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# The $(2^{n+1}-2)$-ray algorithm: a new simplicial algorithm to compute economic equilibria

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# The $(2^{n+1}-2)$-ray algorithm: a new simplicial algorithm to compute economic equilibria*

by

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Abstract

A new variable dimension simplicial restart algorithm is introduced to compute economic equilibria. The number of rays along which the algorithm can leave the starting point differs from the thusfar known algorithms. More precisely, the new algorithm has one ray to each of the $2^{n+1}-2$ faces of the n-dimensional price simplex, whereas the existing algorithms have n+1 rays either to each facet or to each vertex of the unit simplex. The path of points followed by the algorithm can be interpreted as a price adjustment process. Since this process converges for any continuous excess demand function it is a good alternative for the well-known Walras' tatonnement process. Computational results will show that the number of function evaluations is in general less than for the (n+1)-ray algorithms. The examples concern the computation of equilibria in pure exchange economies.

Keywords: excess demand, equilibria, triangulation, vector labelling

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## 1. Introduction

To find a zero point of a continuous excess demand function $z: S^n \to R^{n+1}$ where $S^n = \{x \in R_+^{n+1} \mid \Sigma_j x_j = 1\}$ and $z$ satisfies $z_i(x) > 0$ when $x_i = 0$ and Walras' law $x^T z(x) = 0$ for all $x$, several simplicial algorithms have been developed [Scarf (1967, 1973), Kuhn (1968, 1969), Kuhn and MacKinnon (1975), and van der Laan and Talman (1979)]. In a simplicial subdivision or triangulation of $S^n$ such algorithms search for a simplex which yields an approximate equilibrium by generating a path of adjacent simplices. The simplex with which the algorithm terminates is found within a finite number of steps. The so-called variable dimension algorithm developed in van der Laan and Talman (1979) can be started in an arbitrarily chosen grid point of the subdivision and generates a path of adjacent simplices of varying dimension. When the end simplex does not provide an approximate zero with sufficient accuracy, the subdivision is refined and the algorithm is restarted in or close to the last found approximation. In general, the accuracy of an approximate zero improves when the subdivision is refined.

By generating a sequence of simplices a piecewise linear path is traced on which the piecewise linear approximation $\bar{z}$ to $z$ with respect to the underlying triangulation satisfies certain conditions. For any proper subset $T$ of $I_{n+1} = \{1,\ldots, n+1\}$, let the $|T|$-dimensional convex subset $A(T)$ with $|T|$ the cardinality of $T$ be defined by

$$A(T) = \{x \in S^n \mid x = v + \Sigma_{j\in T} \lambda_j q(j),\ \lambda_j > 0,\ j \in T\}, \qquad (1.1)$$

where $v$ is the arbitrarily chosen starting grid point of the algorithm and where $q(j) = e(j) - e(j-1)$, $j = 1,\ldots,n+1$ ($j-1 = n+1$ if $j = 1$), with $e(j)$ the $j$-th unit vector in $R^{n+1}$, $j \in I_{n+1}$. Clearly $A(T)$ is a $|T|$-dimensional cone in $S^n$ with vertex $v$. In particular there are $n+1$ one-dimensional cones $A(\{i\})$, $i = 1,\ldots, n+1$, which are called the rays of the algorithm. On each $|T|$-dimensional cone $A(T)$, the so-called Q-triangulation of $S^n$ induces a triangulation in $|T|$-dimensional simplices of $A(T)$. A detailed description can be found in van der Laan and Talman (1979). Now the path traced by the variable dimension algorithm is as follows. At the starting point $x = v$, let $i$ be the (unique) index for

which $z_i(v) = \max_h z_h(v)$. Then the algorithm increases $\lambda_i$ away from zero, i.e. the starting point v is left along the ray $A(\{i\})$. Doing so, the i-th component of v is increased and the (i-1)-th component of v is decreased with the same amount until a point x in $A(\{i\})$ is reached for which $\bar{z}_j(x) = \bar{z}_i(x)$, $j \neq i$. Then the algorithm traces a piecewise linear path of points x in $A(\{i,j\})$ by increasing $\lambda_j$ away from zero and keeping $\bar{z}_j(x) = \bar{z}_i(x) = \max_h \bar{z}_h(x)$. In general the algorithm follows for varying $T \subset I_{n+1}$, a piecewise linear path in A(T) such that a point x on the path satisfies

$$\begin{aligned} &\bar{z}_i(x) = \max_h \bar{z}_h(x) && \text{if } i \in T \\ &\text{and} && \qquad (1.2) \\ &\bar{z}_i(x) \leq \max_h \bar{z}_h(x) && \text{if } i \notin T. \end{aligned}$$

So, a point x on this path in A(T) satifies the complementarity property

$$\lambda_i(x) \geq 0, \quad \max_h \bar{z}_h(x) - \bar{z}_i(x) \geq 0$$

and $\lambda_i(x)\,(\max_h \bar{z}_h(x) - \bar{z}_i(x)) = 0$ for $i = 1,\ldots, n+1$ with the nonnegative $\lambda_i(x)$'s uniquely given by

$$x = v + \Sigma_{i \in T}\, \lambda_i(x)\, q(i) \quad \text{and} \quad \lambda_i(x) = 0,\ i \notin T.$$

When for $k \notin T$ an inequality in (1.2) becomes an equality the index k enters T and the algorithm follows a path in $A(T \cup \{k\})$ by increasing $\lambda_k$ away from zero. On the other hand, when for some $j \in T$, $\lambda_j$ becomes zero for a point x on the path in A(T), then the index j is deleted from T and the algorithm continues in $A(T\setminus\{j\})$ by decreasing $\bar{z}_j(x)$ away from $\max_h \bar{z}_h(x)$. As soon as T becomes $I_{n+1}$ we have $\bar{z}_i(x) = \max_h \bar{z}_h(x)$ for all $i \in I_{n+1}$. Because $x^T z(x) = 0$ it follows that an approximate zero of z is found. The algorithm can be restarted in or close to this point with a finer grid in order to improve the accuracy.

An increase of the i-th component for which $\bar{z}_i(x) = \max_h \bar{z}_h(x)$ is obvious, but the decrease of the (i-1)-th component in order to keep the sum of the $x_i$'s equal to one seems to be rather arbitrarily. A more appropriate choice of the cones A(T) seems to be

$$A'(T) = \{x \in S^n \mid x = v + \Sigma_{j\in T}\lambda_j(e(j)-v),\ \lambda_j>0,\ j \in T\}. \quad (1.3)$$

So, the region A'(T) is the convex hull of the starting point v and the vertices e(j), $j \in T$, of $S^n$. Recently, Doup and Talman (1984) gave a triangulation of $S^n$, the so-called V-triangulation, which induces a triangulation of each region A'(T) in $|T|$-dimensional simplices. Taking this triangulation instead of the Q-triangulation, again a piecewise linear path is traced such that for varying T a point on the path in A'(T) satisfies the conditions (1.2). Also this path leads from the starting point v to an approximate solution, but differs from the path induced by the Q-triangulation because the cones differ. For a point x in A'(T) we have

$$\begin{aligned} x_j &= (1-b)\, v_j + \lambda_j, \quad \text{with } \lambda_j \geq 0, \quad \text{if } j \in T \\ \text{and} \qquad & \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (1.4) \\ x_j &= (1-b)\, v_j, \qquad\qquad\qquad\quad \text{if } j \notin T \end{aligned}$$

where $0 \leq b = \Sigma_{j\in T}\lambda_j \leq 1$. So, when v is left along the ray $A'(\{i\})$ for which $z_i(v) = \max_h z_h(v)$, $v_i$ is increased with $\lambda_i(1-v_i)$, and all other components are decreased with $\lambda_i v_h$, $h \neq i$, in order to keep the sum of the components equal to one. So, whereas for the Q-triangulation only $v_{i-1}$ is decreased, in case of the V-triangulation all components $v_h$, $h \neq i$, are decreased proportionally. In particular when the algorithm is applied to find an equilibrium price vector in an economic model, this seems to be much more attractive and appealing.

The variable dimension algorithms described above have the property that the starting point v is left along one of n+1 rays, depending on which component has the largest excess demand value. For the two given alternatives only the underlying triangulations differ. A third alternative is to utilize the so-called U-triangulation [see van der Laan and Talman (1980)]. In this case, the point v is left along a ray on which again one component is increased, say with an amount $\lambda$, and all other components are decreased with $\lambda/n$ [see also Zangwill and Garcia (1981)].

For problems on $R^n$ simplicial algorithms with more than n+1 rays have been shown to be more efficient in the sense that less function

evaluations are needed to reach an approximate solution [see e.g. Kojima and Yamamoto (1984) and van der Laan and Seelen (1983)]. An algorithm on $R^n$ with 2n rays was developed in van der Laan and Talman (1981), with $2^n$ rays in Wright (1981), and with $3^n - 1$ rays in Kojima and Yamamoto (1984).

In this paper a simplicial variable dimension algorithm on $S^n$ is presented with more than n+1 rays, namely one to each proper face of $S^n$. Since a t-dimensional face of $S^n$ is the convex hull of t + 1 vertices e(i) of $S^n$, $0 \leq t \leq n - 1$, the number of proper faces of $S^n$ is equal to $2^{n+1} - 2$. The starting point v will be left along a ray on which the components i of v, having $z_i(v) - z^m(v)$ positive, where $z^m(x) = \sum_{i=1}^{n+1} z_i(x)/(n+1)$, are increased and the components j of v, having $z_j(v) - z^m(v)$ negative, are decreased. This ray points to the face on which the latter components are equal to zero. The algorithm moves along this ray until a point x is reached for which one of the components of $\bar{z}(x)$, say $\bar{z}_h(x)$, is equal to $\bar{z}^m(x)$. Then the algorithm traces a piecewise linear path in a two-dimensional subset of $S^n$ keeping $\bar{z}_h(x)$ equal to $\bar{z}^m(x)$. In general, for varying t, a piecewise linear path in a t-dimensional subset of $S^n$ is traced, on which t-1 components of $\bar{z}(x)$ are equal to $\bar{z}^m(x)$. More precisely, for any x on the path the following holds

$$x_j = (1-b+\lambda)v_j \text{ , with } \lambda > b \quad \text{, if } \bar{z}_j(x) > \bar{z}^m(x)$$

$$x_j = (1-b+\mu_j)v_j \text{, with } 0 \leq \mu_j \leq \lambda, \quad \text{if } \bar{z}_j(x) = \bar{z}^m(x)$$

and

$$x_j = (1-b)v_j \quad \text{, if } \bar{z}_j(x) < \bar{z}^m(x),$$

$j = 1,\ldots, n+1$ and where $0 < b \leq 1$. So, when comparing x with the starting point v, all components j of x for which the relative excess demand $\bar{z}_j(x) - \bar{z}^m(x)$ is positive are a factor $1 - b + \lambda$ larger than $v_j$, all components j of x for which the relative excess demand $\bar{z}_j(x) - \bar{z}^m(x)$ is negative are a factor $1 - b$ smaller than $v_j$, whereas all other components vary between $(1-b)v_j$ and $(1-b+\lambda)v_j$. Again we have a complementarity condition in the sense that for each j either $\bar{z}_j(x) = \bar{z}^m(x)$ or $\mu_j$ is on one of its bounds. An approximation $x^*$ is reached as soon as all components $\bar{z}_j(x^*)$ are equal to $\bar{z}^m(x^*)$. When applied to com-

pute a Walrasian price equilibrium in an economic model, at the starting price vector the prices of the commodities with relative excess demand are proportionally increased and the prices of the commodities with relative excess supply are proportionally decreased. This is rather similar to the classical Walras tatonnement process but then convergence is assured only under strong conditions on z such as Gross-Substitutability or Revealed Preferences. The process described above, however always finds an approximate solution. By taking the grid of the underlying triangulation fine enough, the excess demands and supplies at the approximate solution can be made as close to zero as wanted.

The paper is organized as follows. In section 2 we introduce the subdivision of $S^n$ in cones of varying dimension which underlies the algorithm and discuss how each cone is triangulated by the V-triangulation. In section 3 we formulate the zero point problem and describe the piecewise linear path followed by the algorithm in detail. We also show how $z(x^*)$ differs from zero at an approximate solution point $x^*$. Finally, in section 4 we give some numerical results. These confirm the supposition that a variable dimension algorithm with more than n+1 rays might improve the efficiency of these algorithms.

## 2. The subdivision of $S^n$

To describe the subsets of $S^n$ in which the new simplicial variable dimension algorithm operates, let s be an arbitrary sign vector in $R^{n+1}$ such that at least one component of s is equal to +1 and at least one component of s is equal to -1. Observe that there are $2^{n+1}-2$ of such sign vectors containing no zeroes at all. Furthermore letting

$$I^-(s) = \{i \in I_{n+1} \mid s_i = -1\}$$

$$I^0(s) = \{i \in I_{n+1} \mid s_i = 0\}$$

$$I^+(s) = \{i \in I_{n+1} \mid s_i = +1\},$$

then both $|I^+(s)|$ and $|I^-(s)|$ are at least equal to one. Each such sign vector s will induce a t-dimensional subset A(s) of $S^n$ with $t = |I^0(s)| + 1$. Observe that t lies between 1 and n. Finally, let v be the starting point of the algorithm. We assume that v lies in the interior of $S^n$.

Definition 2.1. Let s be a sign vector with $|I^+(s)|$ and $|I^-(s)|$ positive. The set A(s) is given by

$$A(s) = \{x \in S^n \mid x_i = (1-b+\lambda)v_i,\ i \in I^+(s),\ x_i = (1-b+\mu_i)v_i$$
$$\text{with } 0 < \mu_i < \lambda,\ i \in I^0(s), \text{ and } x_i = (1-b)v_i,\ i \in I^-(s),$$
$$i = 1,\ldots,n+1, \text{ where } \lambda > b \text{ and } 0 < b < 1\}.$$

Further, let $\gamma(s)$ be some permutation of the t-1 elements of $I^0(s)$, say $\gamma(s) = (k_1,\ldots, k_{t-1})$, and let $p(K)$, $K \subset I_{n+1}$, $K \neq \emptyset$, be the (n+1)-vector given by

$$p_i(K) = \begin{cases} v_i(\Sigma_{k\in K} v_k)^{-1}, & i \in K \\ 0, & i \notin K. \end{cases}$$

Definition 2.2. Let s be a sign vector with $|I^+(s)|$ and $|I^-(s)|$ positive. The set $A(s,\gamma(s))$ is given by

$$A(s,\gamma(s)) = \{x \in S^n \mid x = v + b\, q(0) + \sum_{k\in I^0(s)} a(k)q(k), \quad \text{with } 0 < a(k_{t-1}) < \ldots < a(k_1) < b < 1\}, \tag{2.1}$$

where the (n+1)-vector q(0) is given by

$$q(0) = p(I^+(s)) - v,$$

and where for $i = 1,\ldots, t-1$ the (n+1)-vector $q(k_i)$ is given by

$$q(k_i) = p(I^+(s) \cup \{k_1,\ldots,k_i\}) - p(I^+(s) \cup \{k_1,\ldots,k_{i-1}\}).$$

Letting $Q(s,\gamma(s))$ be the $(n+1) \times t$ matrix with first column q(0) and (i+1)-th column $q(k_i)$, $i = 1,\ldots, t-1$, it easily follows that the rank of this matrix is equal to t so that the set $A(s, \gamma(s))$ is a t-dimensional convex subset of A(s). A(s) is the union of $A(s, \gamma(s))$ over all permutations $\gamma(s)$. Some sets are illustrated in figure 1 for n = 3.
The boundary of A(s), which plays an essential role in the algorithm, consists of the (t-1)-dimensional subsets $A(\bar{s})$, with $\bar{s}_i = \pm 1$ for exactly one i in $I^0(s)$ and $\bar{s}_h = s_h$, $h \neq i$, and of a subset of the boundary of $S^n$, viz. the intersection of A(s) and $\bigcap_{k\in I^-(s)} S^n_k$, where $S^n_k = \{x\in S^n \mid x_k = 0\}$, $k \in I_{n+1}$. The boundary of $A(s, \gamma(s))$ which plays an important role in the triangulation, is a collection of (t-1)-dimensional subsets of $S^n$, each of them obtained by setting exactly one inequality in (2.1) to an equality. In the case b is set equal to one we obtain a subset in the boundary of $S^n$, more precisely in $\bigcap_{k\in I^-(s)} S^n_k$. The other subsets in the boundary of $A(s, \gamma(s))$ are obtained when $a(k_1) = b$, $a(k_i) = a(k_{i-1})$ for some $i \in \{2,\ldots, t-1\}$, or $a(k_{t-1}) = 0$. When the sign vector s does not contain zeroes, then A(s) is a one-dimensional line segment having v and the point $p(I^+(s))$ in $\bigcap_{k\in I^-(s)} S^n_k$, as endpoints. Observe that there are $2^{n+1} - 2$ one-dimensional sets A(s), the

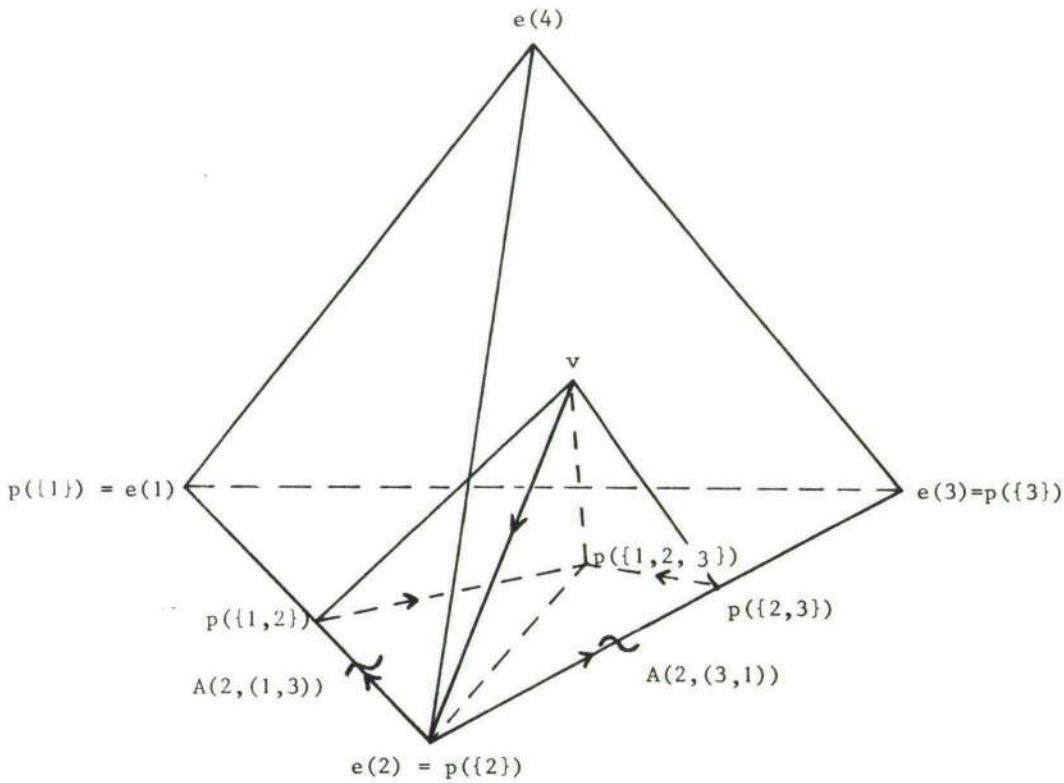


Figure 1. Illustration of A(s), $s = (0, +1, 0, -1)^T$, which is subdivided into A(2, (1,3)) and A(2, (3,1)); dim A(s) = $|I^0(s)| + 1 = 3$.

so-called rays of the algorithm. Along one of these rays, leading from v to one of the $2^{n+1}-2$ faces of $S^n$, the algorithm will leave the starting point.

We are now ready to describe the collection of simplices in which the region A(s) is triangulated by the V-triangulation [see Doup and Talman (1984)]. In fact, each subset A(s, $\gamma(s)$) of A(s) is triangulated in t-simplices and the union of these simplices over all permutations $\gamma(s)$ yields a triangulation of A(s). So let $S^n$ be triangulated according to the V-triangulation with gridsize $m^{-1}$, where m is some positive integer. More specific, we take the V-triangulation with relative projection [see Doup and Talman (1984, section 4)].

Definition 2.3. Let s be a sign vector with $|I^+(s)|$ and $|I^-(s)|$ positive. The set G(s, $\gamma(s)$) is the collection of t-simplices $\sigma(y^1, \pi(s))$ with vertices $y^1,\ldots, y^{t+1}$ such that

(i) $y^1 = v + bm^{-1}q(0) + \sum_{k\in I^0(s)} a(k)m^{-1}q(k)$ for nonnegative integers b and a(k), $k \in I^0(s)$ such that $0 \leq a(k_{t-1}) \leq \ldots \leq a(k_1) \leq b \leq m-1$

(ii) $\pi(s) = (\pi_1,\ldots, \pi_t)$ is a permutation of t elements consisting of 0 and the t-1 elements of $I^0(s)$ such that the following holds: if $a(k_1) = b$ this implies $p > p'$ with p and p' such that $\pi_p = k_1$ and $\pi_{p'} = 0$; if $a(k_i) = a(k_{i-1})$ for some index i in $\{2,\ldots, t-1\}$ this implies $p > p'$ with $\pi_p = k_i$ and $\pi_{p'} = k_{i-1}$

(iii) $y^{i+1} = y^i + m^{-1}q(\pi_i)$, $i = 1,\ldots,t$,

where q(0) and $q(k_i)$, $i = 1,\ldots, t-1$, are defined as before.

It is easy to verify that G(s, $\gamma(s)$) is a triangulation of A(s, $\gamma(s)$), that the union G(s) of G(s, $\gamma(s)$) over all $\gamma(s)$ triangulates A(s), and that the union G of G(s) over all s, with $|I^+(s)|$ and $|I^-(s)|$ positive, induces the V-triangulation of $S^n$ with relative projection and gridsize $m^{-1}$.

Since for varying sign vectors s the algorithm will generate a path of adjacent t-simplices in A(s), we have to know how the parameters $\bar{y}^1$ and $\bar{\pi}(s)$ of a t-simplex $\sigma(\bar{y}^1, \bar{\pi}(s))$ can be obtained from the parameters $y^1$ and $\pi(s)$ of an adjacent simplex $\sigma(y^1, \pi(s))$. If a facet $\tau$ of a simplex $\sigma(y^1, \pi(s))$ in G(s) lies not in bd A(s), then $\tau$ is a facet of just one other t-simplex $\bar{\sigma}(\bar{y}^1, \bar{\pi}(s))$ of G(s). However, this simplex could lie in another subset A(s, $\gamma(s)$) than $\sigma(y^1, \pi(s))$ does, in which case we also have to describe how $\bar{\gamma}(s)$ changes to get $\bar{\sigma}(\bar{y}^1, \bar{\pi}(s))$. If a facet $\tau$ of a t-simplex $\sigma(y^1, \pi(s))$ lies in bd A(s) it is not a facet of another t-simplex in G(s). In this case we will show that $\tau$ either lies in $A(s) \cap ( \bigcap_{k \in I^-(s)} S_k^n )$ or that $\tau = \bar{\sigma}(\bar{y}^1, \pi(\bar{s}))$ is a (t-1)-simplex in $G(\bar{s})$ with $\bar{s}_i = \pm 1$ for exactly one i in $I^0(s)$ and $\bar{s}_h = s_h$, $h \neq i$. So, let $\sigma(y^1, \pi(s))$ be a t-simplex in G(s) and let $\bar{\sigma}(\bar{y}^1, \bar{\pi}(s))$ be a t-simplex sharing with $\sigma(y^1, \pi(s))$ the facet $\tau$ opposite the vertex $y^p$ of $\sigma(y^1, \pi(s))$. Further, suppose that both $\sigma(y^1, \pi(s))$ and $\bar{\sigma}(\bar{y}^1, \bar{\pi}(s))$ lie in the same subset A(s, $\gamma(s)$) of A(s). Then $\bar{y}^1$, $\bar{\pi}(s)$ and $\bar{a}$ are obtained from $y^1$, $\pi(s)$ and a as shown in table 1, where the (n+1)-vector a is given by $a_h = b$, $h \in I^+(s)$, $a_{k_i} = a(k_i)$, $i = 1, \ldots, t-1$, and $a_h = 0$, $h \in I^-(s)$, and where e(0) is the $^i$(n+1)-vector given by $e_i(0) = 1$, if $i \in I^+(s)$ and zero elsewhere.

| | $\bar{y}^1$ | $\bar{\pi}(s)$ | $\bar{a}$ |
|---|---|---|---|
| $p = 1$ | $y^1 + m^{-1} q(\pi_1)$ | $(\pi_2, \ldots, \pi_t, \pi_1)$ | $a + e(\pi_1)$ |
| $1 < p < t+1$ | $y^1$ | $(\pi_1, \ldots, \pi_p, \pi_{p-1}, \ldots, \pi_t)$ | a |
| $p = t+1$ | $y^1 - m^{-1} q(\pi_t)$ | $(\pi_t, \pi_1, \ldots, \pi_{t-1})$ | $a - e(\pi_t)$ |

Table 1. p is the index of the vertex to be replaced

We will now consider the case that the facet $\tau$ opposite a vertex $y^p$, $1 \leq p \leq t+1$, of $\sigma(y^1, \pi(s))$ in G(s, $\gamma(s)$) is not a facet of another simplex in G(s, $\gamma(s)$) but lies in bd A(s, $\gamma(s)$).

Lemma 2.4. Let $\sigma(y^1, \pi(s))$ be a t-simplex in $G(s, \gamma(s))$ and let $\tau$ be the facet opposite vertex $y^p$, $1 \leq p \leq t+1$. Then $\tau$ lies in bd $A(s, \gamma(s))$ iff either one of the following cases occur

(a) $p = 1$ : $\pi_1 = 0$ and $b = m - 1$

(b) $1 < p < t + 1$: $\pi_p = k_1$, $\pi_{p-1} = 0$ and $a(\pi_p) = b$
or
$\pi_p = k_i$ for some $i \in \{2,\ldots,t-1\}$, $\pi_{p-1} = k_{i-1}$ and $a(\pi_p) = a(\pi_{p-1})$

(c) $p = t + 1$ : $\pi_t = k_{t-1}$ and $a(\pi_t) = 0$.

The lemma follows immediately from the definitions 2.2 and 2.3. Suppose that the facet $\tau$ opposite vertex $y^1$ lies on the boundary of $A(s, \gamma(s))$ so that $\pi_1 = 0$ and $b = m - 1$.

Lemma 2.5. $\tau$ is a $(t-1)$-simplex in bd $S^n$. More precisely $\tau$ lies in

$$\bigcap_{k\in I^-(s)} S^n_k.$$

Proof. Since $y^1 = v + b\, m^{-1}q(0) + \sum_{k\in I^0(s)} a(k)m^{-1}\, q(k)$, $b = m - 1$ and $q_k(k_i) = 0$ for all $k \in I^-(s)$, $i = 1,\ldots, t - 1$, we have

$$y^1_k = m^{-1}\, v_k\ ,\ k \in I^-(s).$$

Since $\pi_1 = 0$ and $y^{i+1} = y^i + m^{-1}q(\pi_i)$, $i = 1,\ldots, t$, and again since $q_k(\pi_i) = 0$, $i = 2,\ldots, t$, for all $k \in I^-(s)$, we have

$$y^i_k = 0, \quad k \in I^-(s)$$

for all $i=2,\ldots, t+1$. Therefore $\tau(y^2,\ldots, y^{t+1})$ lies in $\bigcap_{k\in I^-(s)} S^n_k$.

□

Suppose now that the facet $\tau$ opposite vertex $y^p$, for some $1 < p < t+1$, lies on the boundary of $A(s, \gamma(s))$ so that either

$$\pi_p = k_1, \quad \pi_{p-1} = 0 \quad \text{and } a(\pi_p) = b$$

or

$$\pi_p = k_i \text{ for some } i \text{ in } \{2,\ldots, t-1\},\ \pi_{p-1} = k_{i-1} \text{ and } a(\pi_p) = a(\pi_{p-1}).$$

Lemma 2.6. In the case $\pi_p = k_1$, $\pi_{p-1} = 0$ and $a(\pi_p) = b$, the facet $\tau$ opposite vertex $y^p$ lies in bd A(s). Then $\tau$ is a (t-1)-simplex $\bar{\sigma}(\bar{y}^1, \pi(\bar{s}))$ in $G(\bar{s})$ with $|I^0(\bar{s})| = |I^0(s)|-1$. More precisely, $\bar{\sigma}(\bar{y}^1, \pi(\bar{s}))$ is an element of $G(\bar{s}, \gamma(\bar{s}))$ where

$$\bar{s}_h = s_h, \quad h \neq k_1, \text{ and } \bar{s}_{k_1} = 1,$$

$$\gamma(\bar{s}) = (k_2,\ldots, k_{t-1})$$

and where

$$\bar{y}^1 = y^1,\ \pi(\bar{s}) = (\pi_1,\ldots, \pi_{p-2}, \pi_{p-1}, \pi_{p+1},\ldots, \pi_t),$$
$$\text{and } \bar{a} = a.$$

As the following lemma's, the proof of this lemma follows immediately from the definition of $G(s, \gamma(s))$.

Lemma 2.7. In the case $\pi_p = k_i$ for some i in $\{2,\ldots, t-1\}$, $\pi_{p-1} = k_{i-1}$ and $a(\pi_p) = a(\pi_{p-1})$, the facet $\tau$ is a facet of exactly one other t-simplex $\bar{\sigma}(\bar{y}^1, \bar{\pi}(s))$ in G(s). More precisely $\bar{\sigma}(\bar{y}^1, \bar{\pi}(s))$ lies in $G(s, \bar{\gamma}(s))$ where

$$\bar{\gamma}(s) = (k_1,\ldots, k_{i-2}, k_i, k_{i-1}, k_{i+1},\ldots, k_{t-1})$$

and

$$\bar{y}^1 = y^1,\ \bar{\pi}(s) = (\pi_1,\ldots, \pi_{p-2}, \pi_p, \pi_{p-1}, \pi_{p+1},\ldots, \pi_t) \text{ and}$$

$$\bar{a} = a.$$

Suppose now that for $t > 1$ the facet $\tau$ opposite vertex $y^{t+1}$ lies on the boundary of $A(s, \gamma(s))$ so that $\pi_t = k_{t-1}$ and $a(\pi_t) = 0$.

Lemma 2.8. $\tau$ is a (t-1) simplex $\bar{\sigma}(\bar{y}^1, \pi(\bar{s}))$ in $G(\bar{s})$, with $|I^0(\bar{s})| = |I^0(s)| - 1$. More precisely, $\bar{\sigma}(\bar{y}^1, \pi(\bar{s}))$ lies in $G(\bar{s}, \gamma(\bar{s}))$ where

$$\bar{s}_h = s_h,\ h \neq k_{t-1} \text{ and } \bar{s}_{k_{t-1}} = -1,$$

$$\gamma(\bar{s}) = (k_1, \ldots, k_{t-2})$$

and where

$$\bar{y}^1 = y^1,\ \pi(\bar{s}) = (\pi_1, \ldots, \pi_{t-1}) \text{ and } \bar{a} = a.$$

The lemma's above give a complete description of how the parameters of a facet $\tau$ or an adjacent simplex $\bar{\sigma}$ of a t-simplex $\sigma(y^1, \pi(s))$ in $G(s)$ can be obtained from $y^1$, $\pi(s)$ and a.

3. The description of the algorithm

Let z be an excess demand function, i.e. z is a continuous function from $S^n$ into $R^{n+1}$ such that

$$\Sigma_{k=1}^{n+1} x_k z_k(x) = 0 \qquad \text{for all } x \text{ in } S^n$$

$$z_i(x) > 0 \qquad \text{if } x_i = 0,\ i = 1,\ldots,n+1$$

holds. The problem is to find an $x^*$ in $S^n$ such that $z(x^*) = 0$. In a model for an exchange economy such a point $x^*$ yields a vector of prices for which demands equal supplies.

To solve the problem, we could transform z into a continuous function g from $S^n$ into $P^n$, where $P^n = \{x \in R^{n+1} \mid \Sigma_{i=1}^{n+1} x_i = 0\}$, such that $z(x^*) = 0$ if and only if $g(x^*) = 0$. A well-known transformation is the function g defined by

$$g_i(x) = \frac{x_i + \max(0, z_i(x))}{1 + \Sigma_{j=1}^{n+1} \max(0, z_j(x))} - x_i \qquad i = 1,\ldots, n+1.$$

In many text books the existence of equilibrium prices is proved by using the Brouwer fixed point theorem, which garantees that the function $h: S^n \to S^n$ with $h_i(x) = g_i(x) + x_i$, $i = 1,\ldots, n+1$, has a fixed point. Another function, proposed by Todd to be used in simplicial algorithms, is

$$g_i(x) = \frac{x_i + \lambda z_i(x)}{1 + \lambda \Sigma_{j=1}^{n+1} z_j(x)} - x_i \qquad i = 1,\ldots, n+1,$$

with $\lambda$ some positive scalar, small enough to guarantee that $g_i(x) > 0$ if $x_i = 0$. Deleting the denominator we may take $\lambda = 1$ and the transformation becomes

$$g_i(x) = z_i(x) - x_i \Sigma_{j=1}^{n+1} z_j(x) \qquad i = 1,\ldots, n+1$$

which is just $z_i(x)$ itself minus the sum over all components z with

weight $x_i$. Clearly, since $\Sigma_{i=1}^{n+1} x_i = 1$, we have that $\Sigma_{i=1}^{n+1} g_i(x) = 0$. In the following we use a slightly different transformation, which will be motivated later on. In our transformation we replace $x_i$ by $(n+1)^{-1}$ for all i. So, the sum of the components z is equally weighted and we obtain

$$g_i(x) = z_i(x) - z^m(x) \qquad i = 1,\ldots, n+1$$

with $z^m(x) = \Sigma_{j=1}^{n+1} z_j(x)/(n+1)$. Observe that for this transformation it is not guaranteed that $g_i(x) > 0$ if $x_i = 0$. Clearly $g_i(\bar{x}) = 0$ iff $z_i(\bar{x}) = z^m(\bar{x})$ for all i. Since $\bar{x}^T z(\bar{x}) = 0$ this implies $z(\bar{x}) = 0$.

To find an approximate zero of g, we will propose an algorithm which utilizes the sets A(s) of $S^n$ described in the previous section. Unlike other variable dimension algorithms on $S^n$ [see e.g. van der Laan and Talman (1979) and Doup and Talman (1984)], there are $2^{n+1}-2$ rays leaving the starting point. To give the algorithm, we first introduce the concept of an s-complete simplex, where s is a sign vector in $R^{n+1}$, with $s_i \neq 0$ for at least one index i. The vector $\underline{0}$ denotes the (n+1)-vector with all components equal to zero.

Definition 3.1. For s a sign vector, with $|I^+(s)|$ and $|I^-(s)|$ positive, a t-simplex $\sigma(y^1,\ldots, y^{t+1})$ with $t = 1 + |I^0(s)|$ is s-complete if the system of linear equations

$$\Sigma_{i=1}^{t+1} \lambda_i \begin{pmatrix} g(y^i) \\ 1 \end{pmatrix} + \Sigma_{h \notin I^0(s)} \mu_h \begin{pmatrix} -s_h e(h) \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad (3.1)$$

has a nonnegative solution $\lambda_i^*$, i = 1,..., t+1, and $\mu_h^*$, $h \notin I^0(s)$.

A solution $\lambda_i^*$, i = 1,..., t+1, and $\mu_h^*$, $h \notin I^0(s)$ will be denoted by $(\lambda^*, \mu^*)$. Observe that the system (3.1) has $|I^0(s)| + 2 + (n+1-|I^0(s)|) = n+3$ columns, so that in general the solution is not unique, if a solution exists. In the following we call a solution $(\lambda^*, \mu^*)$ a basic solution if at least one of the variables $\lambda_i^*$, i = 1,..., t+1, $\mu_h^*$, $h \notin I^0(s)$, is equal to zero. We now make the following assumption.

Nondegeneracy assumption. For each basic solution to the system (3.1) we have that either at most one of the variables $\lambda_i^*$, i = 1,..., t+1,

$\mu_h^*$, $h \notin I^0(s)$, is equal to zero or that all variables $\mu_h^*$, $h \notin I^0(s)$, are equal to zero.

In the case that $\mu_h^* = 0$ for all $h \notin I^0(s)$ we say that the basic solution $(\lambda^*, \mu^*)$ is complete and that the simplex $\sigma$ is a complete simplex. We will show that a complete simplex induces an approximate zero of z. For s with $|I^+(s)|$ and $|I^-(s)|$ positive, by the nondegeneracy assumption, an s-complete simplex $\sigma$ yields a linesegment of solutions. Each of the two endpoints of such a linesegment is a basic solution of $\sigma$. To each solution $(\lambda^*, \mu^*)$ of an s-complete simplex $\sigma(y^1, \ldots, y^{t+1})$, there corresponds a point $\bar{x} = \Sigma_{i=1}^{t+1} \lambda_i^* y^i$. Since, according to the last equation of system (3.1) $\Sigma_{i=1}^{t+1} \lambda_i^* = 1$, we have that $\bar{x}$ lies in $\sigma$. In particular, if at a basic solution of a t-simplex $\lambda_i^* = 0$, for some i, then the point $\bar{x}$ lies in the facet of $\sigma$ opposite the vertex $y^i$. Such a facet corresponding to a basic solution is called a basic facet of $\sigma$. If at a basic solution $\mu_h^* = 0$, for some $h \notin I^0(s)$, we show that if $\sigma$ is not complete it is a facet of an $\bar{s}$- complete simplex $\bar{\sigma}$ with $\bar{s}_k = s_k$, $k \neq h$, and $\bar{s}_h = 0$. Since $h \notin I^0(s)$, we have that $|I^0(\bar{s})| = |I^0(s)|+1$.

So, each linesegment of solutions to (3.1) induces a linesegment of points $\bar{x}$ in $\sigma$ with two corresponding endpoints, say $x^1$ and $x^2$. The three possible cases which can occur are

a) the two endpoints $x^1$ and $x^2$ lie in two different basic facets of $\sigma$
b) one endpoint, say $x^1$, lies in a basic facet of $\sigma$ and the other endpoint, $x^2$, in the interior of $\sigma$
c) both endpoints $x^1$ and $x^2$ lie in the interior of $\sigma$.

The algorithm will follow linesegments of points in a sequence of adjacent s-complete simplices for varying s by going from basic solution to basic solution. Therefore we will show that a basic solution which is not complete, is a basic solution of exactly one other simplex. Each linesegment of points in an s-complete simplex $\sigma$ can be followed by making a linear programming pivot step in the system (3.1). The algorithm starts in the point v and terminates as soon as a basic solution is complete. The next lemma describes us when a basic solution is complete.

Lemma 3.2. Let s be a sign vector with $|I^+(s)|$ and $|I^-(s)|$ positive. If $\sigma(y^1,\ldots, y^{t+1})$ is an s-complete simplex with a basic solution $(\lambda^*, \mu^*)$ such that $\mu_h^* = 0$ for all $h \in I^+(s)$, then $(\lambda^*, \mu^*)$ is a complete solution. Similarly, a basic solution $(\lambda^*, \mu^*)$ of $\sigma$ is complete if $\mu_h^* = 0$ for all $h \in I^-(s)$.

Proof. First we consider the case $\mu_h^* = 0$ for $h \in I^+(s)$. So, (3.1) has a solution $(\lambda^*, \mu^*)$ such that

$$\Sigma_{i=1}^{t+1} \lambda_i^* \begin{pmatrix} g(y^i) \\ 1 \end{pmatrix} + \Sigma_{h\in I^-(s)} \mu_h^* \begin{pmatrix} e(h) \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

because $s_h = -1$ for all $h \in I^-(s)$. Since $\Sigma_{j=1}^{n+1} g_j(x) = 0$ for all $x \in S^n$, we obtain, by summing up the first n+1 equations, that

$$\Sigma_{h\in I^-(s)} \mu_h^* = 0.$$

This implies that $\mu_h^* = 0$ for all $h \in I^-(s)$ and hence for all $h \notin I^0(s)$. So, $(\lambda^*, \mu^*)$ is complete. In the same way, if $\mu_h^* = 0$ for $h \in I^-(s)$, we obtain

$$\Sigma_{h\in I^+(s)} \mu_h^* = 0,$$

and again $\mu_h^* = 0$ for all $h \notin I^0(s)$.

□

We will show that under the nondegeneracy assumption a complete simplex is found in a finite number of steps. If $\sigma(y^1,\ldots, y^{t+1})$ is complete, then there is a solution $\lambda_i^*$, $i = 1,\ldots,t+1$, such that

$$\Sigma_{i=1}^{t+1} \lambda_i^* \begin{pmatrix} g(y^i) \\ 1 \end{pmatrix} = \begin{pmatrix} \underline{0} \\ 1 \end{pmatrix},$$

with $\Sigma_{i=1}^{t+1} \lambda_i^* = 1$. Hence $\bar{g}(\bar{x}) = \underline{0}$, with $\bar{g}(x)$ the piecewise linear approximation to g with respect to the underlying triangulation, and with $\bar{x} = \Sigma_{i=1}^{t+1} \lambda_i^* y^i$. So, $\bar{x}$ is a zero of $\bar{g}$ and hence an approximate zero of g. However, since we are interested in a zero of z we will give bounds for the accuracy of $\bar{x}$ as an approximate zero of z.

Theorem 3.3. Let $\varepsilon > 0$ and let $S^n$ be triangulated (e.g. by the V-triangulation) with grid size such that for all x and y

$$\max_{k \in I_{n+1}} |z_k(x) - z_k(y)| < \varepsilon$$

if x and y lie in the same simplex. Let $\sigma(y^1, \ldots, y^{t+1})$ be a complete simplex with solution $\lambda_i^*$, $i = 1, \ldots, t+1$. Then

$$\bar{z}^m(\bar{x}) - \varepsilon < z_k(\bar{x}) < \bar{z}^m(\bar{x}) + \varepsilon \qquad k = 1, \ldots, n+1$$

with

$$-\varepsilon < \bar{z}^m(\bar{x}) < \varepsilon,$$

where $\bar{x} = \Sigma_{i=1}^{t+1} \lambda_i^* y^i$.

Proof. Since $\sigma$ is complete we have that $\bar{g}(\bar{x}) = \bar{z}(\bar{x}) - \bar{z}^m(\bar{x})e = 0$, with $\bar{z}$ the piecewise linear approximation to z and $e = (1, \ldots, 1)^T$. Hence

$$\bar{z}_k(\bar{x}) = \bar{z}^m(\bar{x}) \qquad k = 1, \ldots, n+1. \qquad (3.2)$$

Since $\bar{x}^T e = 1$ and $\bar{x}^T z(\bar{x}) = 0$, we get

$$|\bar{z}^m(\bar{x})| = |\bar{x}^T e\, \bar{z}^m(\bar{x})| = |\bar{x}^T \bar{z}(\bar{x})| = |\bar{x}^T(\bar{z}(\bar{x}) - z(\bar{x}))|$$

$$= |\bar{x}^T \Sigma_{i=1}^{t+1} \lambda_i^*(z(y^i) - z(\bar{x}))| < (\bar{x}^T e)\varepsilon = \varepsilon.$$

Furthermore $|z_k(\bar{x}) - \bar{z}^m(\bar{x})| = |z_k(\bar{x}) - \bar{z}_k(\bar{x})| = |\Sigma_{i=1}^{t+1} \lambda_i^*(z_k(\bar{x}) - z_k(y^i))|$ $< \varepsilon$. This proves the theorem.

□

Observe that (3.2) states that all components of $\bar{z}(\bar{x})$ are equal to $\bar{z}^m(\bar{x}) = \Sigma_{j=1}^{n+1} \bar{z}_j(\bar{x})/(n+1)$. Therefore, the accuracy of an approximate zero agrees with the accuracy of an approximate zero of the (n+1)-ray algorithm of Doup and Talman (1984). Notice that a complete simplex as defined above coincides with a complete simplex of this (n+1)-ray algorithm.

It will be shown that for varying s the s-complete simplices in A(s) determine paths of adjacent simplices. One of these paths has the zero-dimensional simplex $\sigma(v)$ as an endpoint and a complete simplex as its other endpoint. For fixed sign vectors s, the s-complete simplices in A(s) determine paths of adjacent simplices, since a basic facet is facet of at most two adjacent simplices. Such a path of adjacent simplices either is a loop in A(s) or has two endpoints. An endpoint either is an s-complete simplex $\sigma$ having a basic facet in the boundary of A(s), or is complete, or is an s-complete simplex $\sigma$ having a basic solution with $\mu_k^* = 0$ for exactly one $k \notin I^0(s)$. We will show that when $\sigma$ has a basic facet in the boundary of A(s), under some boundary-condition on $\bar{z}$, this facet is an endpoint of a path of $\bar{s}$-complete simplices in $A(\bar{s})$, with $\bar{s}_k \neq 0$ for some $k \in I^0(s)$, and $\bar{s}_h = s_h$, $h \neq k$. If $\mu_k^* = 0$ for exactly one $k \notin I^0(s)$, the simplex $\sigma$ is a basic facet of an endpoint of a path of $\bar{s}$-complete simplices in $A(\bar{s})$, with $\bar{s}_k = 0$, and $\bar{s}_h = s_h$, $h \neq k$. Therefore the paths of s-complete simplices in A(s) for varying s can be linked together to paths of adjacent simplices of varying dimension, each of them being either a loop or a path with two endpoints. Exactly one endpoint is the starting point v whereas all other endpoints are complete simplices. We will show that v is the endpoint of exactly one path. This path will be followed by the algorithm leading from v to its other endpoint which then must be a complete simplex.

We will now state the boundary-condition mentioned above.

Condition 3.4. The piecewise linear approximation $\bar{z}$ satisfies that for any $x \in \text{bd } S^n$, there is an index h with $x_h > 0$ such that

$$\bar{z}_h(x) < \max \{\bar{z}_i(x) \mid x_i = 0\}.$$

Since $z_i(x) > 0$ if $x_i = 0$ and $x^T z(x) = 0$ there is an index h with $x_h > 0$ and $z_h(x) \leq 0$. Therefore the condition will be satisfied if the grid size is taken small enough. In general, the condition will hold for any grid size if z is derived from a pure exchange economy.

Now, suppose that $z_h(v) \neq 0$, $h = 1,\ldots,n+1$, which is in line with the nondegeneracy assumption, and let $s^0$ be defined by $s_h^0 = \text{sign } z_h(v)$, $h = 1,\ldots,n+1$. Furthermore, let $\sigma^0$ be the one-dimensional simplex

in $A(s^0)$ having v as one of its vertices, i.e. $\sigma^0 = \sigma(y^1, \pi(s^0))$ with $y^1 = v$ and $\pi(s^0) = (0)$. Clearly, $\sigma^0$ is $s^0$-complete and $\sigma(v)$ is a basic facet of $\sigma^0$. Since $\{v\}$ lies in bd $A(s^0)$, we have that $\sigma^0$ is an endpoint of a path of adjacent $s^0$-complete simplices in $A(s^0)$. Now, let us consider an endpoint $\sigma(y^1,\ldots, y^{t+1})$ of a path of s-complete t-simplices in A(s) with common basic facets, such that $\sigma$ has a basic facet $\tau$ in bd A(s) (unequal to $\{v\}$). Observe that for some $\gamma(s)$, $\sigma$ lies in $A(s,\gamma(s))$ and is represented by a leading vertex $y^1$ and a permutation $\pi(s)$ . We first prove that $\tau$ does not lie in bd $S^n$.

Lemma 3.5. Let $\tau$ be a basic facet of an s-complete simplex $\sigma$ in A(s) with a solution $(\lambda^*,\mu^*)$. Then $\tau$ does not lie in bd $S^n$ if condition 3.4 is satisfied.

Proof. Suppose $\tau$ lies in bd $S^n$ and condition 3.4 is satisfied. Then, by definition of A(s) we have that $\tau \subset S^n_k$ iff $k \in I^-(s)$. So, if for some $x \in \tau$, $x_h > 0$, then $h \in I^0(s) \cup I^+(s)$. Now let $x^1,\ldots, x^t$ be the vertices of $\tau$. Then we have

$$\Sigma_{i=1}^t \lambda_i^*\begin{pmatrix} g(x^i) \\ 1 \end{pmatrix} + \Sigma_{h\in I^+(s)} \mu_h^*\begin{pmatrix} -e(h) \\ 0 \end{pmatrix} + \Sigma_{h\in I^-(s)} \mu_h^*\begin{pmatrix} e(h) \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

So, with $\bar{x} = \Sigma_{i=1}^t \lambda_i^* x^i$ we get

$$\bar{g}_h(\bar{x}) = \bar{z}_h(\bar{x}) - \bar{z}^m(\bar{x}) - \mu_h^* = 0 \qquad \text{if } h \in I^+(s)$$

$$\bar{g}_h(\bar{x}) = \bar{z}_h(\bar{x}) - \bar{z}^m(\bar{x}) \qquad = 0 \qquad \text{if } h \in I^0(s)$$

$$\bar{g}_h(\bar{x}) = \bar{z}_h(\bar{x}) - \bar{z}^m(\bar{x}) + \mu_h^* = 0 \qquad \text{if } h \in I^-(s).$$

Hence $\bar{z}_h(\bar{x}) \geq \bar{z}^m(\bar{x})$ for $h \in I^+(s) \cup I^0(s)$, and $\bar{z}_h(\bar{x}) \leq \bar{z}^m(\bar{x})$ for $h \in I^-(s)$. Since $h \in I^+(s) \cup I^0(s)$ for all h with $\bar{x}_h > 0$ and $h \in I^-(s)$ if $\bar{x}_h = 0$ this contradicts condition 3.4.

□

The lemma proves that a basic facet $\tau$ of an s-complete simplex $\sigma$ does not lie in bd $S^n$. Hence either there is an other s-complete sim-

plex $\bar{\sigma}$ in A(s) having the facet $\tau$ as a basic facet, or $\tau$ is a simplex in bd A(s), not in bd $S^n$. From the lemma's 2.4, 2.6 and 2.8 we immediately obtain the next corollary, where $(k_1,\ldots, k_{t-1})$ is the permutation $\gamma(s)$ of the elements of $I^0(s)$.

Corollary 3.6. Let $\tau$ be a basic facet $\tau(y^1,\ldots, y^{p-1}, y^{p+1}, \ldots, y^{t+1})$ of an s-complete simplex $\sigma(y^1,\ldots, y^{t+1})$ in A(s, $\gamma(s)$) which lies in bd A(s). Then we have

$2 < p < t+1$, and $\tau$ is an $\bar{s}$-complete (t-1)-simplex in $A(\bar{s})$ with $\bar{s}_h = s_h$, $h \neq k_1$, and $\bar{s}_{k_1} = 1$

or

$p = t+1$, and $\tau$ is an $\bar{s}$-complete (t-1)-simplex in $A(\bar{s})$ with $\bar{s}_h = s_h$, $h \neq k_{t-1}$, and $\bar{s}_{k_{t-1}} = -1$.

The corollary implies that $\tau$ is an endpoint of a path of adjacent $\bar{s}$-complete simplices in $A(\bar{s})$ with $\bar{s}_h = s_h$, $h \neq k$ and $\bar{s}_k \neq 0$ for some k in $I^0(s)$.

Finally we consider the case that $\sigma(y^1, \pi(s))$ is an s-complete simplex in A(s) having a basic solution $(\lambda^*, \mu^*)$ with $\mu_k^* = 0$ for some $k \notin I^0(s)$. If $\mu_h^* = 0$ for all $h \notin I^0(s)$ then by definition $\sigma$ is complete. If $\sigma$ is not complete either $k \in I^+(s)$ and $|I^+(s)| \geq 2$ or $k \in I^-(s)$ and $|I^-(s)| \geq 2$. In both cases $\sigma$ lies in the boundary of $A(\bar{s})$ with $\bar{s}_h = s_h$, $h \neq k$, and $\bar{s}_k = 0$, and is a facet of a unique simplex $\bar{\sigma}$ in $A(\bar{s})$. This simplex $\bar{\sigma}$ is $\bar{s}$-complete and has $\sigma$ as a basic facet in the boundary of $A(\bar{s})$. So $\bar{\sigma}$ is an endpoint of a path of $\bar{s}$-complete simplices in $A(\bar{s})$. It remains to characterize $\bar{\sigma}$. Suppose that $\sigma = \sigma(y^1, \pi(s))$ and lies in A(s, $\gamma(s)$) with $\gamma(s) = (k_1,\ldots, k_{t-1})$ a permutation of the elements of $I^0(s)$. Then, from definition 2.3 we obtain the following corollary.

Corollary 3.7. If the basic solution $(\lambda^*, \mu^*)$ with $\mu_k^* = 0$ for some $k \notin I^0(s)$ corresponding to the s-complete simplex $\sigma$ is not complete, then $\sigma$ is a basic facet of the $\bar{s}$-complete simplex $\bar{\sigma}(\bar{y}^1, \pi(\bar{s}))$ in $A(\bar{s}, \gamma(\bar{s}))$ with

$$\bar{s}_h = s_h \text{ for } h \neq k \text{ and } \bar{s}_k = 0,$$

$$\gamma(\bar{s}) = \begin{cases} (k, k_1, \ldots, k_{t-1}) & \text{if } k \in I^+(s) \\ (k_1, \ldots, k_{t-1}, k) & \text{if } k \in I^-(s), \end{cases}$$

and

$$\bar{y}^1 = y^1, \quad \pi(\bar{s}) = \begin{cases} (\pi_1, \ldots, \pi_p, k, \pi_{p+1}, \ldots, \pi_t) \text{ with } \pi_p = 0 & \text{if } k \in I^+(s) \\ (\pi_1, \ldots, \pi_t, k) & \text{if } k \in I^-(s), \end{cases}$$

and $\bar{a} = a$.

All of this together implies that each s-complete t-simplex in A(s) lies on exactly one path. Since the total number of simplices $\sigma(y^1, \pi(s))$ in G(s) is finite for each s, all paths are finite and there is exactly one path from $\sigma(v)$ to a complete simplex $\sigma^*$ which can be followed within a finite number of steps. Moreover, the solutions to (3.1) on this path determine a piecewise linear path from v to $x^*$, with $x^*$ in $\sigma^*$. This path can be followed by performing alternating linear programming pivot steps in system (3.1) and replacement steps according to table 1. If for some s, an s-complete facet $\tau$ in $A(\bar{s})$, $\bar{s}_h = s_h$, $h \neq k$, and $\bar{s}_k = \pm 1$ for some k in $I^0(s)$ is generated, then $(-\bar{s}_k e^T(k), 0)^T$ is reintroduced in system (3.1) with respect to $\tau$. On the other hand, when an $\bar{s}$-complete t-simplex $\sigma$ in A(s), $\bar{s}_h = s_h$, $h \neq k$ and $\bar{s}_k = 0$, for some $k \notin I^0(s)$ is generated and $\sigma$ is not complete, a linear programming pivot step in the system (3.1) with respect to $\sigma$ is made with $(g^T(y^{\bar{p}}), 1)^T$ with $y^{\bar{p}}$ the unique vertex of the (t+1)-simplex in $A(\bar{s})$ having $\sigma$ as facet opposite this vertex. The steps of the algorithm which generates the path from v to an approximate solution $x^*$ are described as follows where $\bar{p}$ is the index of the vertex of $\sigma$ whose label is to be calculated.

Step 0. [Initialization]. Let s be given by $s_i = 1$ if $g_i(v) > 0$ and $s_i = -1$ if $g_i(v) < 0$, $i \in I_{n+1}$. Set $t = 1$, $y^1 = v$, $\pi(s) = (0)$, $\sigma = \sigma(y^1, \pi(s))$, $\bar{p} = 2$, $a = \underline{0}$, $\lambda_1 = 1$, $\mu_h = g_h(v)$, $h \in I^+(s)$ and $\mu_h = -g_h(v)$, $h \in I^-(s)$.

Step 1. Calculate $g(y^{\bar{p}})$. Perform a linear programming step by bringing $(g^T(y^{\bar{p}}), 1)^T$ in the linear system

$$\sum_{\substack{i=1 \\ i\neq \bar{p}}}^{t+1} \lambda_i \begin{pmatrix} g(y^i) \\ 1 \end{pmatrix} + \sum_{h\notin I^0(s)} \mu_h \begin{pmatrix} -s_h\, e(h) \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ \bar{1} \end{pmatrix} .$$

If for some $h \notin I^0(s)$, $\mu_h$ becomes equal to zero, then go to step 3. Otherwise the facet $\tau(y^1, \ldots, y^{p-1}, y^{p+1}, \ldots, y^{t+1})$ is s-complete for some $p \neq \bar{p}$, i.e. $\lambda_p$ becomes equal to zero.

Step 2. If $1 < p < t+1$, and if $\pi_p = k_1$, $\pi_{p-1} = 0$, and $a(\pi_p) = b$, then the dimension is decreased. Set $t = t-1$, and adapt s, $\gamma(s)$, $\sigma$ and a according to lemma 2.6. Go to step 4 with $r = k_1$.

If $1 < p < t+1$, and if for some $i \geq 2$, $\pi_p = k_i$, $\pi_{p-1} = k_{i-1}$, and $a(\pi_p) = a(\pi_{p-1})$, then $\gamma(s)$ and $\sigma$ are adapted according to lemma 2.7. Return to step 1 with $\bar{p}$ the index of the new vertex of $\sigma$.

If $p = t+1$, $\pi_t = k_{t-1}$, and $a(\pi_t) = 0$, then the dimension is decreased. Set $t = t-1$, and adapt s, $\gamma(s)$, $\sigma$ and a according to lemma 2.8.

Go to step 4 with $r = k_{t-1}$.

In all other cases $\sigma$ and a are adapted according to table 1. Return to step 1 with $\bar{p}$ the index of the new vertex of $\sigma$.

Step 3. [Increase dimension]. If $\mu_h = 0$ for all $h \notin I^0(s)$, then $\sigma$ is complete and the algorithm terminates. Otherwise s, $\gamma(s)$, $\sigma$ and a are adapted according to corollary 3.7. Set $t = t+1$, and return to step 1 with $\bar{p}$ the index of the new vertex of $\sigma$.

Step 4. [Decrease dimension]. Perform a linear programming step by bringing $(-s_r\, e^T(r), 0)^T$ in the linear system

$$\sum_{i=1}^{t+1} \lambda_i \begin{pmatrix} g(y^i) \\ 1 \end{pmatrix} + \sum_{\substack{h\notin I^0(s) \\ h\neq r}} \mu_h \begin{pmatrix} -s_h\, e(h) \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ \bar{1} \end{pmatrix} .$$

If for some $h \notin I^0(s)$, $\mu_h$ becomes equal to zero, go to step 3. Otherwise return to step 2 with p the index of the vertex for which $\lambda_p$ becomes zero.

The algorithm starts in v by increasing proportionally the components of v with positive g-value and decreasing proportionally the components of v with negative g-value, until for some h, $\bar{g}_h(x)$ becomes

equal to zero. Then $x_h$ is not further increased proportionally if $\bar{g}_h$ was positive and $x_h$ is not further decreased proportionally if $\bar{g}_h$ was negative. In general the algorithm generates points in t-dimensional regions of $S^n$ such that for some $\lambda > b > 0$ and for $j = 1,\ldots,n+1$

$$x_j = (1-b+\lambda)\, v_j \qquad \text{if } \bar{g}_j(x) \geq 0$$

$$x_j = (1-b+\mu_j)\, v_j \text{ with } 0 < \mu_j < \lambda \qquad \text{if } \bar{g}_j(x) = 0$$

$$x_j = (1-b)v_j \qquad \text{if } \bar{g}_j(x) \leq 0,$$

where t-1 is the number of indices j with $\bar{g}_j(x) = 0$. So, the $x_j$'s with positive $\bar{g}_j(x)$ are a factor $1-b-\lambda$ larger than $v_j$, and the $x_j$'s with negative $\bar{g}_j(x)$ are a factor $1-b$ smaller than $v_j$, whereas the $x_j$'s with $\bar{g}_j(x)$ equal to zero, vary between $(1-b)v_j$ and $(1-b+\lambda)v_j$.

When applied to a pure exchange model the path of points generated by the algorithm yields a price adjustment process similar to Walras' tatonnement process in the sense that prices of goods with positive excess demand are increased and those of goods with negative excess demand are decreased. The increase and decrease, however is not proportional to the excess demand as in the tatonnement process but proportional to the starting prices.

We remark that it is also possible to take $z(x)$ instead of $g(x) = z(x)-z^m(x)$. However, there is a reason why we have chosen $g(x)$ instead of $z(x)$. Taking z, the system of linear equations becomes

$$\Sigma_{i=1}^{t+1} \lambda_i \begin{pmatrix} z(y^i) \\ 1 \end{pmatrix} + \Sigma_{h \notin I^0(s)} \mu_h \begin{pmatrix} -s_h \, e(h) \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Now, since $\Sigma_j x_j z_j(x) = 0$ and not $\Sigma_j z_j(x) = 0$, we do not have that $\mu_h^* = 0$ for all $h \notin I^0(s)$ as soon as $\mu_h^* = 0$ for all $h \in I^+(s)$ or for all $h \in I^-(s)$. So, in general a complete simplex, in the sense that $\mu_h^* = 0$ for all $h \notin I^0(s)$, does not exist. Therefore taking z instead of g we have to stop the algorithm as soon as $\mu_h^* = 0$ for either all $h \in I^+(s)$ or for all $h \in I^-(s)$. Suppose that $\mu_h^* = 0$ for all $h \in I^-(s)$. Then the system becomes

$$\bar{z}_h(\bar{x}) = 0 \qquad \text{for all } h \notin I^+(s)$$

and

$$\bar{z}_h(\bar{x}) = \mu_h^* > 0 \qquad \text{for all } h \in I^+(s).$$

This does not guarantee that $\bar{x}$ is a good approximate zero of z. As a matter of fact, examples can be constructed that $\bar{x}$ approximates a point x such that $z_h(x) = 0$ for all $h \notin I^+(s)$ and $x_h = 0$ for all $h \in I^+(s)$. Although it can be proved that the sequence of approximate zeroes $\bar{x}^1$, $\bar{x}^2$,..., with $\bar{x}^k$ the approximate zero after the k-th cycle of the algorithm, has a subsequence converging to a zero of z, this difficulty motivates the choice of g instead of z. In case of the computation of economic equilibria the just mentioned problem will not arise if monotonicity of the preferences is assumed.

4. Computational results.

The algorithm presented in section 3 has been applied to the three pure exchange economies given in Scarf (1967), and a pure exchange economy with fifteen commodities and five consumers presented in the appendix. The algorithm is compared with the algorithm described in Doup and Talman (1984). Note that in both algorithms $S^n$ is triangulated by the V-triangulation and that the latter algorithm only has $n+1$ rays, whereas the new algorithm has $2^{n+1}-2$ rays. Both algorithms are started in the barycenter of $S^n$ with an initial grid size of $m^{-1} = 1$. When a complete simplex is found the grid is refined and the algorithm is restarted in the approximate solution. The grid is refined with different factors of incrementation. The grid refinement is stopped when the excess demands at the approximate solution are less than $10^{-9}$ in case of the three Scarf economies, and $10^{-8}$ in case of the economy presented in this section. Both algorithms are run with the labelling on z and the new algorithm is also run with the labelling on g.

Throughout this section we will use the following notations.

$FE_1$: the number of function evaluations in the $(n+1)$-ray algorithm

$FE_2$: the number of function evaluations in the $(2^{n+1}-2)$-ray algorithm with the labelling on z

$FE_3$: the number of function evaluations in the $(2^{n+1}-2)$-ray algorithm with the labelling on g.

The data of the three Scarf economies is given in Scarf (1967) and the data of the fourth economy is given in the appendix.

Economy 1: 5 commodities and 3 consumers, i.e. n = 4.

| factor | $FE_1$ | $FE_2$ | $FE_3$ |
|---|---|---|---|
| 2 | 50 | 49 | 50 |
| 3 | 52 | 49 | 54 |
| 4 | 50 | 47 | 55 |
| 5 | 45 | 51 | 56 |
| 6 | 41 | 50 | 64 |
| 7 | 56 | 61 | 65 |
| 8 | 67 | 73 | 70 |
| 9 | 66 | 74 | 78 |
| 10 | 64 | 70 | 75 |

Table 2. The number of function evaluations for the two algorithms with different grid refinement factors.

Economy 2: 8 commodities and 5 consumers, i.e. n = 7.

| factor | $FE_1$ | $FE_2$ | $FE_3$ |
|---|---|---|---|
| 2 | 96 | 80 | 94 |
| 3 | 81 | 64 | 80 |
| 4 | 71 | 72 | 69 |
| 5 | 79 | 74 | 82 |
| 6 | 101 | 96 | 86 |
| 7 | 91 | 86 | 78 |
| 8 | 79 | 79 | 81 |
| 9 | 95 | 82 | 84 |
| 10 | 107 | 103 | 98 |

Table 3. The number of function evaluations for the two algorithms with different grid refinement factors.

Economy 3: 10 commodities and 5 consumers, i.e. n = 9.

| factor | $FE_1$ | $FE_2$ | $FE_3$ |
|---|---|---|---|
| 2 | 135 | 103 | 102 |
| 3 | 133 | 90 | 105 |
| 4 | 126 | 80 | 95 |
| 5 | 133 | 72 | 92 |
| 6 | 137 | 71 | 88 |
| 7 | 149 | 73 | 100 |
| 8 | 152 | 78 | 115 |
| 9 | 154 | 82 | 91 |
| 10 | 143 | 71 | 98 |

Table 4. The number of function evaluations for the two algorithms with different grid refinement factors.

Economy 4: 15 commodities and 5 consumers, i.e. n = 14.

| factor | $FE_1$ | $FE_2$ | $FE_3$ |
|---|---|---|---|
| 2 | 238 | 189 | 169 |
| 3 | 209 | 148 | 163 |
| 4 | 198 | 187 | 155 |
| 5 | 217 | 190 | 184 |
| 6 | 217 | 212 | 184 |
| 7 | 178 | 242 | 181 |

Table 5. The number of function evalutions for the two algorithms with different grid refinement factors.

Concluding the computational results suggest that the number of function evaluations for the new algorithm is lower than for the algorithm described in Doup and Talman (1984), especially when n becomes larger.

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Appendix

The excess demand function $z: S^n \to R^{n+1}$ is given by

$$z_j(p) = \sum_{h=1}^{H} \left\{ \frac{a_{h,j} \sum_{k=1}^{n+1} w_{h,k} p_k}{\frac{b_h}{p_j} \sum_{k=1}^{n+1} a_{h,k} p_k} - w_{h,j} \right\}, \quad j = 1,\ldots, n+1,$$

where H is the number of consumers. The elements $a_{h,j}$, $w_{h,j}$, $h = 1,\ldots, 5$, $j = 1,\ldots, 10$, and $b_h$, $h = 1,\ldots, 5$ for economy 4 are the same as for economy 3, and the remaining elements $a_{h,j}$ and $w_{h,j}$, $h = 1,\ldots, 5$, $j = 11,\ldots, 15$ are given in table 6 and table 7.

| h \ j | 11 | 12 | 13 | 14 | 15 |
|---|---|---|---|---|---|
| 1 | 2.5 | 0.8 | 1.4 | 4.0 | 3.6 |
| 2 | 1.0 | 1.0 | 1.0 | 1.0 | 1.0 |
| 3 | 2.3 | 4.5 | 3.0 | 0.9 | 7.9 |
| 4 | 11.0 | 12.0 | 13.0 | 14.0 | 15.0 |
| 5 | 3.0 | 6.0 | 0.8 | 7.0 | 12.0 |

Table 6. The elements $a_{h,j}$, $h = 1,\ldots, 5$, $j = 11,\ldots, 15$, for economy 4.

| h \ j | 11 | 12 | 13 | 14 | 15 |
|---|---|---|---|---|---|
| 1 | 7.9 | 3.1 | 5.3 | 4.0 | 2.0 |
| 2 | 8.0 | 7.0 | 6.0 | 5.0 | 4.0 |
| 3 | 10.0 | 3.0 | 7.0 | 5.0 | 1.5 |
| 4 | 6.0 | 4.6 | 2.0 | 11.0 | 0.4 |
| 5 | 4.8 | 6.1 | 3.2 | 9.4 | 0.9 |

Table 7. The elements $w_{h,j}$, $h = 1,\ldots, 5$, $j = 11,\ldots, 15$, for economy 4.

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