligned}\operatorname{sgn}(\textstyle\sum_{i \notin \{i_1,\cdots,i_l\}} X_i) &= \operatorname{sgn}(\textstyle\sum_{i \in \{i_1,\cdots,i_l\}} X_i)\} \\ &\geqq \tfrac{1}{2} \textstyle\sum_{l \geqq l_0} P\{|X_i| > \eta\gamma(n_k) \text{ for exactly } l \text{ indices } 1 \leqq i \leqq n_k\} \\ &\quad P\{|\textstyle\sum_1^l Y_i(n_k, \eta)| > A\gamma(n_k)\} \geqq \tfrac{1}{4} d(\delta, l_0) \text{ (see (5.56), (5.57))}\,.\end{aligned}$$

Consequently, for any $A \geqq 1$,

$$P\{|S_n| > A\gamma(n) \text{ i.o.}\} \geqq \limsup_{k\to\infty} P\{|S_{n_k}| > A\gamma(n_k)\} > 0\,, \tag{5.58}$$

and by the Hewitt-Savage 0–1 law

$$P\{|S_n| > A\gamma(n) \text{ i.o.}\} = 1\,,$$

which contradicts (5.11). Thus (4.15) must hold and we now show that (4.16) must hold as well. Again assume the contrary and let

$$n_k H(\gamma(n_k)) \geqq \delta^2(\gamma(n_k))^2$$

for some $\delta > 0$ and $n_k \uparrow \infty$. Let

$$Z_i = Z_i(n_k) = X_i I[|X_i| \leqq \gamma(n_k)]\,.$$

Then for any $\eta > 0$

$$\begin{aligned}\frac{n_k E|Z_1|^3}{\{n_k\sigma^2(Z_1)\}^{\frac{3}{2}}} &= n_k^{-\frac{1}{2}}[H(\gamma(n_k))]^{-\frac{3}{2}} \int_{|y| \leqq \gamma(n_k)} |y|^3\, dF(y) \\ &\leqq n_k^{-\frac{1}{2}}[H(\gamma(n_k))]^{-\frac{3}{2}}[\eta\gamma(n_k) \int_{|y| \leqq \eta\gamma(n_k)} y^2\, dF(y) \\ &\qquad + (\gamma(n_k))^3 \int_{|y| > \eta\gamma(n_k)} dF(y)] \\ &\leqq n_k^{-\frac{1}{2}}[H(\gamma(n_k))]^{-\frac{3}{2}}[\eta\gamma(n_k)H(\gamma(n_k)) + (\gamma(n_k))^3 P\{|X_1| > \eta\gamma(n_k)\}] \\ &\leqq \frac{\eta}{\delta} + \frac{1}{\delta^3} n_k P\{|X_1| > \eta\gamma(n_k)\}\,.\end{aligned} \tag{5.59}$$

By (4.15) the last term in the last member of (5.59) tends to 0 as $k \to \infty$ for every $\eta > 0$, so that

$$\frac{n_k E|Z_1|^3}{\{n_k \sigma^2(Z_1)\}^{\frac{3}{2}}} \to 0 \qquad (k \to \infty)\,.$$

Thus, by Lyapounov's theorem ([16], Problem 8.10.17 or Chapter XVI. 5)

$$\liminf_{k\to\infty} P\{|\sum_1^{n_k} Z_i| \geqq A\gamma(n_k)\} \geqq \lim_{k\to\infty} P\{|\sum_1^{n_k} Z_i| \geqq (A/\delta)[n_k H(\gamma(n_k))]^{\frac{1}{2}}\}$$
$$= (2\pi)^{-\frac{1}{2}} \int_{|t|\geqq A\delta^{-1}} e^{-\frac{1}{2}t^2}\, dt > 0 .$$

Since, by (4.15)

$$P\{S_{n_k} \neq \sum_1^{n_k} Z_i\} \leqq n_k P\{|X_1| > \gamma(n_k)\} \to 0$$

we conclude again that (5.58) must hold, which is not possible. Thus also (4.16) has been proved. Finally (5.52) follows from Chebyshev's inequality

$$P\{|S_n| \geqq \varepsilon\gamma(n)\} \leqq P\{S_n \neq \sum_1^n Z_i(n)\} + \frac{\sigma^2(\sum_1^n Z_i(n))}{\varepsilon^2(\gamma(n))^2}$$
$$\leqq nP\{|X_1| > \gamma(n)\} + \frac{nH(\gamma(n))}{\varepsilon^2(\gamma(n))^2}$$

and (4.15), (4.16).

REMARK 9. Lemma 4 proves (4.15) and (4.16) and as stated before, the remainder of Theorem 7 is proved pretty much as Theorems 1 and 3 of [17]. The most important deviation from Feller's proof is in the choice of subsequences n_k in the analogues of Lemmas 4.1 and 5.1 of [17] (we do not understand anyway how Feller chooses n_k for his Lemma 4.1). For both lemmas we pick $\{n_k\} = \{n_k(\varepsilon)\}$ as follows: In each interval $((1+\varepsilon)^{k-1}, (1+\varepsilon)^k]$ choose the largest $\gamma(n)$, if at least one γ falls in this interval. Let the sequence of γ's thus selected be $\gamma(n_1) < \gamma(n_2) < \gamma(n_3) \cdots$. The sequence of indices $n_1, n_2, \cdots$ is the required sequence.

We also point out that we cannot quite follow Feller's argument about normalization by $\rho_n(2 \log\log \rho_n)^{\frac{1}{2}}$ so that we only formulated a theorem involving sequences $\gamma(n)$ which behave roughly like roots of the equation

$$(\gamma(n))^2[\log\log\gamma(n)]^{-1} = AnH(\gamma(n)) ,$$

for some $0 < A < \infty$, respectively sequences $\{\gamma(n)\}$ satisfying (4.21) or (4.23). Fristedt [18] defines the analogues of $\gamma(n)$ in a slightly different manner, and his choice could probably be adapted to the random walk case as well.

REFERENCES

[1] BINMORE, K. G. and KATZ, M. (1968). A note on the strong law of large numbers. *Bull. Amer. Math. Soc.* **74** 941–943.

[2] BREIMAN, L. (1968). *Probability*. Addison-Wesley, Reading.

[3] BRETAGNOLLE, J. et DACUNHA-CASTELLE, D. (1967). Sur une classe de marches aléatoires. *Ann. Inst. H. Poincaré Sect. B* **3** 403–431.

[4] CHUNG, K. L. (1954). Contribution to the theory of Markov chains. *Trans. Amer. Math. Soc.* **76** 397–419.

[5] CHUNG, K. L. (1960). *Markov Chains with Stationary Transition Probabilities*. Springer-Verlag, Berlin.

[6] CHUNG, K. L. and ERDÖS, P. (1951). Probability theorems assuming only the first moment I. Memoir No. 6, *Amer. Math. Soc.*

[7] CHUNG, K. L. and FUCHS, W. H. J. (1951). On the distribution of values of sums of random variables. Memoir No. 6, *Amer. Math. Soc.*

[8] DOEBLIN, W. (1938). Sur deux problèmes de M. Kolmogoroff concernant les chaines dénombrables. *Bull. Soc. Math. France* **66** 210–220.

[9] DOEBLIN, W. (1939). Sur les sommes d'un grand nombre de variables aléatoires indépendantes. *Bull. Sci. Math.* **63** 23–32, 35–64.

[10] DOEBLIN, W. et LÉVY, P. (1936). Sur les sommes de variables aléatoires indépendantes à dispersion bornées inférieurement. *C. R. Acad. Sci. Paris* **202** 2027–2029.

[11] DOOB, J. L. (1953). *Stochastic Processes.* Wiley, New York.

[12] DOOB, J. L., SNELL, J. L. and WILLIAMSON, R. E. (1960). Application of boundary theory to sums of independent random variables. *Contributions to Probability and Statistics* (I. Olkin *et al.*, eds.), 182–197. Stanford Univ. Press.

[13] ESSÉEN, C. G. (1966). On the Kolmogorov-Rogozin inequality for the concentration function. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete* **5** 210–216.

[14] ESSÉEN, C. G. (1968). On the concentration function of a sum of independent random variables. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete* **9** 290–308.

[15] FELLER, W. (1968). *An Introduction to Probability Theory and its Applications,* **1** (3rd ed.). Wiley, New York.

[16] FELLER, W. (1966). *idem,* **2**. Wiley, New York.

[17] FELLER, W. (1968). An extension of the law of the iterated logarithm to variables without variance. *J. Math. Mech.* **18** 343–356.

[18] FRISTEDT, B. (1970). Upper functions for symmetric processes with stationary, independent increments. Math. Res. Center Report No. 1110, Univ. of Wisconsin.

[19] HARRIS, T. E. and ROBBINS, H. (1953). Ergodic theory of Markov chains admitting an invariant measure. *Proc. Nat. Acad. Sci.* **39** 860–864.

[20] HARTMAN, P. and WINTNER, A. (1941). On the law of the iterated logarithm. *Amer. J. Math.* **63** 169–176.

[21] HERMANN, H. (1965). Glättungseigenschaften der Faltung. *Wiss. Z. Friedrich Schiller Univ. Jena, Math.-Natur. Reihe* **14** 5 221–234.

[22] HERZ, C. S. (1965). Les théorèmes de renouvellement. *Ann. Inst. Fourier* **15** 169–188.

[23] HEYDE, C. C. (1969). A note concerning behavior of iterated logarithm type. *Proc. Amer. Math. Soc.* **23** 85–90.

[24] HOEFFDING, W. (1961). On sequences of sums of independent random vectors. *Proc. Fourth Berkeley Symp. Math. Statist. Prob.* **2** 213–226. Univ. of California Press.

[25] KATZ, M. L. (1963). The probability in the tail of a distribution. *Ann. Math. Statist.* **34** 312–318.

[26] KEMENY, J. G. (1959). A probability limit theorem requiring no moments. *Proc. Amer. Math. Soc.* **10** 607–612.

[27] KEMENY, J. G. and SNELL, J. L. (1961). Potentials for denumerable Markov chains. *J. Math. Anal. Appl.* **3** 196–260.

[28] KEMENY, J. G. and SNELL, J. L. (1961). On Markov chain potentials. *Ann. Math. Statist.* **32** 709–715.

[29] KERSTAN, J. und MATTHES, K. (1965). Gleichverteilungseigenschaften von Faltungspotenzen auf lokalkompakten abelschen Gruppen. *Wiss. Z. Friedrich Schiller Univ. Jena, Math.-Natur. Reihe* **14** 5 457–462.

[30] KESTEN, H. (1963). Ratio theorems for random walks II. *J. Analyse. Math.* **11** 323–379.

[31] KESTEN, H. (1967). The Martin boundary for recurrent random walks on countable groups. *Proc. Fifth Berkeley Symp. Math. Statist. Prob.* **2** 51–74. Univ. of California Press.

[32] KESTEN, H. (1969). A sharper form of the Doeblin-Lévy-Kolmogorov-Rogozin inequality for concentration functions. *Math. Scand.* **25** 133–144.

[33] KESTEN, H. (1970). A ratio limit theorem for symmetric random walk. *J. Analyse. Math.* **23** 199–213.

[34] KESTEN, H. (1970). The limit points of a normalized random walk. *Ann. Math. Statist.* **41** 1173–1205.

[35] KESTEN, H. and SPITZER, F. (1963). Ratio theorems for random walks I. *J. Analyse. Math.* **11** 285–322.

[36] KINGMAN, J. F. C. and OREY, S. (1964). Ratio limit theorems for Markov chains. *Proc. Amer. Math. Soc.* **15** 907–910.

[37] KNOPP, K. (1951). *Theory and Application of Infinite Series*, (2nd ed.). Blackie and Son, London.

[38] KOLMOGOROV, A. N. (1928). Über die Summen durch den Zufall bestimmter unabhängiger Grössen. *Math. Ann.* **99** 309–319; (1930). **102** 484–488.

[39] KOLMOGOROV. A. N. (1958–60). Sur les propriétés des fonctions de concentration de M. P. Lévy. *Ann. Inst. H. Poincare Sect. B* **16** 27–34.

[40] LECAM, L. (1965). On the distribution of sums of independent random variables. *Bernoulli, Bayes, Laplace*, (J. Neyman and L. LeCam, eds.) 179–202. Springer, New York.

[41] LÉVY, P. (1954). *Théorie de l'Addition des Variables Aléatoires*, (2nd ed.). Gauthier-Villars, Paris.

[42] NEVEU, J. (1963). Sur le théorème ergodique de Chung-Erdös. *C. R. Acad. Sci. Paris* **257** 2953–2955.

[43] OREY, S. (1961). Strong ratio limit property. *Bull. Amer Math. Soc.* **67** 571–574.

[44] OREY, S. (1962). An ergodic theorem for Markov chains. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete* **1** 174–176.

[45] ORNSTEIN, D. (1967). A limit theorem for independent random variables. *Proc. Fifth Berkeley Symp. Math. Statist. Prob.* **2** 213–216. Univ. of California Press. 1967.

[46] ORNSTEIN, D. (1969). Random walks I. *Trans. Amer. Math. Soc.* **138** 1–43.

[47] ORNSTEIN, D. (1969). Random walks II. *Trans. Amer. Math. Soc.* **138** 45–60.

[48] PORT, S. C. and STONE, C. J. (1967). Hitting time and hitting places for non-lattice recurrent random walks. *J. Math. Mech.* **17** 35–58.

[49] ROBBINS, H. (1953). On the equidistribution of sums of independent random variables. *Proc. Amer. Math. Soc.* **4** 786–799.

[50] ROGOZIN, B. A. (1961). An estimate for concentration functions. *Teor. Verojatnost. i Primenen* **6** 103–105 (English translation in *Theor. Probability Appl.* **6** (1961) 94–96).

[51] ROGOZIN, B. A. (1961). On the increase of dispersion of sums of independent random variables. *Teor. Verojatnost. i Primenen* **6** 106–108 (English translation in *Theor. Probability Appl.* **6** (1961) 97–99).

[52] ROGOZIN, B. A. (1968). On the existence of exact upper sequences. *Teor. Verojatnost. i Primenen* **13** 701–707 (English translation in *Theor. Probability Appl.* **13** (1968) 667–672).

[53] SAZONOV, V. V. (1966). On multi-dimensional concentration functions. *Teor. Verojatnost. i Primenen* **11** 683–690 (English translation in *Theor. Probability Appl.* **11** (1966) 603–609).

[54] SPITZER, F. (1962). Hitting probabilities. *J. Math. Mech.* **11** 593–614.

[55] SPITZER, F. (1964). *Principles of Random Walk.* Van Nostrand, Princeton.

[56] STONE, C. J. (1966). Ratio limit theorems for random walks on groups. *Trans. Amer. Math. Soc.* **125** 86–100.

[57] STONE, C. J. (1967). On local and ratio limit theorems. *Proc. Fifth Berkeley Symp. Math. Statist. Prob.* **2** 217–224. Univ. of California Press.

[58] STONE, C. J. (1969). On the potential operator for one-dimensional recurrent random walks. *Trans. Amer. Math. Soc.* **136** 413–426.

[59] STONE, C. J. (1969). The growth of a random walk. *Ann. Math. Statist.* **40** 2203–2206.

[60] STRASSEN, V. (1964). An invariance principle for the law of the iterated logarithm. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete* **3** 211–226.

[61] STRASSEN, V. (1966). A converse to the law of the iterated logarithm. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete* **4** 265–268.

[62] TOMKINS, R. J. (1971). Some iterated logarithm results related to the central limit theorem. *Trans. Amer. Math. Soc.* **156** 185–192.