# The 2-Page Crossing Number of $\boldsymbol{K}_{\boldsymbol{n}}$ 

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Received: 24 June 2012 / Revised: 14 February 2013 / Accepted: 18 April 2013 /
Published online: 4 June 2013
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Abstract Around 1958, Hill described how to draw the complete graph $K_{n}$ with

$$
Z(n):=\frac{1}{4}\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor\left\lfloor\frac{n-2}{2}\right\rfloor\left\lfloor\frac{n-3}{2}\right\rfloor
$$

crossings, and conjectured that the crossing number $\operatorname{cr}\left(K_{n}\right)$ of $K_{n}$ is exactly $Z(n)$. This is also known as Guy's conjecture as he later popularized it. Towards the end of the century, substantially different drawings of $K_{n}$ with $Z(n)$ crossings were found. These drawings are 2-page book drawings, that is, drawings where all the vertices are on a line $\ell$ (the spine) and each edge is fully contained in one of the two half-planes (pages) defined by $\ell$. The 2 -page crossing number of $K_{n}$, denoted by $\nu_{2}\left(K_{n}\right)$, is the minimum number of crossings determined by a 2-page book drawing of $K_{n}$. Since $\operatorname{cr}\left(K_{n}\right) \leq \nu_{2}\left(K_{n}\right)$ and $\nu_{2}\left(K_{n}\right) \leq Z(n)$, a natural step towards Hill's Conjecture is the

[^0]weaker conjecture $\nu_{2}\left(K_{n}\right)=Z(n)$, popularized by Vrt'o. In this paper we develop a new technique to investigate crossings in drawings of $K_{n}$, and use it to prove that $\nu_{2}\left(K_{n}\right)=Z(n)$. To this end, we extend the inherent geometric definition of $k$-edges for finite sets of points in the plane to topological drawings of $K_{n}$. We also introduce the concept of $\leq \leq k$-edges as a useful generalization of $\leq k$-edges and extend a powerful theorem that expresses the number of crossings in a rectilinear drawing of $K_{n}$ in terms of its number of $\leq k$-edges to the topological setting. Finally, we give a complete characterization of crossing minimal 2-page book drawings of $K_{n}$ and show that, up to equivalence, they are unique for $n$ even, but that there exist an exponential number of non homeomorphic such drawings for $n$ odd.

Keywords Crossing number • Topological drawing • Complete graph • Book drawing • 2-Page drawing

Mathematics Subject Classification (2000) $05 \mathrm{C} 10 \cdot 68 \mathrm{R} 10 \cdot 52 \mathrm{C} 10 \cdot 57 \mathrm{R} 15$

## 1 Introduction

In a drawing of a graph in the plane, each vertex is represented by a point and each edge is represented by a simple open arc (i.e., the image of an open interval in the plane, say $\{(x, 0): 0<x<1\}$, under a homeomorphism of the plane), such that if $u v$ is an edge, then the closure (in the plane) of the arc $\alpha$ representing $u v$ consists precisely of $\alpha$ and the points representing $u$ and $v$. It is further required that no arc representing an edge contains a point representing a vertex and that any two edges intersect only finitely many times. A crossing in a drawing $D$ of a graph $G$ is a pair ( $x,\{\alpha, \beta\}$ ), where $\alpha, \beta$ are arcs representing different edges and $\{x\} \in \alpha \cap \beta$ is a point in the plane where $\alpha$ and $\beta$ intersect transversally ${ }^{1}$. The crossing number $\operatorname{cr}(D)$ of $D$ is the number of crossings in $D$, and the crossing number $\operatorname{cr}(G)$ of $G$ is the minimum $\operatorname{cr}(D)$, taken over all drawings $D$ of $G$.

A drawing is good if (i) no three distinct arcs representing edges meet at a common point; (ii) if two edges are adjacent, then the arcs representing them do not intersect each other; and (iii) an intersection point between two arcs representing edges is a crossing. It is well-known (and easy to prove) that every graph has a crossing-minimal drawing which is good (moreover, (ii) and (iii) hold in every crossing-minimal drawing). Thus, when our aim (as in this paper) is to estimate the crossing number of a graph, we may assume that all drawings under consideration are good. As usual, for simplicity we often make no distinction between a vertex and the point representing it, or between an edge and the arc representing it. No confusion should arise from this practice.

Around 1958, Hill conjectured that

$$
\begin{equation*}
\operatorname{cr}\left(K_{n}\right)=Z(n):=\frac{1}{4}\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor\left\lfloor\frac{n-2}{2}\right\rfloor\left\lfloor\frac{n-3}{2}\right\rfloor . \tag{1}
\end{equation*}
$$

[^1]This conjecture appeared in print a few years later in papers by Guy [17] and Harary and Hill [19]. Hill described drawings of $K_{n}$ with $Z(n)$ crossings, which were later corroborated by Blažek and Koman [8]. These drawings show that $\operatorname{cr}\left(K_{n}\right) \leq Z(n)$. The best known general lower bound is $\lim _{n \rightarrow \infty} \operatorname{cr}\left(K_{n}\right) / Z(n) \geq 0.8594$, due to de Klerk et al. [15]. For more on the history of this problem we refer the reader to the excellent survey by Beineke and Wilson [6].

One of the major motivations for investigating crossing numbers is their application to VLSI design. With this motivation in mind, Chung, Leighton and Rosenberg [11] analyzed embeddings of graphs in books. We recall that a book consists of a line (the spine) and $k \geq 1$ half-planes (the pages), such that the boundary of each page is the spine. In a book embedding, every vertex lies on the spine, and each edge lies on a single page. Book embeddings of graphs have been extensively studied [7,16]. Now if the book has $k$ pages, and crossings among edges are allowed, the result is a $k$-page book drawing.

Here we concentrate on 2-page book drawings. The 2-page crossing number $v_{2}(G)$ of a graph $G$ is the minimum of $\operatorname{cr}(D)$ taken over all 2-page book drawings $D$ of $G$. As in the general case, it can be proven that this minimum is achieved by good 2-page book drawings. Alternative terminologies for the 2-page crossing number are circular crossing number [20] and fixed linear crossing number [12]. We may regard the pages as the closed half-planes defined by the spine, and so every 2-page book drawing can be realized as a plane drawing; it follows that $\operatorname{cr}(G) \leq \nu_{2}(G)$ for every graph $G$.

In 1964, Blažek and Koman [8] found 2-page book drawings of $K_{n}$ with $Z(n)$ crossings, thus showing that $\nu_{2}\left(K_{n}\right) \leq Z(n)$ (see also Guy et al. [18], Damiani et al. [13], Harborth [20], and Shahrokhi et al. [23]). Once these constructions were known, the conjecture that $\nu_{2}\left(K_{n}\right)=Z(n)$ is implicit in the conjecture given by Eq. (1) since $\operatorname{cr}\left(K_{n}\right) \leq \nu_{2}\left(K_{n}\right)$. However, the only explicit reference to this weaker conjecture is, as far as we know, from Vrt'o [24].

Buchheim and Zhang [9] reformulated the problem of finding $\nu_{2}\left(K_{n}\right)$ as a maximum cut problem on associated graphs, and then solved exactly this maximum cut problem for all $n \leq 13$, thus confirming Eq. (1) for 2-page book drawings for all $n \leq 14$ (the case $n=14$ follows from the case $n=13$ by an elementary counting argument). Very recently, De Klerk and Pasechnik [14] used this max cut reformulation to find the exact value of $\nu_{2}\left(K_{n}\right)$ for all $n \leq 21$ and $n=24$, and moreover, by using semidefinite programming techniques, to obtain the asymptotic bound $\lim _{n \rightarrow \infty} \nu_{2}\left(K_{n}\right) / Z(n) \geq 0.9253$. All the results reported in $[9,14]$ are computeraided.

In this paper we prove that $\nu_{2}\left(K_{n}\right)=Z(n)$. The main technique for the proof is the extension of the concept of $k$-edge of a finite set of points to topological drawings of the complete graph. We do this in a way such that the identities proved by Ábrego and Fernández-Merchant [1] and Lovász et al. [22], that express the crossing number of a rectilinear drawing of $K_{n}$ in terms of the $k$-edges or the $\leq k$-edges of its set of vertices, are also valid in the topological setting.

We recall that a drawing $D$ is rectilinear if the edges of $D$ are straight line segments, and the rectilinear crossing number $\overline{\operatorname{cr}}(G)$ of a graph $G$ is the minimum of $\operatorname{cr}(D)$ taken over all rectilinear drawings $D$ of $G$. Although the exact value of $\overline{\operatorname{cr}\left(K_{n}\right) \text { is known }}$ only for $n \leq 27$ and $n=30[4,10]$, there are fairly good estimates of its asymptotic
behavior (cf. [4, 2]):

$$
0.379972<\frac{277}{729} \leq \lim _{n \rightarrow \infty} \frac{\overline{\operatorname{cr}}\left(K_{n}\right)}{\binom{n}{4}} \leq \frac{83247328}{218791125}<0.380488
$$

For a survey on the rectilinear crossing number of $K_{n}$, we refer the reader to [5].
The remarkable recent progress on the estimation of $\overline{\operatorname{cr}}\left(K_{n}\right)$ has been prompted by the close relationship between this parameter and the number of $k$-edges in a rectilinear drawing of $K_{n}$. An edge $p q$ of $D$ is a $k$-edge if the line spanned by $p q$ divides the remaining set of vertices into two subsets of cardinality $k$ and $n-2-k$. Thus a $k$-edge is also an $(n-2-k)$-edge. Denote by $E_{k}(D)$ the number of $k$-edges of $D$ and let $E_{\leq k}(D)=\sum_{j=0}^{k} E_{j}(D)$. The following identity $[1,22]$ has been key to the recent developments on the rectilinear crossing number of $K_{n}$.

$$
\begin{equation*}
\overline{\operatorname{cr}}(D)=3\binom{n}{4}-\sum_{k=0}^{\lfloor n / 2\rfloor-1} k(n-2-k) E_{k}(D) \tag{2}
\end{equation*}
$$

In Sect. 2 we generalize the concept of $k$-edge to arbitrary (that is, not necessarily rectilinear) good drawings of $K_{n}$, and Theorem 1 extends Eq. (2) to these drawings. Although half-planes are not well defined for arbitrary good drawings of $K_{n}$, we can use the orientation of the triangles defined by three points: the edge $p q$ is a $k$-edge of the topological drawing if the set of triangles adjacent to $p q$ is divided, according to their orientation, into two subsets with cardinality $k$ and $n-k-2$. It was proved by Ábrego and Fernández-Merchant [1] and by Lovász et al. [22] that the inequality $E_{\leq k}(D) \geq 3\binom{k+2}{2}$ holds for every rectilinear drawing $D$ of $K_{n}$. This inequality and (2) imply that $\overline{\mathrm{cr}}\left(K_{n}\right) \geq Z(n)$ [1]. In contrast to the rectilinear case, the inequality $E_{\leq k}(D) \geq 3\binom{k+2}{2}$ does not hold in general for topological drawings $D$ of $K_{n}$, not even for 2-page book drawings, as can be seen in Fig. 4. This shows the relevance of introducing the new parameter $E_{\leq \leq k}(D)$, which we bound from below in Theorem 2. In Sect. 3 we use Theorems 2 and 1 to show that $\nu_{2}\left(K_{n}\right)=Z(n)$.

Two drawings $D$ and $D^{\prime}$ are plane-homeomorphic if there is a homeomorphism of the plane that sends $D$ to $D^{\prime}$. Typically, drawings on the plane are seen as drawings on the sphere under the one-point compactification of the sphere. In this context, when the drawings $D$ and $D^{\prime}$ on the plane correspond to the drawings $D_{S}$ and $D_{S}^{\prime}$ on the sphere, we say that $D$ and $D^{\prime}$ are sphere-homeomorphic if there is a homeomorphism of the sphere that sends $D_{S}$ to $D_{S}^{\prime}$. For crossing number purposes, it is enough and natural to consider sphere-homeomorphic drawings. However, it is impossible to define $k$-edges for drawings of the complete graph on the sphere (the way we do it on the plane) because it is impossible to orient simple closed curves on the sphere. Therefore, we use plane-homeomorphic drawings in Sects. 2 and 3 to prove that $\nu_{2}\left(K_{n}\right)=Z(n)$, and sphere-homeomorphic drawings to analyze the crossing optimal 2-page drawings of $K_{n}$ in Sect. 4. We give a complete characterization of their structure, showing that, up to sphere-homeomorphism, crossing optimal drawings are unique for $n$ even. In contrast, for $n$ odd we provide a family of size $2^{(n-5) / 2}$ of non sphere-homeomorphic crossing optimal drawings. We conclude with some open questions and directions for future research in Sect. 5. An extended abstract of this paper appeared in SoCG [3].

## 2 Crossings and $\boldsymbol{k}$-Edges

In this section we generalize the concept of $k$-edges and extend Eq. (2) to the topological setting. So far this concept has only been used in the geometric setting of finite sets of points in the plane. Here, we extend it to topological drawings of $K_{n}$. Let $D$ be a good drawing of $K_{n}$, let $\overrightarrow{p q}$ be a directed edge of $D$, and $r$ a vertex of $D$ other than $p$ or $q$. We denote by $p q r$ the oriented, closed curve defined by concatenating the (oriented) edges $p q, q r$ and $r p$. Note that $p q r$ is simple because the three edges $p q, q r$, and $r p$ do not self-intersect and do not intersect each other, since $D$ is good. An oriented, simple, and closed curve in the plane is oriented counterclockwise if the bounded region it encloses is on the left-hand side of the curve. We say that $r$ is on the left (respectively, right) side of $\overrightarrow{p q}$ if $p q r$ is oriented counterclockwise (respectively, clockwise). We say that the edge $p q$ is a $k$-edge of $D$ if it has exactly $k$ points of $D$ on one side (left or right), and thus $n-2-k$ points on the other side. Hence, as in the geometric setting, a $k$-edge is also an $(n-2-k)$-edge. Note that the direction of the edge $p q$ is no longer relevant and every edge of $D$ is a $k$-edge for some unique $k$ such that $0 \leq k \leq\lfloor n / 2\rfloor-1$. Let $E_{k}(D)$ be the number of $k$-edges of $D$.

First we show that there are essentially 3 topological good drawings of $K_{4}$.
Lemma 1 Any good drawing of $K_{4}$ in the plane is plane-homeomorphic to one of the three drawings shown in Fig. 1.

Proof We first observe that in a good drawing of $K_{4}$ there is at most one crossing. Let $p, q, r$, and $s$ be the vertices and assume that the edges $p r$ and $q s$ cross at $x$. The edge $r s$ cannot intersect the edge $p q$ because $p q x$ is a closed curve and the drawing is good. Similarly, the edge $q r$ does not intersect the edge $p s$, and the other possible pairs of edges share a vertex and thus their corresponding arcs do not intersect because the drawing is good. Thus, in a good drawing of $K_{4}$ there is always a hamiltonian cycle of non crossed edges: if we suppose that the only possible crossing is between the edges $p r$ and $q s$, then $p q, q r, r s$ and $s p$ form such a cycle, and if there are no crossings, then these edges form the cycle as well. Finally, once this cycle is drawn, there are only three possibilities to draw the edges $p r$ and $q s$ : both in the bounded face, both in the unbounded face, or one in each face. These correspond to the three drawings in Fig. 1. Clearly Drawing $A$ is not plane-homeomorphic to the other two because it has no crossings. To see that Drawings $B$ and $C$ are not plane-homeomorphic note that a homeomorphism of the plane taking one drawing to the other would need to map


A


B


C

Fig. 1 The three non plane-homeomorphic good drawings of $K_{4}$, with 3, 2, and 2 separations. The edge of each separation is shown bold (Color figure online)
the closure of the bounded face (a compact set) to the closure of the unbounded face, which is impossible. However, we note that Drawings $B$ and $C$ are indeed spherehomeomorphic.

We now extend Eq. (2) to the topological case.
Theorem 1 For any good drawing $D$ of $K_{n}$ in the plane the following identity holds,

$$
\operatorname{cr}(D)=3\binom{n}{4}-\sum_{k=0}^{\lfloor n / 2\rfloor-1} k(n-2-k) E_{k}(D)
$$

Proof In a good drawing of $K_{n}$, we say that an edge pq separates the vertices $r$ and $s$ if the orientations of the triangles $p q r$ and $p q s$ are opposite. In this case, we say that the set $\{p q, r, s\}$ is a separation.

We denote by $T_{A}, T_{B}$, and $T_{C}$ the number of induced subdrawings of $D$ of type $A$, $B$, and $C$, respectively. Then

$$
\begin{equation*}
T_{A}+T_{B}+T_{C}=\binom{n}{4} \tag{3}
\end{equation*}
$$

and since the subdrawings of types $B$ or $C$ are in one-to-one correspondence with the crossings of $D$, it follows that

$$
\begin{equation*}
\operatorname{cr}(D)=T_{B}+T_{C} . \tag{4}
\end{equation*}
$$

We count the number of separations in $D$ in two different ways: First, each subdrawing of type $A$ has 3 separations (the edge in each separation is bold in Fig. 1), and each subdrawing of types $B$ or $C$ has 2 separations. This gives a total of $3 T_{A}+2 T_{B}+2 T_{C}$ separations in $D$. Second, each $k$-edge belongs to exactly $k(n-2-k)$ separations. Summing over all $k$-edges for $0 \leq k \leq\lfloor n / 2\rfloor-1$ gives a total of $\sum_{k=0}^{\lfloor n / 2\rfloor-1} k(n-2-k) E_{k}(D)$ separations in $D$. Therefore

$$
\begin{equation*}
3 T_{A}+2 T_{B}+2 T_{C}=\sum_{k=0}^{\lfloor n / 2\rfloor-1} k(n-2-k) E_{k}(D) \tag{5}
\end{equation*}
$$

Finally, subtracting Eq. (5) from three times Eq. (3) we get

$$
T_{B}+T_{C}=3\binom{n}{4}-\sum_{k=0}^{\lfloor n / 2\rfloor-1} k(n-2-k) E_{k}(D)
$$

and thus by Eq. (4) we obtain the claimed result.
In order to prove $\nu_{2}\left(K_{n}\right)=Z(n)$, we need to rewrite the expression in the previous theorem. For $0 \leq k \leq\lfloor n / 2\rfloor-1$ and $D$ a good drawing of $K_{n}$, we define the set of
$\leq k$-edges of $D$ as all $j$-edges in $D$ for $j=0, \ldots, k$. The number of $\leq k$-edges of $D$ is denoted by

$$
E_{\leq k}(D):=\sum_{j=0}^{k} E_{j}(D)
$$

Similarly, we denote the number of $\leq \leq k$-edges of $D$ by

$$
E_{\leq \leq k}(D):=\sum_{j=0}^{k} E_{\leq j}(D)=\sum_{j=0}^{k} \sum_{i=0}^{j} E_{i}(D)=\sum_{i=0}^{k}(k+1-i) E_{i}(D) .
$$

To avoid special cases we define $E_{\leq \leq-1}(D)=E_{\leq \leq-2}(D)=0$.
The following result restates Theorem 1 in terms of the number of $\leq \leq k$-edges. In the next section, we bound $E_{\leq \leq k}(D)$ from below for 2-page book drawings $D$ of $K_{n}$.

Proposition 1 Let D be a good drawing of $K_{n}$. Then

$$
\operatorname{cr}(D)=2 \sum_{k=0}^{\lfloor n / 2\rfloor-2} E_{\leq \leq k}(D)-\frac{1}{2}\binom{n}{2}\left\lfloor\frac{n-2}{2}\right\rfloor-\frac{1}{2}\left(1+(-1)^{n}\right) E_{\leq \leq\lfloor n / 2\rfloor-2}(D) .
$$

Proof First note that for $2 \leq k \leq\lfloor n / 2\rfloor-1$ we have that $E_{\leq \leq k}(D)-E_{\leq \leq k-1}(D)=$ $E_{\leq k}(D)$ and $E_{\leq k}(D)-E_{\leq k-1}(D)=E_{k}(D)$. Thus

$$
E_{k}(D)=E_{\leq \leq k}(D)-2 E_{\leq \leq k-1}(D)+E_{\leq \leq k-2}(D) .
$$

Note that this equation also holds for $k=0$ and $k=1$. We now rewrite the last term in Theorem 1 as follows.

$$
\begin{aligned}
& \sum_{k=0}^{\lfloor n / 2\rfloor-1} k(n-2-k) E_{k}(D) \\
&= \sum_{k=0}^{\lfloor n / 2\rfloor-1} k(n-2-k)\left[E_{\leq \leq k}(D)-2 E_{\leq \leq k-1}(D)+E_{\leq \leq k-2}(D)\right] \\
&= \sum_{k=0}^{\lfloor n / 2\rfloor-3}(k(n-2-k)-2(k+1)(n-3-k)+(k+2)(n-4-k)) E_{\leq \leq k}(D) \\
&+\left(\left\lfloor\frac{n}{2}\right\rfloor-1\right)\left(n-1-\left\lfloor\frac{n}{2}\right\rfloor\right) E_{\leq \leq\lfloor n / 2\rfloor-1}(D)+\left(-2\left(\left\lfloor\frac{n}{2}\right\rfloor-1\right)\left(n-1-\left\lfloor\frac{n}{2}\right\rfloor\right)\right. \\
&\left.+\left(\left\lfloor\frac{n}{2}\right\rfloor-2\right)\left(n-\left\lfloor\frac{n}{2}\right\rfloor\right)\right) E_{\leq \leq\lfloor n / 2\rfloor-2}(D) \\
&=-2 \sum_{k=0}^{\lfloor n / 2\rfloor-3} E_{\leq \leq k}(D)+\left(\left\lfloor\frac{n}{2}\right\rfloor-1\right)\left(n-1-\left\lfloor\frac{n}{2}\right\rfloor\right) E_{\leq \leq \leq n / 2\rfloor-1}(D) \\
& \quad+\left(-2\left(\left\lfloor\frac{n}{2}\right\rfloor-1\right)\left(n-1-\left\lfloor\frac{n}{2}\right\rfloor\right)+\left(\left\lfloor\frac{n}{2}\right\rfloor-2\right)\left(n-\left\lfloor\frac{n}{2}\right\rfloor\right)\right) E_{\leq \leq \leq n / 2\rfloor-2}(D) .
\end{aligned}
$$

Since $E_{\leq \leq\lfloor n / 2\rfloor-1}(D)=E_{\leq \leq\lfloor n / 2\rfloor-2}(D)+E_{\leq\lfloor n / 2\rfloor-1}(D)=E_{\leq \leq\lfloor n / 2\rfloor-2}(D)+\binom{n}{2}$, it follows by Theorem 1 that

$$
\begin{aligned}
\operatorname{cr}(D)= & 3\binom{n}{4}-\sum_{k=0}^{\lfloor n / 2\rfloor-1} k(n-2-k) E_{k}(D)=3\binom{n}{4}+2 \sum_{k=0}^{\lfloor n / 2\rfloor-3} E_{\leq \leq k}(D) \\
& +\left(n+1-2\left\lfloor\frac{n}{2}\right\rfloor\right) E_{\leq \leq\lfloor n / 2\rfloor-2}(D)-\left(\left\lfloor\frac{n}{2}\right\rfloor-1\right)\left(n-1-\left\lfloor\frac{n}{2}\right\rfloor\right)\binom{n}{2} \\
= & 2 \sum_{k=0}^{\lfloor n / 2\rfloor-3} E_{\leq \leq k}(D)-\frac{1}{2}\binom{n}{2}\left\lfloor\frac{n-2}{2}\right\rfloor+\left\{\begin{array}{l}
E_{\leq \leq\lfloor n / 2\rfloor-2}(D) \text { if } n \text { is even, } \\
2 E_{\leq \leq\lfloor n / 2\rfloor-2}(D) \text { if } n \text { odd, },
\end{array}\right.
\end{aligned}
$$

which is equivalent to the claimed result.

## 3 The 2-Page Crossing Number

We are concerned with 2-page book drawings of $K_{n}$. Obviously any line can be chosen as the spine, and for the rest of the paper we will assume that the spine is the $x$-axis. Moreover, using a suitable homeomorphism of the plane, we will assume that the vertices are precisely the points with coordinates $(1,0),(2,0), \ldots,(n, 0)$. Furthermore, because each edge is completely contained in a page (the upper or lower half plane), the crossings cannot happen on the spine. This means that the local redrawing rules used to transform any drawing of a graph into a good drawing without increasing the number of crossings preserve the property of being a 2-page book drawing. Therefore, we only consider good 2-page book drawings of $K_{n}$.

Consider a good 2-page book drawing $D$ of $K_{n}$, and label the vertices $1,2, \ldots, n$ from left to right. Because $D$ is a good drawing, it is readily seen that none of the edges $12,23, \ldots,(n-1) n, n 1$ is crossed. Thus we may choose to place each of these edges in either the upper closed halfplane (page) or in the lower closed halfplane (page). Moreover, we may choose to place each of the edges $12,23, \ldots,(n-1) n$ completely on the spine, and this is the convention we shall adopt for the rest of the paper. The edge $n 1$ may be placed indistinctly in the upper page or in the lower page, and for the rest of the paper we adopt the convention that it is placed in the upper page. Furthermore, because we are only concerned with good drawings, we assume without loss of generality that the rest of the edges are semicircles.

Color the edges above or on the spine blue and below the spine red, respectively. We construct an upper triangular matrix which corresponds to the coloring of these edges, see Fig. 2. We call this the 2-page matrix of $D$ and denote it by $M(D)$. Label the columns of the 2-page matrix with $2, \ldots, n$ from left to right and the rows with $1,2, \ldots, n-1$ from top to bottom. For $i<j$ an entry $(i, j)$ (row,column) in the 2-page matrix $M(D)$ is a point with the same color as the edge $i j$ in the drawing $D$.

Remark 1 It follows from the convention laid out above that for every 2-page book drawing $D$, the entries $(1,2),(2,3), \ldots,(n-1, n)$ and $(1, n)$ in $M(D)$ are all blue.

We start by proving some basic properties of the 2-page matrix.


Fig. 2 Two-colored diagram for a 2-page book drawing $D$ of $K_{8}$ and the corresponding 2-page matrix $M(D)$. Solid dots and lines represent blue edges. Open dots and dashed lines represent red edges. From our convention to place the edges $12,23, \ldots,(n-1) n$ on the spine and the edge $1 n$ in the upper page, it follows that all the entries in the main diagonal, as well as the upper right corner entry, are blue (Color figure online)

Lemma 2 Let $D$ be a 2-page book drawing of $K_{n}$ and $1 \leq i<j \leq n$. Let $k$ denote the number of points in $M(D)$ with the same color as entry $(i, j)$ that are located above or to the right of entry ( $i, j$ ). Then the edge $i j$ is a $k$-edge. (It is possible to have $k>\lfloor n / 2\rfloor-1$.)

Proof Let $1 \leq i<j \leq n$ and assume that the edge $i j$ is blue (red). We prove that the number of points $l$ in $D$ to the left (right) of $i j$ is exactly $k$. For $l \notin\{i, j\}$ the triangle $i j l$ is oriented counter-clockwise (clockwise) if and only if either $l<i$ and the edge $l j$ is blue (red), or $l>j$ and the edge $i l$ is blue (red). In the first case these edges correspond to blue (red) points above the entry $(i, j)$, and in the second case to blue (red) points to the right of the entry $(i, j)$, respectively.

In view of Lemma 2 we say that the point in the entry $(i, j)$ of the 2-page matrix of $D$ represents a $k$-edge if $i j$ is a $k$-edge (or an $(n-2-k)$-edge) in $D$.

Lemma 3 Fork $<n / 2-1$ andfor $1 \leq j \leq k+1$, in the 2-page matrix of a drawing $D$ of $K_{n}$ there are at least $2(k+2-j)$ points in row $j$ representing $\leq k$-edges. Similarly, for $n-k \leq j \leq n$ there are at least $2(k+1-n+j)$ points in column $j$ representing $\leq k$-edges.

Proof For $1 \leq j \leq k+1$, in row $j$ the rightmost $k+2-j$ points of each color represent $\leq k$-edges as they have at most $k+1-j$ points of their color to the right and at most $j-1$ on top. So if each color appears at least $k+2-j$ times in row $j$, we have guaranteed $2(k+2-j) \leq k$-edges in row $j$. If one of the colors appears fewer than $k+2-j$ times, so that there are $k+2-j-e$ blue points in row $j$ for some $1 \leq e \leq k+2-j$, then there are $n-j-(k+2-j-e)=n-2-k+e$ red points in this row. In this case we claim that also the leftmost $e$ red points in this row represent $\leq k$-edges. In fact, for $1 \leq i \leq e$, the $i$ th red point (from the left) in row $j$, has exactly $n-2-k+e-i$ red points to the right and perhaps more red points above. Since $e \geq i$ implies $n-2-k+e-i \geq n-2-k$, this $i$ th red point also represents $\mathrm{a} \leq k$-edge. The equivalent result holds for the rightmost $k+1$ columns.

Lemma 4 For $0 \leq j<n / 2-1$, in the 2-page matrix of a drawing $D$ of $K_{n}$ there are two points in column $n$ which correspond to $j$-edges in $D$. For $n$ even there exists one point in column $n$ corresponding to an $(n / 2-1)$-edge in $D$.

Proof We follow the lines of the proof of Lemma 3. Consider the points in column $n$ in order from top to bottom. By Lemma 2 the $i$ th vertex of a color corresponds to an $(i-1)$-edge. Thus, if there are at least $j+1$ vertices for each color we are done. Otherwise assume without loss of generality that there are $j+1-e$ blue points in column $n$ for some $1 \leq e \leq j+1$. Then there are $n-1-(j+1-e)=n-j+e-2$ red points in this column. For $1 \leq i \leq\lfloor n / 2\rfloor$ the $i$ th red point corresponds to an ( $i-1$ )-edge, and for $\lfloor n / 2\rfloor+1 \leq i \leq n-j+e-2$ the $i$ th red point corresponds to an $(i-1)=(n-i-1)$-edge. Thus we get two red points corresponding to $j$-edges for $i=j+1$ and $i=n-j-1$. Finally, observe that these two points are different for $j<n / 2-1$. For $n$ even we get only one such point for $j=n / 2-1$.

The next theorem gives a lower bound on the number of $\leq \leq k$-edges, which will play a central role in deriving our main result. We need the following definitions. Let $D$ be a good drawing of $K_{n}$. Let $l$ be a vertex of $K_{n}$, and let $D^{\prime}$ be the (evidently, also good) drawing of $K_{n-1}$ obtained by deleting from $D$ the vertex $l$ and its adjacent edges. Note that a $k$-edge $i j$ in $D^{\prime}$ is a $k$-edge or a $(k+1)$-edge in $D$. Indeed, if $i j$ has exactly $k$ points to its right in $D^{\prime}$ (an equivalent argument holds if the $k$ points are on its left), then there are $k$ or $k+1$ points to the right of $i j$ in $D$ depending on whether $l$ is to the left or to the right, respectively, of $i j$. We say that a $k$-edge in $D$ is ( $D, D^{\prime}$ )-invariant if it is also a $k$-edge in $D^{\prime}$. Whenever it is clear what $D$ and $D^{\prime}$ are, we simply say that an edge is invariant. A $\left(D, D^{\prime}\right)$-invariant $\leq k$-edge is a ( $D, D^{\prime}$ )-invariant $j$-edge for some $0 \leq j \leq k \leq n / 2-1$. Denote by $E_{\leq k}\left(D, D^{\prime}\right)$ the number of ( $D, D^{\prime}$ )-invariant $\leq k$-edges.

Theorem 2 Let $n \geq 3$. For every 2-page book drawing $D$ of $K_{n}$ and $0 \leq k<n / 2-1$, we have

$$
\begin{equation*}
E_{\leq \leq k}(D) \geq 3\binom{k+3}{3} \tag{6}
\end{equation*}
$$

Proof We proceed by induction on $n$. The induction base $n=3$ holds trivially. For $n \geq 4$, consider a 2-page book drawing $D$ of $K_{n}$ with horizontal spine and label the vertices from left to right with $1,2, \ldots, n$. Remove the point $n$ and all incident edges to obtain a 2-page book drawing $D^{\prime}$ of $K_{n-1}$. To bound $E_{\leq \leq k}(D)$, recall that

$$
\begin{equation*}
E_{\leq \leq k}(D)=\sum_{j=0}^{k}(k+1-j) E_{j}(D) \tag{7}
\end{equation*}
$$

All edges incident to $n$ are in $D$ but are not in $D^{\prime}$. In fact, by Lemma 4, there are two $j$-edges adjacent to the vertex $n$ for each $0 \leq j \leq k \leq\lfloor n / 2\rfloor-2$. These edges contribute with $2 \sum_{j=0}^{k}(k+1-j)=2\binom{k+2}{2}$ to Eq. (7). We next compare Eq. (7) to

$$
\begin{equation*}
E_{\leq \leq k-1}\left(D^{\prime}\right)=\sum_{j=0}^{k-1}(k-j) E_{j}\left(D^{\prime}\right) \tag{8}
\end{equation*}
$$

Any edge contributing to Eq. (8) also contributes to Eq. (7), but possibly with a different value. As observed before, a $j$-edge in $D^{\prime}$ is a $j$-edge or a $(j+1)$-edge in $D$. A $j$-edge in $D^{\prime}$ contributes to Eq. (8) with $k-j$. A $j$-edge and a $(j+1)$-edge in $D$ contribute to Eq. (7) with $k+1-j$ and $k-j$, respectively. This is a gain of +1 or 0 , respectively, towards $E_{\leq \leq k}(D)$ when compared to $E_{\leq \leq k-1}\left(D^{\prime}\right)$. Finally, a $k$-edge in both $D$ and $D^{\prime}$ does not contribute to Eq. (8) and contributes to Eq. (7) with +1 . Therefore

$$
E_{\leq \leq k}(D)=E_{\leq \leq k-1}\left(D^{\prime}\right)+2\binom{k+2}{2}+E_{\leq k}\left(D, D^{\prime}\right) .
$$

By induction hypothesis, $E_{\leq \leq k-1}\left(D^{\prime}\right) \geq 3\binom{k+2}{3}$ and thus

$$
\begin{aligned}
E_{\leq \leq k}(D) & \geq 3\binom{k+2}{3}+2\binom{k+2}{2}+E_{\leq k}\left(D, D^{\prime}\right) \\
& =3\binom{k+3}{3}-\binom{k+2}{2}+E_{\leq k}\left(D, D^{\prime}\right) .
\end{aligned}
$$

We finally prove that

$$
\begin{equation*}
E_{\leq k}\left(D, D^{\prime}\right) \geq\binom{ k+2}{2} \tag{9}
\end{equation*}
$$

In fact, we prove that for each $1 \leq j \leq k+1$ there are at least $k+2-j$ points in row $j$ of $M(D)$ that represent $\left(D, D^{\prime}\right)$-invariant $\leq k$-edges. Suppose that the edge $j n$ is blue (the equivalent argument holds when $j n$ is red). Then any red point in row $j$ with $i \leq k$ red points above or to its right in $M(D)$ represents a ( $D, D^{\prime}$ )-invariant $i$-edge; and any blue point in row $j$ with $i \geq n-2-k$ blue points above or to its right represents a $\left(D, D^{\prime}\right)$-invariant $(n-2-i)$-edge. Thus, the first $k+2-j$ red points from the right in row $j$ (if they exist) represent ( $D, D^{\prime}$ )-invariant $\leq k$-edges as they have at most $k+2-j-1$ red points to the right and at most $j-1$ red points above in both $M(D)$ and $M\left(D^{\prime}\right)$. If there are fewer than $k+2-j$ red points in row $j$ of $M(D)$, say $k+2-j-e$ for some $1 \leq e \leq k+2-j$, then the first $e$ blue points in row $j$ of $M(D)$ from the left represent $\leq k$-edges, because they have at least $n-j-e \geq n-j-k-2+j=n-k-2$ blue points to their right. Hence there are at least $k+2-j-e$ red points and at least $e$ blue points (for a total of at least $k+2-j$ points) that represent ( $D, D^{\prime}$ )-invariant $\leq k$-edges in row $j$ of $M(D)$. Summing over all $1 \leq j \leq k+1$, we get that

$$
E_{\leq k}\left(D, D^{\prime}\right) \geq \sum_{j=1}^{k+1}(k+2-j)=\binom{k+2}{2}
$$

We now summarize the conditions that guarantee that equality is achieved in Theorem 2. This remark is used in the next section to understand the structure of the crossing optimal drawings.

Remark 2 Let $D$ be a 2-page book drawing of $K_{n}$ and $0 \leq k<n / 2-1$. Moreover $D^{\prime}$ is defined as in the proof of Theorem 2. Then $E_{\leq \leq k}(D)=3\binom{k+3}{3}$ if and only if

1. $E_{\leq \leq k-1}\left(D^{\prime}\right)=3\binom{k+2}{3}$ and
2. $E_{\leq k}\left(D, D^{\prime}\right)=\binom{k+2}{2}$, which is equivalent to simultaneously satisfying that
(a) For each $1 \leq j \leq k+1$ there are exactly $k+2-j$ entries in row $j$ of $M(D)$ that represent ( $D, D^{\prime}$ )-invariant $\leq k$-edges and
(b) For each $k+2 \leq j \leq n-1$ there are no entries in row $j$ of $M(D)$ that represent ( $D, D^{\prime}$ )-invariant $\leq k$-edges.

We are now ready to prove our main result, namely that the 2-page crossing number of $K_{n}$ is $Z(n)$.

Theorem 3 For every positive integer $n, \nu_{2}\left(K_{n}\right)=Z(n)$.
Proof The cases $n=1$ and $n=2$ are trivial. Let $n \geq 3$. As we mentioned above, 2-page book drawings with $Z(n)$ crossings were constructed by Blažek and Koman [8] (see also Guy et al. [18], Damiani et al. [13], Harborth [20], and Shahrokhi et al. [23].) These drawings show that $\nu_{2}\left(K_{n}\right) \leq Z(n)$. For the lower bound, let $D$ be a 2-page book drawing of $K_{n}$. Using Proposition 1 and Theorem 2, we obtain

$$
\begin{aligned}
\operatorname{cr}(D) & \geq 2 \sum_{k=0}^{\lfloor n / 2\rfloor-2} 3\binom{k+3}{3}-\frac{1}{2}\binom{n}{2}\left\lfloor\frac{n-2}{2}\right\rfloor-\frac{3}{2}\left(1+(-1)^{n}\right)\binom{\left\lfloor\frac{n}{2}\right\rfloor+1}{3} \\
& =6\binom{\left\lfloor\frac{n}{2}\right\rfloor+2}{4}-\frac{1}{2}\binom{n}{2}\left\lfloor\frac{n-2}{2}\right\rfloor-\frac{3}{2}\left(1+(-1)^{n}\right)\binom{\left\lfloor\frac{n}{2}\right\rfloor+1}{3} \\
& =\left\{\begin{array}{c}
\frac{1}{64}(n-1)^{2}(n-3)^{2} \text { if } n \text { odd, } \\
\frac{1}{64} n(n-2)^{2}(n-4) \text { if } n \text { is even, } \\
\end{array}=Z(n) .\right.
\end{aligned}
$$

## 4 Crossing Optimal Configurations

In all this section, $D$ denotes a good 2-page book drawing of $K_{n}$ (satisfying the conditions laid out at the beginning of Sect. 3) and $M(D)$ its 2-page matrix. We say that $D$ is crossing optimal if $\nu_{2}(D)=Z(n)$. Theorem 4 in Sect. 4.3 describes the general structure of the crossing optimal 2-page book drawings of $K_{n}$. We use it to prove that, up to sphere homeomorphism, there is a unique crossing optimal 2-page book drawing of $K_{n}$ when $n$ is even and, in contrast, there exists an exponential number of non sphere-homeomorphic crossing optimal 2-page book drawings of $K_{n}$ when $n$ is odd.


Fig. 3 A 2-page drawing of $K_{8}$ and its strip diagram (Color figure online)

### 4.1 Equivalent Drawings

Let $D$ be a 2-page book drawing of $K_{n}$. Recall that we are assuming that the vertices of $D$ are the points $\{(i, 0): 1 \leq i \leq n\}$. Consider the following transformation $f$ that results in the 2-page book drawing $f(D)$ of $K_{n}$ : move the vertex $(1,0)$ to the point $(n, 0)$, and for every $2 \leq k \leq n$ move the vertex $(k, 0)$ to the vertex $(k-1,0)$. That is, if an edge $1 j$ was drawn above (below) the spine in $D$, then the edge $(j-1) n$ is drawn above (below) the spine in $f(D)$; for all other edges $i j$ with $1<i<j \leq n$, if $i j$ was drawn above (below) the spine in $D$, then the edge $(i-1)(j-1)$ is drawn above (below) the spine in $f(D)$. Note that $D$ and $f(D)$ have the same number of crossings, and $f^{n}(D)=D$. There are two other natural crossing-preserving transformations of a drawing $D$ : A vertical reflection $g(D)$ about the line with equation $x=n / 2$ and a horizontal reflection $h(D)$ about the spine (or $x$-axis). In $g(D)$ an edge $i j$ is drawn above (below) the spine if the edge $(n+1-j)(n+1-i)$ is drawn above (below) the spine in $D$. In $h(D)$ an edge $i j$ is drawn above (below) the spine if the edge $i j$ is drawn below (above) the spine in $D$. Note that $g^{2}(D)=h^{2}(D)=D$. Given a 2-page book drawing $D$, any drawing $D^{\prime}$ obtained from $D$ by compositions of these transformations is said to be equivalent to $D$. Indistinctively, we say that the matrices $M(D)$ and $M\left(D^{\prime}\right)$ are equivalent. All drawings obtained this way are sphere-homeomorphic and thus they all have the same number of crossings as $D$. The group spanned by these transformations is isomorphic to the direct sum of the dihedral group $D_{2 n}$ and the group with 2 elements $\mathbb{Z}_{2}$. The set $\{f, g, h\}$ is a set of generators such that $g^{2}=h^{2}=f^{n}=1, g \circ f=f^{-1} \circ g, h \circ f=f \circ h$, and $g \circ h=h \circ g$. Thus the $4 n$ transformations in the group can be parametrized by $h^{a} \circ g^{b} \circ f^{i}$ with $i \in\{0,1, \ldots, n-1\}$ and $a, b \in\{0,1\}$.

Now we describe these transformations in the 2-page matrix diagram of $D$ : To obtain $M(f(D))$ from $M(D)$, we simply rotate 90 degrees clockwise the first row of $M(D)$ and use it as the $n$th column of $M(f(D))$. The diagram $M(g(D))$ is obtained from $M(D)$ by reflecting with respect to the diagonal $\{(i, n+1-i): 1 \leq i \leq\lfloor n / 2\rfloor\}$.


Fig. 4 A 2-page book drawing of $K_{8}$ with four 0-edges (namely 17, 18, 27, and 28) and four 1-edges (namely 1516,38 , and 48 ). This shows that the inequality $E_{\leq k}(D) \geq 3\binom{k+2}{2}$, which holds for every geometric drawing $D$ of $K_{n}$, does not necessarily hold if $D$ is a topological drawing (Color figure online)

Finally, $M(h(D))$ is obtained by switching the color of every point except those that join consecutive vertices on the spine or the point $(1, n)$. We can place $M(D)$ and $M(f(D))$ together so that the part they have in common overlaps. Doing this for $M\left(f^{m}(D)\right)$ for all integers $m$ we obtain a periodic double infinite strip with period $n$ and with a horizontal section that is $n-1$ units wide. We call this the strip diagram of $D$, or of $f^{m}(D)$ for any integer $m$. (See Fig. 3.) Any right triangular region with the same dimensions as $M(D)$ obtained from the strip diagram of $D$ by a horizontal and a vertical cut is the matrix diagram of a drawing equivalent to $D$ and thus it has the same number of crossings as $D$.

### 4.2 Properties of Crossing Optimal Drawings

In contrast to the rectilinear case, the inequality $E_{\leq k}(D) \geq 3\binom{k+2}{2}$ does not hold in general for topological drawings $D$ of $K_{n}$, not even for general 2-page book drawings, as can be seen in Fig. 4. However, the inequality $E_{\leq k}(D) \geq 3\binom{k+2}{2}$ does hold for crossing optimal drawings of $K_{n}$, where in fact the following stronger result is true.

Proposition 2 Let $D$ be a 2-page book drawing of $K_{n}$ and $I_{n}=\{k \in \mathbb{Z}: 0 \leq k \leq$ $\lfloor n / 2\rfloor-2\}$. The following are equivalent: (i) $\operatorname{cr}(D)=Z(n)$, (ii) $E_{k}(D)=3(k+1)$ for all $k \in I_{n}$, (iii) $E_{\leq k}(D)=3\binom{k+2}{2}$ for all $k \in I_{n}$, and (iv) $E_{\leq \leq k}(D)=3\binom{k+3}{3}$ for all $k \in I_{n}$.

Proof Parts (i) and (iv) are equivalent as equality is achieved in Theorem 3 if and only if equality is achieved in Theorem 2 for all $k \in I_{n}$. The implications (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) follow directly from the definitions of $E_{\leq k}(D)$ and $E_{\leq \leq k}(D)$, using the identity $\sum_{m=0}^{r}\binom{m}{s}=\binom{r+1}{s+1}$. It remains to show that (iv) implies (ii), which we do by applying induction on $k$. For the induction base note that $E_{\leq \leq 0}(D)=E_{\leq 0}(D)=E_{0}(D)=3$. For $1 \leq k \leq\lfloor n / 2\rfloor-2$, the identities $E_{j}(D)=3(j+1)$ for all $0 \leq j \leq k-1$ and $E_{\leq \leq k}(D)=3\binom{k+3}{3}$ imply that

$$
\begin{aligned}
3\binom{k+3}{3} & =E_{\leq \leq k}(D)=\sum_{j=0}^{k}(k+1-j) E_{j}(D) \\
& =E_{k}(D)+3 \sum_{j=0}^{k-1}(k+1-j)(j+1)
\end{aligned}
$$

and thus

$$
\begin{aligned}
E_{k}(D) & =3\binom{k+3}{3}-3 \sum_{j=0}^{k-1}(k+1-j)(j+1) \\
& =3\binom{k+3}{3}-\frac{1}{2} k(k+1)(k+5)=3(k+1)
\end{aligned}
$$

We now give a more detailed analysis on the crossing optimal 2-page book drawings of $K_{n}$. We start with a couple of definitions. Consider the entry $(i, j)$ of $M(D)$. We order the entries in row $i$ to the left of $(i, j)$ as follows: first all entries, from right to left, whose color differs to that of $(i, j)$, followed by all other entries (those with the same color as $(i, j))$ from left to right. This is called the order associated to $(i, j)$. Observe that this is the order in which the edges $i l(i<l<j)$ appear in the 2-page drawing, ordered bottom to top if the edge $i j$ is blue and top to bottom if the edge $i j$ is red. Let $c$ be an integer such that $0 \leq c \leq n-1$. Denote by $D_{c}$ the subgraph of $D$ obtained by deleting the $c$ right-most vertices of $D$, or equivalently, $M\left(D_{c}\right)$ is obtained by deleting the last $c$ columns of $M(D)$.

Lemma 5 Suppose that $l \geq i+m+1$ for some integers $1 \leq i<l<j \leq n$ and $1 \leq m<j-i$. The entry $(i, l)$ is one of the first $m$ entries in the order associated to $(i, j)$ if and only if $(i, l)$ and $(i, j)$ have different colors.

Proof Note that if $(i, l)$ and $(i, j)$ have the same color, then all entries to the left of $(i, l)$ come before $(i, l)$ in the order associated to $(i, j)$.

Lemma 6 Let p be an integer, $0 \leq p \leq\lfloor n / 2\rfloor-2$. Suppose that $E_{\leq k}\left(D, D_{1}\right)=\binom{k+2}{2}$ for all $0 \leq k \leq p$. Then $M(D)$ satisfies that for $1 \leq i \leq p+1$ in row $i$ there is exactly one $\left(D, D_{1}\right)$-invariant $k$-edge for each $i-1 \leq k \leq p$, and there are no $\left(D, D_{1}\right)$ invariant $(\leq i-2)$-edges. In all other rows there are no $\left(D, D_{1}\right)$-invariant $\leq p$-edges.

Proof In what follows all invariant edges are ( $D, D_{1}$ )-invariant edges. We prove by induction that for each $0 \leq k \leq p$, row $i$ of $M(D)$ contains exactly one invariant $k$-edge for $1 \leq i \leq k+1$ and no invariant $k$-edge for $k+2 \leq i \leq n-1$. For $k=0$ there is a unique invariant 0 -edge and it appears in row 1 . This edge corresponds to the first entry in the order associated to $(1, n)$ in $M(D)$. For $k>0$, we use Remark 2, $E_{\leq k}\left(D, D_{1}\right)=\binom{k+2}{2}$, and $E_{\leq k-1}\left(D, D_{1}\right)=\binom{k+1}{2}$. For $k+2 \leq i \leq n-1$, Remark 2.2 b (for $k$ ) implies that there are no invariant $k$-edges in row $i$. For $1 \leq i \leq k$, Remark 2.2a (first for $k$ and then for $k-1$ ), implies that there is exactly $(k+2-i)-(k+1-i)=1$ invariant $k$-edge in row $i$ (invariant $\leq k$-edges that are not $(\leq k-1)$-edges). Finally, for $i=k+1$ Remarks 2.2 a and 2.2 b (for $k$ ) imply that there is exactly $k+2-(k+1)=1$
invariant $\leq k$-edge and no invariant ( $\leq k-1$ )-edge in row $i$. Therefore, there is exactly one invariant $k$-edge in row $k+1$.

Lemma 7 Let $p$ be an integer such that $0 \leq p \leq\lfloor n / 2\rfloor-2$.
(i) Suppose that for some $1 \leq i \leq p+1$ row $i$ of $M(D)$ has exactly one $\left(D, D_{1}\right)$ invariant $k$-edgefor each $i-1 \leq k \leq p$ andno $\left(D, D_{1}\right)$-invariant $\leq(i-2)$-edges. If the entry $(i, n)$ in $M(D)$ is blue (red), then the mth entry in row $i$ in the order associated to $(i, n)$ has at least $\min \{p+2-m, i-1\}$ red (blue) entries above for every $1 \leq m \leq \min \{p+1, n-i-1\}$.
(ii) Suppose that for some $i \geq p+2$ row $i$ of $M(D)$ does not have $\left(D, D_{1}\right)$-invariant $\leq p$-edges. If the entry $(i, n)$ in $M(D)$ is blue (red), then the mth entry in row $i$ in the order associated to $(i, n)$ has at least $p+2-m$ red (blue) entries above for every $1 \leq m \leq \min \{p+1, n-i-1\}$.

Proof In what follows invariant edges refer to ( $D, D_{1}$ )-invariant edges. Denote by $\left(i, e_{m}\right)$ the $m$ th entry in the order associated to $(i, n)$. Note that if $\left(i, e_{m}\right)$ and $(i, n)$ have opposite colors and the number of points above plus the number of points to the right of $\left(i, e_{m}\right)$ with the same color as $\left(i, e_{m}\right)$ is at most $p$, then $\left(i, e_{m}\right)$ is an invariant $\leq p$-edge. Similarly, if $\left(i, e_{m}\right)$ and $(i, n)$ have the same color and the number of points above plus the number of points to the right of $\left(i, e_{m}\right)$ with the same color as $\left(i, e_{m}\right)$ is more than $n-2-p$, then $\left(i, e_{m}\right)$ is an invariant $\leq p$-edge.

Suppose that the entry $(i, n)$ of $M(D)$ is blue (red).
(i) If $\left(i, e_{1}\right)$ is red (blue), then it does not have red entries to its right and it has at most $i-1$ red (blue) entries above. Since $i-1 \leq p$, then $\left(i, e_{1}\right)$ is an invariant $(\leq i-1)$ edge. Because there are no invariant ( $\leq i-2$ )-edges in row $i$, it follows that $\left(i, e_{1}\right)$ is the unique invariant $(i-1)$-edge in row $i$ and thus all $i-1$ entries above it are red (blue). Similarly, if the ( $i, e_{1}$ ) is blue (red), then all entries in row $i$ are blue (red) and $\left(i, e_{1}\right)=(i, i+1)$. Hence ( $i, e_{1}$ ) has $n-i-1$ blue (red) entries to its right and perhaps some other blue (red) entries above. Since $n-i-1 \geq n-(p+1)-1 \geq n-2-p$, then $\left(i, e_{1}\right)$ is an invariant $(\leq i-1)$-edge. Because there are no invariant $(\leq i-2)$-edges in row $i$, it follows that $\left(i, e_{1}\right)$ is the unique invariant $(i-1)$-edge in row $i$ and thus all $i-1$ entries above it are red (blue).

For $2 \leq m \leq p+2-i$ assume that the entry $\left(i, e_{m^{\prime}}\right)$ is an invariant $\left(i-2+m^{\prime}\right)$-edge for every $1 \leq m^{\prime} \leq m-1$. Note that $i-1 \leq i-2+m^{\prime} \leq p-1$.

If $\left(i, e_{m}\right)$ is red (blue), then $\left(i, e_{m^{\prime}}\right)$ is red (blue) for every $1 \leq m^{\prime} \leq m-1$. So ( $i, e_{m}$ ) has exactly $m-1$ red (blue) entries to its right and at most $i-1$ red (blue) entries above, that is, $\left(i, e_{m}\right)$ is an invariant $\leq(i-2+m)$-edge. By hypothesis there is a unique invariant $k$-edge for every $i-1 \leq k \leq p$ and among the first $m-1$ entries there is exactly one invariant $k$-edge for each $i-1 \leq k \leq i-2+(m-1)=i-3+m$. So $\left(i, e_{m}\right)$ is the unique invariant $(i-2+m)$-edge (note that $1 \leq i-2+m \leq p$ ) and thus all the entries above it are red (blue).

If ( $i, e_{m}$ ) is blue (red), then there are exactly $n-i+m$ blue (red) entries to its right and perhaps some others above it. Since $n-i+m \geq n-i-(p+2-i)=n-p+2$, then $\left(i, e_{m}\right)$ is an invariant $\leq(i-2+m)$-edge. As before ( $i, e_{m}$ ) must be an invariant ( $i-2+m$ )-edge and thus it must have only red (blue) entries above.

We have already determined the unique invariant $k$-edge for each $1 \leq k \leq p$. So there are no more invariant $\leq p$-edges in row $i$. For $p+3-i \leq m \leq \min \{p+1$, $n-i-1\}$, we prove that the entry $\left(i, e_{m}\right)$ has at least $p+2-m=\min \{p+2-m, i-1\}$ red (blue) entries above.

If $\left(i, e_{m}\right)$ is red (blue), then it has $m-1$ red (blue) entries to its right. If $\left(i, e_{m}\right)$ had less than $p+2-m$ (note that $p+2-m \leq i-1$ ) red (blue) entries above, then it would be an invariant $\leq p$-edge (because $(m-1)+(p+1-m)=p$ ) getting a contradiction.

If $\left(i, e_{m}\right)$ is blue (red), then it has $n-i-m$ blue (red) entries to its right. If ( $i, e_{m}$ ) had less than $p+2-m$ red (blue) entries above, then it would have a total of at least $n-i-m+(i-1)-(p+1-m)=n-2-p$ blue (red) entries above or to its right, and thus it would be an invariant $\leq p$-edge getting a contradiction.
(ii) The proof is the same as for the case $p+3-i \leq m \leq \min \{p+1, n-i-1\}$ in (i) as we only used that the $m$ th entry in that range was not an invariant $\leq p$-edge.

Lemma 8 If $D$ is crossing optimal, then for $0 \leq j \leq\lfloor n / 2\rfloor-2$ we have

$$
E_{\leq \leq k}\left(D_{j}\right)=3\binom{k+3}{3} \text { for all } 0 \leq k \leq\lfloor n / 2\rfloor-2-j .
$$

Proof Since $D$ is crossing optimal, equality must be achieved in the proof of Theorem 3, that is, $E_{\leq \leq k}(D)=3\binom{k+3}{3}$ for all $0 \leq k \leq\lfloor n / 2\rfloor-2$. By Remark 2.1 we have that $E_{\leq \leq k-1}\left(D_{1}\right)=3\binom{k+2}{3}$ for all $0 \leq k \leq\lfloor n / 2\rfloor-2$. In other words, $E_{\leq \leq k}\left(D_{1}\right)=3\binom{k+3}{3}$ for all $0 \leq k \leq\lfloor n / 2\rfloor-3$.

In general, for $0 \leq j \leq\lfloor n / 2\rfloor-2, E_{\leq \leq k}\left(D_{j}\right)=3\binom{k+3}{3}$ for all $1 \leq k \leq$ $\lfloor n / 2\rfloor-2-j$ implies that $E_{\leq \leq k-1}\left(D_{j+1}\right)=3\binom{k+2}{3}$ for all $1 \leq k \leq\lfloor n / 2\rfloor-2-j$ by Remark 2.1. In other words, $E_{\leq \leq k}\left(D_{j+1}\right)=3\binom{k+3}{3}$ for $1 \leq k \leq\lfloor n / 2\rfloor-3-j$.

Lemma 9 If $D$ is crossing optimal, then in $M(D)$ the $m$ th entry in the order associated to $(i, j)$ has at least $\min \{j-\lceil n / 2\rceil-m, i-1\}$ entries above with different color than $(i, j)$ for all $1 \leq m \leq \min \{j-\lfloor n / 2\rfloor-1, j-i-1\}$.

Proof Consider the entry $(i, j)$ of $M(D)$. Because $D$ is crossing optimal, it follows from Lemma 8 that
$E_{\leq \leq k}\left(D_{n-j}\right)=3\binom{k+3}{3}$ for all $0 \leq k \leq\lfloor n / 2\rfloor-2-(n-j)=j-2-\lceil n / 2\rceil$.
Consider row $i$ of $D_{n-j}$. (Note that $D_{n-j}$ has $j-1$ rows.) If $1 \leq i \leq j-1-\lceil n / 2\rceil$, then by Lemma 6 for $p=j-2-\lceil n / 2\rceil$, the 2-page matrix $M\left(D_{n-j}\right)$ satisfies that in row $i$ there is exactly one ( $D_{n-j}, D_{n-j+1}$ )-invariant $k$-edge for each $i-1 \leq k \leq$ $j-2-\lceil n / 2\rceil$ and there are no $\left(D_{n-j}, D_{n-j+1}\right)$-invariant $(\leq j-2-\lceil n / 2\rceil)$-edges. Then by Lemma 7(i) if the entry $(i, j)$ in $M(D)$ (actually in $M\left(D_{n-j}\right)$ but we look at it as a submatrix of $M(D))$ is blue (red), then the $m$ th entry in the order associated to $(i, j)$ has at least $\min \{j-\lceil n / 2\rceil-m, i-1\}$ red (blue) entries above.

If $j-\lceil n / 2\rceil \leq i \leq j-1$, then by Lemma 6 for $p=j-2-\lceil n / 2\rceil$, the 2-page matrix $M\left(D_{n-j}\right)$ satisfies that in row $i$ there are no ( $\left.D_{n-j}, D_{n-j+1}\right)$-invariant
( $\leq j-2-\lceil n / 2\rceil$ )-edges. Then by Lemma 7(ii) if the entry $(i, j)$ in $M(D)$ is blue (red), then the $m$ th entry in the order associated to $(i, j)$ has at least $j-\lceil n / 2\rceil-m=$ $\min \{j-\lceil n / 2\rceil-m, i-1\}$ red (blue) entries above.

Corollary 1 If $D$ is crossing optimal, then for $2 \leq i \leq\lceil n / 2\rceil$ and $\lceil n / 2\rceil+2 \leq j \leq n$, each of the first $j-\lceil n / 2\rceil-1$ entries in the order associated to $(i, j)$ has at least one entry above with different color than ( $i, j$ ).

Proof Let $1 \leq m \leq j-\lceil n / 2\rceil-1$. Since $\lfloor n / 2\rfloor$ and $i$ are at most $\lceil n / 2\rceil$, then $m \leq \min \{j-\lfloor n / 2\rfloor-1, j-i-1\}$. Also $m \leq j-\lceil n / 2\rceil-1$ and $i \geq 2$ imply that $\max \{j-\lceil n / 2\rceil-m, i-1\} \geq 1$. Thus by Lemma 9 , the $m$ th entry in row $i$ in the order associated to $(i, j)$ has at least one entry above with different color than $(i, j)$.

Corollary 2 If $D$ is crossing optimal, then for $n \geq 3,2 \leq i \leq\lfloor n / 2\rfloor-1$, and $\lceil n / 2\rceil+i \leq j \leq n$, all entries above the first $j-i+1-\lceil n / 2\rceil$ entries in the order associated to $(i, j)$ have different color than $(i, j)$.

Proof Let $1 \leq m \leq j-i+1-\lceil n / 2\rceil$. Since $i \geq 2$ and $n \geq 3$, then $m \leq \min \{j-$ $\lfloor n / 2\rfloor-1, j-i-1\}$. Also $m \leq j-i+1-\lceil n / 2\rceil$ implies that $\max \{j-\lceil n / 2\rceil-$ $m, i-1\} \geq i-1$. Thus by Lemma 9 , the $m$ th entry in row $i$ in the order associated to $(i, j)$ has at least $i-1$ entries above, (i.e., all entries above it) with different color than $(i, j)$ in $M(D)$.

Lemma 10 Suppose that $D$ is crossing optimal and $0 \leq k \leq\lfloor n / 2\rfloor-2$. Then all $\leq k$-edges of $D$ belong to the union of the first $k+1$ rows and the last $k+1$ columns of $M(D)$.

Proof Suppose by contradiction that the entry $(i, j)$ of $M(D)$ represents a $k$-edge and is not in the first $k+1$ rows $(i \geq k+2)$ or in the last $k+1$ columns $(j \leq n-k-1)$. Since $D$ is crossing optimal, by Remark 2.1 the entry $(i, j)$ is not ( $D, D_{1}$ )-invariant, that is, $i j$ is a $(k-1)$-edge in $D_{1}$. Also, $E_{\leq \leq k-1}\left(D_{1}\right)=3\binom{k+2}{2}$ by Remark 2.1. In general, assume that $(i, j)$ represents a $(k-l)$-edge in $D_{l}$, not in the first $k-l+1$ rows of $M\left(D_{l}\right)$, and that $E_{\leq \leq k-l}\left(D_{l}\right)=3\binom{k-l+3}{2}$. Then, by Remark 2.2 b, we have that $i j$ is a $(k-l-1)$-edge in $D_{l+1}$ and, by Remark 2.1, $E_{\leq \leq k-l-1}\left(D_{l+1}\right)=3\binom{k-l+2}{2}$. In particular, $(i, j)$ is a 0 -edge in $M\left(D_{k}\right)$ that is not in the last column of $M\left(D_{k}\right)$ (column $n-k$ of $M(D)$ ). Since by Lemma 3 there are at least three 0 -edges in the first column and row of $M\left(D_{k}\right)$ and $i \geq 2$, then $E_{\leq \leq 0}\left(D_{k}\right) \geq 4$. But $E_{\leq \leq 0}\left(D_{k}\right)$ must be 3, by Lemma 8, getting a contradiction.

We extend the standard terminology from the geometrical setting, and call a $(\lfloor n / 2\rfloor-1)$-edge a halving edge.

Lemma 11 Suppose that $D$ is crossing optimal, then the entries $(\lfloor n / 2\rfloor,\lceil n / 2\rceil+1)$, $(\lfloor n / 2\rfloor,\lfloor n / 2\rfloor+1)$, and $(\lceil n / 2\rceil,\lceil n / 2\rceil+1)$ of $M(D)$ are halving edges.

Proof This follows from Lemma 10 as all $\leq(\lfloor n / 2\rfloor-2)$-edges of $D$ belong to the union of the first $\lfloor n / 2\rfloor-1$ rows (top to bottom) and the last $\lfloor n / 2\rfloor-1$ columns (left to right) of $D$. The entries $(\lfloor n / 2\rfloor,\lceil n / 2\rceil+1),(\lfloor n / 2\rfloor,\lfloor n / 2\rfloor+1)$, and $(\lceil n / 2\rceil,\lceil n / 2\rceil+1)$ are not in the first $\lfloor n / 2\rfloor-1$ rows or in the last $\lfloor n / 2\rfloor-1$ columns.


Fig. 5 A halving line in a drawing equivalent to $D$ seen in the matrix $M(D)$ (Color figure online)

Lemma 11 guarantees that the entry $(i, i+1)$ in general, and the entry $(i, i+2)$ when $n$ is odd, are halving lines in some drawing equivalent to $D$. The next result states what this means in $D$. We state it only for $1 \leq i \leq\lfloor n / 2\rfloor$ (but it can be stated for $\lceil n / 2\rceil \leq i \leq n$ as well) as it is the only case we explicitly use later in the paper.

Lemma 12 Let $1 \leq i \leq\lfloor n / 2\rfloor$. If $D$ is crossing optimal, then $M(D)$ satisfies that the number of blue entries in

$$
\begin{align*}
& \{(r, i+1): 1 \leq r \leq i-1\} \cup\{(i, c): i+2 \leq c \leq i+\lceil n / 2\rceil\}  \tag{10}\\
& \cup\{(i+1, c): i+\lceil n / 2\rceil+1 \leq c \leq n\}
\end{align*}
$$

is either $\lfloor n / 2\rfloor-1$ or $\lceil n / 2\rceil-1$. If $n$ is odd, then the number of entries in

$$
\begin{align*}
& \{(r, i+2): 1 \leq r \leq i-1\} \cup\{(i, c): i+3 \leq c \leq i+\lceil n / 2\rceil\}  \tag{11}\\
& \cup\{(i+2, c): i+\lceil n / 2\rceil+1 \leq c \leq n\}
\end{align*}
$$

with the same color as the entry $(i, i+2)$ is either $\lfloor n / 2\rfloor-1$ or $\lfloor n / 2\rfloor$.
Proof In the strip diagram of $D$, the entry $(i, i+1)$ of $M(D)$ corresponds to the entry $(\lfloor n / 2\rfloor,\lfloor n / 2\rfloor+1)$ of $M\left(f^{i-\lfloor n / 2\rfloor}(D)\right)$, see Fig. 5 (left). Applying Lemma 11 to $M\left(f^{i-\lfloor n / 2\rfloor}(D)\right)$ and noticing that the entries of $M(D)$ in (10) correspond to the entries above plus the entries below the entry $(\lfloor n / 2\rfloor,\lfloor n / 2\rfloor+1)$ of $M\left(f^{i-\lfloor n / 2\rfloor}(D)\right)$ gives the result. The proof of the second part is similar, see Fig. 5 (right).

Lemma 13 If $D$ is crossing optimal, then there exists a drawing $D^{\prime}$ equivalent to $D$ such that in $M\left(D^{\prime}\right)$ the $\lceil n / 2\rceil$ entries $(1, n),(2, n), \ldots$, and $(\lceil n / 2\rceil, n)$ are blue and the $\lfloor n / 2\rfloor-1$ entries $(1,\lceil n / 2\rceil+1)$, $(1,\lceil n / 2\rceil+2), \ldots,(1, n-1)$ are red.

Proof For each integer $m$, let $e_{m}$ be the largest integer such that the last $e_{m}$ entries in row $\lfloor n / 2\rfloor$ of $M\left(f^{m}(D)\right)$ have the same color. (These entries are $\left(\lfloor n / 2\rfloor, n-e_{m}+1\right)$, $\ldots,(\lfloor n / 2\rfloor, n)$.) Similarly, let $e_{m}^{\prime}$ be the largest integer such that the first $e_{m}^{\prime}$ entries in column $\lceil n / 2\rceil+1$ of $M\left(f^{m}(D)\right)$ have the same color. (These entries are $(1,\lceil n / 2\rceil+1)$, $\ldots,\left(e_{m}^{\prime},\lceil n / 2\rceil+1\right)$.) Let $E=\max \left\{e_{m}, e_{m}^{\prime}: m \in \mathbb{Z}\right\}$. We claim that $E=\lceil n / 2\rceil$. Indeed, suppose that $E \leq\lceil n / 2\rceil-1$ and without loss of generality assume that


Fig. 6 The even and odd cases in Theorem 4. The crosses in the odd case represent points whose color is not fixed (Color figure online)
$E=e_{m_{0}}$ for some integer $m_{0}$. (If $E=e_{m_{0}}^{\prime}$, start with $g(D)$ instead of $D$.) Then entry ( $\lfloor n / 2\rfloor, n-e_{m_{0}}$ ) has a different color than the entries to its right, namely, $\left(\lfloor n / 2\rfloor, n-e_{m_{0}}+1\right), \ldots,(\lfloor n / 2\rfloor, n)$. By Lemma 9 (for $i=\lfloor n / 2\rfloor$ and $\left.j=n\right)$ the entry $\left(\lfloor n / 2\rfloor, n-e_{m_{0}}\right)$ has at least $\min \{n-\lceil n / 2\rceil-1,\lfloor n / 2\rfloor-1\}=\lfloor n / 2\rfloor-1$ entries above with the same color as $\left(\lfloor n / 2\rfloor, n-e_{m_{0}}\right)$. But this means that $e_{m_{0}-1+\lfloor n / 2\rfloor-e_{m_{0}}}^{\prime} \geq$ $e_{m_{0}}+1=E+1$, a contradiction.

Because $E=e_{m_{0}}=\lceil n / 2\rceil$, all entries in row $\lfloor n / 2\rfloor$ of $M\left(f^{m_{0}}(D)\right)$ are blue. By Lemma 11 all entries above the entry $(\lfloor n / 2\rfloor,\lfloor n / 2\rfloor+1)$ in column $\lfloor n / 2\rfloor+1$ of $M\left(f^{m_{0}}(D)\right)$ are red. This implies that $D^{\prime}=f^{m_{0}+\lfloor n / 2\rfloor}(D)$ satisfies the statement.

### 4.3 The Structure of Crossing Optimal Drawings

We are finally ready to investigate the structure of crossing optimal drawings. The next result is the workhorse behind Theorems 5 and 7, the main results in this section. To help comprehension, we refer the reader to Fig. 6.

Theorem 4 Let $n \geq 6, e=0$ for $n$ even and $e=1$ for $n$ odd, and let $D$ be a crossing optimal 2-page book drawing of $K_{n}$. Then there exists a drawing $D^{\prime}$ equivalent to $D$ such that $M\left(D^{\prime}\right)$ satisfies:

1. for $4+e \leq s \leq\lfloor n / 2\rfloor+1$ and $n+2+e \leq s \leq n+\lfloor n / 2\rfloor+1$ the entry $(r, s-r)$ is blue for all $\max \{1, s-n\} \leq r \leq(s-5) / 2$;
2. for $\lceil n / 2\rceil+2+e \leq s \leq n$ and $n+\lceil n / 2\rceil+2+e \leq s \leq 2 n-2-e$ the entry $(r, s-r)$ is red for all $\max \{1, s-n\} \leq r \leq(s-5) / 2$ (except for $(1, n)$, which by convention is blue);
3. for $n$ odd, the entries $(1,\lceil n / 2\rceil+1)$ and $(\lfloor n / 2\rfloor,\lceil n / 2\rceil+1)$ are red, and the entries $(2, n)$ and $(\lceil n / 2\rceil,\lceil n / 2\rceil+2)$ are blue.


Fig. 7 The regions $T_{U}(D), R(D)$, and $T_{L}(D)$ (Color figure online)

## Proof Let

$$
\begin{aligned}
T_{U}(D) & =\{(r, c) \in M(D): 2 \leq c \leq\lceil n / 2\rceil, 1 \leq r \leq c-1\}, \\
R(D) & =\{(r, c) \in M(D):\lceil n / 2\rceil+1 \leq c \leq n, 1 \leq r \leq\lceil n / 2\rceil\}, \text { and } \\
T_{L}(D) & =\{(r, c) \in M(D):\lceil n / 2\rceil+1 \leq c \leq n,\lceil n / 2\rceil+1 \leq r \leq c-1\} .
\end{aligned}
$$

We shall prove the theorem first for those entries that lie on $R(D)$, then for those that lie on $T_{U}(D)$, and finally for those that lie on $T_{L}(D)$.

The entries in $R(D)$
We refer the reader to Fig. 7. By Lemma 13, we can assume that in $M(D)$

$$
\begin{equation*}
\text { the entries }(1, n),(2, n), \ldots,(\lfloor n / 2\rfloor, n) \text { are blue } \tag{12}
\end{equation*}
$$

(in fact ( $\lceil n / 2\rceil, n$ ) can also be assumed to be blue but we do not use this fact) and

$$
\begin{equation*}
\text { the entries }(1,\lceil n / 2\rceil+1), \ldots,(1, n-1) \text { are red. } \tag{13}
\end{equation*}
$$

Moreover, we can assume that

$$
\begin{equation*}
\text { the entry }(2, n-1) \text { is red. } \tag{14}
\end{equation*}
$$

(If it is blue, then $M(h \circ g(D))$ satisfies (12), (13), and (14)).
We now prove that for each $r$ such that $2 \leq r \leq\lfloor n / 2\rfloor$,
the entries $(r,\lceil n / 2\rceil+1),(r,\lceil n / 2\rceil+2), \ldots,(r, 2\lfloor n / 2\rfloor-r+1)$ are red
and
the entries $(r, 2\lceil n / 2\rceil-r+2),(r, 2\lceil n / 2\rceil-r+3), \ldots,(r, n)$ are blue

Observe that if $r=2$ and $n$ is even, then (16) only concerns the entry $(2, n)$, which is blue by (12). (For $r=2$ and $n$ odd, (16) is an empty claim.) Thus we only need to take care of the base case $r=2$ for (15). Since (by (14)) the entry ( $2, n-1$ ) is red, by Corollary 1 the first $\lfloor n / 2\rfloor-2$ entries in the order associated to $(2, n-1)$ have a blue point above. By (13) the only candidates to have blue points above them are the $\lceil n / 2\rceil-2$ entries $(2,3),(2,4), \ldots,(2,\lceil n / 2\rceil)$. (Note that the order associated to the entry $(i, j)$ only applies to entries in row $j$ to the left of entry $(i, j)$.) Thus the $\lceil n / 2\rceil-2$ entries $(1,3),(1,4), \ldots,(1,\lceil n / 2\rceil)$ are blue if $n$ is even, and at most one of them, say $\left(1, c_{1}\right)$, is red if $n$ is odd. Moreover, by Lemma 5 the entries $(2,\lceil n / 2\rceil+1),(2,\lceil n / 2\rceil+2), \ldots,(2, n-2)$ are red.

For the inductive step, suppose that for some $3 \leq t \leq\lfloor n / 2\rfloor$, each row $r$ with $2 \leq r \leq t-1$ satisfies the result. We now prove (15) and (16) for $r=t$. Suppose that the entry $(t, 2\lceil n / 2\rceil-t+2)$ is red. Then by Corollary 1 each of the first $\lceil n / 2\rceil-t+1$ entries in the order associated to $(t, 2\lceil n / 2\rceil-t+2)$ has at least one blue entry above. Since the entries $(t,\lceil n / 2\rceil+1), \ldots,(t, 2\lfloor n / 2\rfloor-t+2)$ have all red above, the only candidates are the $\lceil n / 2\rceil-t$ entries $(t, t+1),(t, t+2), \ldots,(t,\lceil n / 2\rceil)$ and the entry $2\lfloor n / 2\rfloor-t+3=$ $2\lceil n / 2\rceil-t+1$ for odd $n$. But, by Lemma 5 , to be a candidate this last entry should be blue, which is impossible because it would be the first entry in the order associated to $(t, 2\lceil n / 2\rceil-t+2)$ with at most one blue entry above, contradicting Lemma 9. Since there are not enough candidates, then the entry $(t, 2\lceil n / 2\rceil-t+2)$ is blue.

Now consider the blue entry $(t, n)$. By Corollary 2 the first $\lfloor n / 2\rfloor-t+1$ entries in the order associated to $(t, n)$ have all entries above them red. The only candidates are $\left(t, c_{1}\right)$ if it exists, $(t,\lceil n / 2\rceil+1), \ldots,(t, 2\lceil n / 2\rceil-t+1)$. For $n$ even, there are $\lceil n / 2\rceil-t+1=\lfloor n / 2\rfloor-t+1$ candidates because $\left(t, c_{1}\right)$ does not exists, and thus all of them are red by Lemma 5. For $n$ odd, there are at most 2 more candidates than we need. By Lemma 5 any blue entry $(t, c)$ with $c \geq\lfloor n / 2\rfloor+2$ is not a candidate. Thus at most two of the last $\lceil n / 2\rceil-t+1$ candidates are blue. Suppose that one of the entries $(t,\lceil n / 2\rceil+1),(t,\lceil n / 2\rceil+2), \ldots,(t, 2\lfloor n / 2\rfloor-t+1)$ is blue. Then there exists $\lceil n / 2\rceil+1 \leq c \leq 2\lceil n / 2\rceil-t$ such that $(t, c)$ is blue and $(t, c+1)$ is red. Then $(t, c)$ is the first entry in the order associated to $(t, c+1)$ and all entries above it are red, contradicting Corollary 1. Thus (15) holds and, by Lemma 5 for $(i, j)=(t, n)$, the rest of (16) holds too.

Note that (15) is vacuous if $r=\lfloor n / 2\rfloor$ and $n$ is odd. On the other hand, we argue that it is possible to assume that

$$
\begin{equation*}
\text { for odd } n \text {, the entry }(\lfloor n / 2\rfloor,\lceil n / 2\rceil+1) \text { is red. } \tag{17}
\end{equation*}
$$

Indeed, suppose that it is blue. Then, by Lemma $11,(\lfloor n / 2\rfloor,\lceil n / 2\rceil+1)$ is a blue halving entry with $\lfloor n / 2\rfloor-1$ red entries above and thus all $\lfloor n / 2\rfloor-1$ entries to its right are blue. Hence, by Lemma 11, ( $\lfloor n / 2\rfloor,\lceil n / 2\rceil$ ) is halving with $\lfloor n / 2\rfloor$ blue entries to its right and thus all $\lfloor n / 2\rfloor-1$ entries above are red. Note that $M\left(f^{\lfloor n / 2\rfloor}(D)\right)$ satisfies (12), (13), and (14) and its entry $(\lfloor n / 2\rfloor,\lceil n / 2\rceil+1)$ is red. Then we start with $f^{\lfloor n / 2\rfloor}(D)$ instead of $D$.

We now prove that the version of (16) for $r=\lceil n / 2\rceil$ also holds:
the entries $(\lceil n / 2\rceil,\lceil n / 2\rceil+2),(\lceil n / 2\rceil,\lceil n / 2\rceil+3), \ldots,(\lceil n / 2\rceil, n)$ are blue.


Fig. 8 The upper triangle $T_{U}(D)$ for even and odd $n$ in the proof of Theorem 4 (Color figure online)

Note that (18) only needs to be proved for odd $n$, since for even $n$ this is the case $r=$ $\lfloor n / 2\rfloor$ in (16). Using (12) and (15) it follows that all the entries above ( $\lceil n / 2\rceil,\lceil n / 2\rceil+$ $1)$ are red. By Lemma $11(\lceil n / 2\rceil,\lceil n / 2\rceil+1)$ is a halving entry, and so it follows that all the entries to its right are blue. This proves (18).

We now prove that for $2 \leq r \leq\lfloor n / 2\rfloor-1$

$$
\begin{equation*}
\text { for odd } n \text {, the entry }(r, n-r+1) \text { is red. } \tag{19}
\end{equation*}
$$

Note that (17) is a version of (19) for $r=\lfloor n / 2\rfloor$. Observe that $M\left(f^{\lceil n / 2\rceil}(D)\right)$ satisfies (12) and (13). If $(2, n-1)$ is red in $M\left(f^{\lceil n / 2\rceil}(D)\right)$, then the diagonal $(r, n-r)$ with $1 \leq r \leq\lfloor n / 2\rfloor-1$ in $M\left(f^{\lceil n / 2\rceil}(D)\right)$ is red by (15). This corresponds to the diagonal $(r, n-r+1)$ with $2 \leq r \leq\lfloor n / 2\rfloor$ in $M(D)$. So now assume that the entry $(2, n-1)$ is blue in $M\left(f^{\lceil n / 2\rceil}(D)\right)$, which corresponds to $(\lfloor n / 2\rfloor,\lceil n / 2\rceil+2)$ being blue in $M(D)$. In this case, we can assume that $(1,\lceil n / 2\rceil)$ is blue. (Otherwise start with $M\left(h \circ g \circ f^{\lceil n / 2\rceil}(D)\right)$ instead of $D$, which satisfies (12), (13), (14), ( $\left.\lfloor n / 2\rfloor,\lceil n / 2\rceil+1\right)$ is red, and ( $1,\lceil n / 2\rceil$ ) is blue.) Now, by Lemma 11, $(\lfloor n / 2\rfloor,\lfloor n / 2\rfloor+1)$ is a halving entry with $\lfloor n / 2\rfloor$ of the entries in (10) blue, then all others must be red, i.e., $(2,\lceil n / 2\rceil),(3,\lceil n / 2\rceil), \ldots,(\lfloor n / 2\rfloor-1,\lceil n / 2\rceil)$ are red. Assume by contradiction that $(r, n-r+1)$ is blue for some $2 \leq r \leq\lfloor n / 2\rfloor-1$. Then $(r, n-r+2)$ is blue, otherwise $(r, n-r+1)$ would be the first entry in the order associated to $(r, n-r+2)$ with no blue entry above, contradicting Corollary 1 . But now the red entry $(r,\lceil n / 2\rceil)$ is the $(\lceil n / 2\rceil-r)$ th entry in the order associated to the blue entry $(r, n)$ with a blue entry above, contradicting Corollary 2 and proving (19).

We finally observe that (12), (13), (14), (15), (16), (17), (18), and (19) prove Theorem 4 for the entries in $R(D)$.
The entries in $T_{U}(D)$
We refer the reader to Fig. 8. We prove by induction on $c$ that for $1 \leq c \leq\left\lfloor\frac{1}{2}\lceil n / 2\rceil\right\rfloor$,

$$
\begin{equation*}
\text { the entries }(c+e,\lceil n / 2\rceil+2-c), \ldots,(\lfloor n / 2\rfloor-c,\lceil n / 2\rceil+2-c) \text { are red, } \tag{20}
\end{equation*}
$$

and
the entries $(1,\lceil n / 2\rceil+2-c), \ldots,(c-1-e,\lceil n / 2\rceil+2-c)$ are blue.
We have proved it for $c=1$. Suppose that the result holds for all $1 \leq c \leq d-1$ and we now prove it for $c=d$. By Lemma 12 for $i=\lceil n / 2\rceil+1-d$, and since by (16) the $\lfloor n / 2\rfloor-d$ entries $\{(i, b) \mid 2\lceil n / 2\rceil-i+2 \leq b \leq i+\lceil n / 2\rceil\} \cup\{(i+1, b) \mid$ $i+\lceil n / 2\rceil+1 \leq b \leq n\}$ in (10) are blue, then $(i, i+1)$ has at most $d-1+e$ blue entries above. Suppose by contradiction that $(r, i+1)$ is blue for some $d+e \leq r \leq\lfloor n / 2\rfloor-d$. Then $(r, i+1)$ is the first entry in the order associated to $(r, n-r+1)$ and has at most $\lceil n / 2\rceil-1-(\lfloor n / 2\rfloor-d)-1=d-2+e$ blue entries above. By Lemma $9,(r, i+1)$ has at least $\min \{\lfloor n / 2\rfloor-r, r-1\}$ blue entries above and thus $\min \{\lfloor n / 2\rfloor-r, r-1\} \leq d-2+e$. But $r-1>d-2+e$ because $r \geq d+e$, and $\lfloor n / 2\rfloor-r \geq d>d-2+e$ because $r \leq\lfloor n / 2\rfloor-d$. Thus (20) holds for $c=d$.

Look at $(i, i+1)$ again. The $\lfloor n / 2\rfloor-1-3 e$ entries $\{(r, i+1) \mid d+e \leq r \leq$ $i-1-e\} \cup\{(i, b) \mid i+2+e \leq b \leq n-i+1\}$ in (10) are red and thus, by Lemma 12 , at most other $4 e$ entries are red. For $n$ even, $4 e=0$ and thus (21) holds. For $n$ odd, suppose by contradiction that $(d-e, i+1)$ has a red entry above. We prove that in this case the entries $(d-e, i+1),(d-e+1, i+1)$, and $(i-1, i+1)$ are red. Since ( $d-e, n+1-d+e$ ) is red, then by Corollary 2 the first $\lfloor n / 2\rfloor+2-2 d+2 e$ entries in the order associated to ( $d-e, n+1-d+e$ ) have only blue entries above. If $(d-e, i+1)$ were blue, then it would be one of the first two entries in the order associated to $(d-e, n+1-d+e)$ with at least one red point above. This means that $1 \geq\lfloor n / 2\rfloor+2-2 d+2 e$ contradicting that $d \leq\left\lfloor\frac{1}{2}\lceil n / 2\rceil\right\rfloor$. Thus $(d-e, i+1)$ is red. Similarly, $(d-e+1, i+1)$ cannot be blue as it would be the first entry in the order associated to ( $d-e+1, n-d+e$ ), which by Lemma 9 should have at most one red entry above, but $(d-e+1, i+1)$ has now at least 2 red entries above. Now $(i-1, i+1)$ is the first entry for $(i-1, n+2-i)$ and, by (20), it has at least $\lceil n / 2\rceil+1-2 d+e$ red entries above, i.e., at most $d-2-e$ blue entries above. But by Lemma 9, the first entry in the order associated to the red entry $(i-1, n+2-i)$ has at least $\min \{d-1, i-2\}$ blue entries above. Thus $\min \{d-1, i-2\} \leq d-2-e$, but $d-1>d-2-e$ and $i-2>d-2-e$ because $d \leq\left\lfloor\frac{1}{2}\lceil n / 2\rceil\right\rfloor$, getting a contradiction. Hence $(i-1, i+1)$ is red. By Lemma 12 at most $\lfloor n / 2\rfloor$ of the entries in (11) are red, yet we already have $\lceil n / 2\rceil$ red entries (namely, at least the $\lceil n / 2\rceil+1-2 d+e$ above $(i-1, i+1)$ mentioned before and the $2 d-2$ entries $\{(i-1, b) \mid i+2 \leq b \leq n-i+2\}$ to its right), getting a contradiction. Thus (21) holds for $c=d$.

Now we prove that for $2 \leq c \leq\left\lceil\frac{1}{2}\lceil n / 2\rceil\right\rceil+1$,
the entries $(1, c),(2, c), \ldots,(c-2-e, c)$ are blue.
Since $(c-1, c)$ is one of the $\lfloor n / 2\rfloor+5-2 c$ entries in the order associated to the red entry $(c-1, n+2-c)$ (we have shown that the $\lfloor n / 2\rfloor-1-e$ entries immediately to the left of $(n+2-c)$ are red), then $(c-1, c)$ has at most one red entry above by Lemma 9. Suppose by contradiction that ( $r, c$ ) is red for some $1 \leq r \leq c-2-e$. Then $(r+1, c)$ is blue. Since $(r+1, n-r)$ is red, then by Corollary 2 the first $\lfloor n / 2\rfloor-2 r$ entries in the order associated to $(r+1, n-r)$ have only blue entries above. But


Fig. 9 The lower triangle $T_{L}(D)$ versus the upper triangle $T_{U}\left(f^{\lceil n / 2\rceil}(D)\right)$ for even and odd $n$ in the proof of Theorem 4 (Color figure online)
$(r+1, c)$ is one of the first $\lfloor n / 2\rfloor-2 r$ entries and has the red entry $(r, c)$ above, getting a contradiction.

We finally note that (20), (21), and (22) prove Theorem 4 for the entries in $T_{U}(D)$. The entries in $T_{L}(D)$

We refer the reader to Fig. 9. Consider $f^{\lceil n / 2\rceil}(D)$. When $n$ is even, see Fig. 9 (left), $R(D)$ and $R\left(f^{\lceil n / 2\rceil}(D)\right.$ ) are identical and thus our previous arguments show that $T_{U}(D)$ and $T_{U}\left(f^{\lceil n / 2\rceil}(D)\right)=T_{L}(D)$ are identical too, concluding the proof in this case. When $n$ is odd, see Fig. 9 (right), $R(D)$ and $R\left(f^{\lceil n / 2\rceil}(D)\right)$ are slightly different: for $2 \leq r \leq\lfloor n / 2\rfloor$ the diagonal entries $(r, n+1-r)$ are red in $R(D)$ and unfixed in $R\left(f^{\lceil n / 2\rceil}(D)\right)$, and for $3 \leq r \leq\lfloor n / 2\rfloor$ the diagonal entries $(r, n+2-r)$ are unfixed in $R(D)$ and blue in $R\left(f^{\lceil n / 2\rceil}(D)\right)$. Also the last row of $R(D)$ is blue and the last row of $R\left(f^{\lceil n / 2\rceil}(D)\right)$ is unfixed. However, the last column of $T_{U}\left(f^{\lceil n / 2\rceil}(D)\right)$ is red and this is what allows us to mimic the arguments used for (20), (21), and (22) to show that $T_{L}(D)$, which corresponds to $T_{U}\left(f^{\lceil n / 2\rceil}(D)\right)$ minus its last column, satisfies the statement. More precisely, it can be proved by induction on $c$ that for $1 \leq c \leq\left\lfloor\frac{1}{2}\lceil n / 2\rceil\right\rfloor$, in $M\left(f^{\lceil n / 2\rceil}(D)\right)$
the entries $(c+1,\lceil n / 2\rceil+1-c), \ldots,(\lfloor n / 2\rfloor-c-1,\lceil n / 2\rceil+1-c)$ are red
and

$$
\begin{equation*}
\text { the entries }(1,\lceil n / 2\rceil+1-c), \ldots,(c-2,\lceil n / 2\rceil+1-c) \text { are blue. } \tag{24}
\end{equation*}
$$

We omit the proofs of (23) and (24), as they very closely resemble the proofs of (20) and (21). Similarly, it can be proved by induction that for $2 \leq c \leq\left\lceil\frac{1}{2}\lceil n / 2\rceil\right\rceil$,
in $M\left(f^{\lceil n / 2\rceil}(D)\right)$
the entries $(1, c),(2, c), \ldots,(c-3, c)$ are blue.
The proof of (25) is also omitted, as it very closely resembles the proof of (22). We finally note that (23), (24), and (25) prove Theorem 4 for the entries in $T_{L}(D)$.

### 4.4 The Number of Crossing Optimal Drawings

Theorem 4 completely determines $M\left(D^{\prime}\right)$ when $n$ is even, which means that in this case there is essentially only one crossing optimal drawing.

Theorem 5 For n even, up to sphere-homeomorphism, there is a unique crossing optimal 2-page book drawing of $K_{n}$.

Proof The result is easily seen to hold for $n=2$ and $n=4$. For $n \geq 6$ Theorem 4 completely determines $M\left(D^{\prime}\right)$. Note that this matrix corresponds to the drawings by Blažek and Koman [8].

In contrast to the even case, for $n$ odd there is an exponential number of non spherehomeomorphic crossing optimal 2-page book drawings of $K_{n}$. For any odd integer $n \geq 5$, we construct $2^{(n-5) / 2}$ non-equivalent crossing optimal drawings of $K_{n}$. In fact, these $2^{(n-5) / 2}$ drawings are pairwise non homeomorphic. To prove this, we need the next two results.

Theorem 6 For every $n \geq 13$ odd, every crossing optimal 2-page book drawing of $K_{n}$ has exactly one Hamiltonian cycle of non-crossed edges, namely the one obtained from the edges on the spine and the 1 n edge.

Proof Assume $n \geq 13$ is odd. To show that $123 \ldots n$ is the only non-crossed Hamiltonian cycle, we show that all other edges are crossed at least once. Assume that $D$ has the form described in Theorem 4. Let $(r, c)$ be an entry of $M(D)$ that does not represent an edge on the spine or the $1 n$ edge. Let

$$
(r, c)^{+}=\left\{\begin{array}{ll}
(r+1, c+1) & \text { if } c<n, \\
(1, r+1), & \text { if } c=n,
\end{array} \text { ond }(r, c)^{-}= \begin{cases}(r-1, c-1) & \text { if } r>1, \\
(c-1, n), & \text { if } r=1 .\end{cases}\right.
$$

Note that the edges corresponding to $(r, c)^{+}$and $(r, c)^{-}$cross the edge $r c$ if they have the same color as $(r, c)$.

First assume that $3 \leq c-r \leq n-3$. Suppose that $(r, c)$ is a blue entry specified by Theorem 4. If $5 \leq r+c \leq\lfloor n / 2\rfloor-1$ or if $n+3 \leq r+c \leq n+\lfloor n / 2\rfloor-1$, then note that the entry $(r, c)^{+}$is also blue according to Theorem 4, and thus the edges corresponding to $(r, c)$ and $(r, c)^{+}$cross each other.

Because $n \geq 13$, if $\lfloor n / 2\rfloor \leq r+c \leq\lfloor n / 2\rfloor+1$ or $n+\lfloor n / 2\rfloor \leq r+c \leq n+\lfloor n / 2\rfloor+1$, then $5 \leq\lfloor n / 2\rfloor-2 \leq r+c-2 \leq\lfloor n / 2\rfloor+1$ or $n+3 \leq n+\lfloor n / 2\rfloor-2 \leq r+c-2 \leq$ $n+\lfloor n / 2\rfloor+1$, respectively. Thus the entry $(r, c)^{-}$is also blue according to Theorem 4, and thus the edges corresponding to $(r, c)$ and $(r, c)^{-}$cross each other.

A similar argument shows that for every red entry $(r, c)$ specified by Theorem 4, either $(r, c)^{+}$or $(r, c)^{-}$is also a red edge.

Second, assume that $c-r=n-2$, that is $(r, c) \in\{(1, n-1),(2, n)\}$. If $(r, c)=$ ( $1, n-1$ ), then $(r, c)$ is red and because $(2 n-4) \geq n+\lceil n / 2\rceil+2$ for $n \geq 13$, it follows that $r c$ crosses the edge corresponding to $(n-3, n)$, which is red. If $(r, c)=(2, n)$, then $(r, c)$ is blue and because $\lfloor n / 2\rfloor \geq 4$ for $n \geq 13$, it follows that $r c$ crosses the edge corresponding to $(1,4)$, which is blue.

Suppose now that the color of $(r, c)$ is not determined by Theorem 4. First assume that $r+c \in\{\lfloor n / 2\rfloor+2,\lceil n / 2\rceil+2, n+\lfloor n / 2\rfloor+2, n+\lceil n / 2\rceil+2\}$. Again, by Theorem 4 note that $(r, c)^{-}$is blue and $(r, c)^{+}$is red. Similarly, if $r+c=n+2$, then $(r, c)^{-}$is red and $(r, c)^{+}$is blue. Thus regardless of its color, the edge $r c$ will cross one of the two edges corresponding to these two entries.

Finally, assume $c-r=2$. From Theorem 4, the number of red entries of the form $(t, r+1)$ or $(r+1, d)$, with $1 \leq t \leq r$ and $r+3 \leq d \leq n$ is at least $\lfloor n / 2\rfloor-5 \geq 1$. A similar statement holds for the number of blue entries of the same form. Thus there is at least one blue edge (not on the spine) and at least one red edge incident to $r+1$. One of these two edges will necessarily cross the edge $r c$ regardless of its color.

Note that for $n \leq 11$ the above approach cannot guarantee that there are no additional non-crossed edges. For example for $n=11$ the element $(1,10)$ cannot be determined. However, these small cases can be handled by exhaustive enumeration, which shows that for crossing optimal drawings there are no such edges for $n=11$ and no alternative Hamiltonian cycles for $n=9$. For $n=5,7$ there exist alternative Hamiltonian cycles of non-crossed edges, but they do not lead to additional equivalences between the crossing optimal drawings.

Corollary 3 If $D$ and $D^{\prime}$ are sphere-homeomorphic crossing optimal 2-page book drawings of $K_{n}$, then $M(D)$ and $M\left(D^{\prime}\right)$ are equivalent.

Proof If $n$ is even the result is trivial by Theorem 5. If $n$ is odd and $n \leq 11$, then using Theorem 4 we exhaustively found all equivalence classes of crossing optimal drawings. There are $1,4,9$, and 25 equivalence classes for $n=5,7,9$, and 11 , respectively. We verified that all of these equivalence classes were non sphere-homeomorphic. If $n \geq 13$ and $D$ and $D^{\prime}$ are crossing optimal 2-page book drawings, then by the previous theorem both $D$ and $D^{\prime}$ have only one non-crossed Hamiltonian cycle. Thus if H is a homeomorphism of the sphere sending $D_{S}$ to $D_{S}^{\prime}$, then $H$ must send the Hamiltonian cycle $123 \ldots n$ to itself. It follows that $H$ restricted to this cycle is the composition of a rotation of the cycle with either the identity, or the function that reverses the order of the cycle. Moreover, once the edges on the spine are fixed, the drawing is determined by the colors of the remaining edges. Thus either $H$ is determined by its action on the cycle, or else $H$ switches the blue edges not on the spine with the red edges. In other words, $M\left(D^{\prime}\right)=M\left(\left(h^{a} \circ g^{b} \circ f^{i}\right)(D)\right)$ for some $i \in\{0,1,2, \ldots, n-1\}$ and $a, b \in\{0,1\}$. Thus $M(D)$ and $M\left(D^{\prime}\right)$ are equivalent.

Theorem 7 Forn odd, there are at least $2^{(n-5) / 2}$ pairwise non sphere-homeomorphic crossing optimal 2-page book drawings of $K_{n}$.

Fig. 10 The $2^{8}$ crossing optimal drawings (only $2^{7}$ non-equivalent) for $n=19$ in Theorem 7. They are obtained by assigning arbitrary colors to the crosses in this matrix (Color figure online)


Proof As usual let $1,2, \ldots, n$ be the vertices of $K_{n}$. Let $r c$ be an edge of $K_{n}$ that is not on the Hamiltonian cycle $H=12 \ldots n$, we color $r c$ red or blue according to the following rule: if $r+c \equiv s(\bmod n)$ for some integer $2 \leq s \leq(n+1) / 2$, then we color $r c$ blue, if $r+c \equiv s(\bmod n)$ for some integer $(n+5) / 2 \leq s \leq n+1$, then we color $r c$ red. Finally, if $r+c \equiv(n+3) / 2(\bmod n)$, then we color $r c$ either red or blue. See (Fig. 10.)

We first argue that all of these colorings yield crossing optimal drawings of $K_{n}$ regardless of the color of the $(n-3) / 2$ edges $r c$ for which $r+c \equiv(n+3) / 2$ $(\bmod n)$.

For every $1 \leq s \leq n$, let $I_{s}=\{r c$ edge: $r c \notin H$ and $r+c \equiv s(\bmod n)\}$. Note that $\left|I_{s}\right|=(n-3) / 2$ for all $s$ and $\bigcup_{s=1}^{n} I_{s}$ is the complete set of edges not in $H$. Moreover note that each $I_{s}$ is a matching of pairwise non-crossing edges.

Let $r c$ be an edge such that $r+c \equiv(n+3) / 2(\bmod n)$. Assume without loss of generality that $r<c$. If $t d$ is an edge that crosses $r c$, then $t$ and $d$ are cyclically separated from $r$ and $c$; that is, we may assume that $r<t<c$ and $d<r$ or $d>c$. To facilitate the case analysis we may assume that the edges that could cross $r c$ are the edges $t d$ such that $r<t<c<d<n+r$, with the understanding that $d$ represents the point $d-n$ when $d>n$. Let $C=\{t d$ edge: $r<t<c<d<n+r\}$ and consider the function $T: C \rightarrow C$ defined by $T(t d)=t^{\prime} d^{\prime}$ where $t^{\prime}=r+c-t$ and $d^{\prime}=r+c+n-d$. Note that $T$ is well defined because $r<t^{\prime}<c<d^{\prime}<n+r$ and $T$ is one-to-one on $C$. Moreover, note that

$$
\begin{aligned}
t^{\prime}+d^{\prime} & \equiv r+c+n+r+c-t-d \quad(\bmod n) \\
& \equiv 2(r+c)-(t+d) \quad(\bmod n) \\
& \equiv(n+3)-(t+d) \equiv 3-(t+d) \quad(\bmod n),
\end{aligned}
$$

so $t+d \equiv s(\bmod n)$ with $2 \leq s \leq(n+1) / 2$ if and only if $t^{\prime}+d^{\prime} \equiv 3-(t+d) \equiv$ $n+3-s(\bmod n)$ and $(n+5) / 2 \leq n+3-s \leq n+1$. Thus $t d$ and $T(t d)$ have different colors, which means that $C$ contains as many red edges as blue edges. Hence $r c$ crosses the same number of edges independently of its color. This shows that all the drawings we have described have the same number of crossings. Finally, we note that the drawing for which all the arbitrary edges have the same color corresponds to the construction originally found by Blažek and Koman [8] having exactly $Z(n)=\frac{1}{64}(n-1)^{2}(n-3)^{2}$ crossings. Hence all the other drawings described are crossing optimal as well.

Table 1 The number of non sphere-homeomorphic crossing optimal 2-page book drawings of $K_{n}$ for odd $n, 5 \leq n \leq 37$

| $n$ | Drawings | $n$ | Drawings | $n$ | Drawings |
| :--- | :---: | :--- | :---: | :--- | :---: |
| 5 | 1 | 17 | 324 | 29 | 38,944 |
| 7 | 4 | 19 | 748 | 31 | 84,064 |
| 9 | 9 | 21 | 1,672 | 33 | 180,288 |
| 11 | 25 | 23 | 3,736 | 35 | 385,216 |
| 13 | 58 | 25 | 8,208 | 37 | 819,328 |
| 15 | 142 | 27 | 17,968 |  |  |

We now argue that every drawing constructed here is equivalent to exactly one other drawing. Let $D$ and $D^{\prime}$ be two of the crossing optimal drawings we just constructed and suppose that $M(D)$ and $M\left(D^{\prime}\right)$ are equivalent. Thus there exists a transformation $F: D \rightarrow D^{\prime}$ such that $F=h^{a} \circ g^{b} \circ f^{i}$ with $i \in\{0,1,2, \ldots, n-1\}$ and $b, a \in\{0,1\}$. First observe that under $f, g$, or $h$, the absolute value difference of the number of red minus blue edges remains invariant. Thus the drawing $D$ in which all of the edges in $I_{(n+3) / 2}$ are red can only be equivalent to the drawing $D^{\prime}$ in which all of those edges are blue. These two are indeed equivalent under the function $F=h \circ g \circ f^{(n+1) / 2}$. Now suppose that the edges $I_{(n+3) / 2}$ in $D$ and in $D^{\prime}$ are not all of the same color. Note that $f, g$, and $h$ send $I_{m}$ into another $I_{m^{\prime}}$, and if $I_{m}$ is monochromatic (all edges of $I_{m}$ have the same color) in $D$, then $I_{m^{\prime}}$ is monochromatic in $f(D), g(D)$, and $h(D)$. Since $I_{m}$ is monochromatic in $D$ if and only if $m \neq(n+3) / 2$, then $F$ must send $I_{(n+3) / 2}$ to itself. If $b=0, r c \in I_{(n+3) / 2}$, and $r^{\prime} c^{\prime}$ is the image of $r c$ under $F$, then $r^{\prime}+c^{\prime} \equiv r-i+c-i(\bmod n)$. Thus $r^{\prime}+c^{\prime} \equiv r+c(\bmod n)$ if and only if $i=0$. Because the edges $I_{1}$ in $D$ are blue and the edges $I_{1}$ in $h(D)$ are red, it follows that $a=0$ and thus $F$ is the identity. Last, if $b=1, r c \in I_{(n+3) / 2}$, and $r^{\prime} c^{\prime}$ is the image of $r c$ under $F$, then $r^{\prime}+c^{\prime} \equiv(n+1-(c-i))+(n+1-(r-i)) \equiv 2+2 i-(r+c)$ $(\bmod n)$. Thus $r^{\prime}+c^{\prime} \equiv r+c(\bmod n)$ if and only if $i=(n+1) / 2$. Because the edges $I_{1}$ in both $D$ and $h\left(f^{(n+1) / 2}(D)\right)$ are blue, it follows that $a=1$ and thus $F=h \circ g \circ f^{(n+1) / 2}$. It can be verified that indeed $F(D)$ is one of the drawings we constructed here, and thus exactly half of the drawings we described are pairwise non-equivalent. Therefore, by Corollary 3 we have constructed exactly $2^{(n-5) / 2}$ non sphere-homeomorphic drawings of $K_{n}$.

The above theorem gives a lower bound of $2^{(n-5) / 2}$ for the number of non spherehomeomorphic crossing optimal drawings. As in the crossing optimal drawings of Theorem 4 there are $\frac{5}{2}(n-5)$ entries with non-fixed colors, we get an upper bound of $2^{5(n-5) / 2}$ non sphere-homeomorphic crossing optimal drawings. We were able to determine the exact numbers of non sphere-homeomorphic crossing optimal drawings for $n \leq 37$ (see Table 1) by using exhaustive enumeration. The obtained results suggest an asymptotic growth of roughly $2^{0.54 n}$, rather close to our lower bound.

## 5 Concluding Remarks

Our approach to determine $k$-edges in the topological setting is to define the orientation of three vertices by the orientation of the corresponding triangle in a good drawing
of the complete graph. It is natural to ask whether this defines an abstract order type. To this end, the setting would have to satisfy the axiomatic system described by Knuth [21]. But it is easy to construct an example which does not fulfill these axioms, that is, our setting does not constitute an abstract order type as described by Knuth [21]. It is an interesting question for further research how this new concept compares to the classic order type, both in terms of theory (realizability, etc.) and applications.

We believe that the developed techniques of generalized orientation, $k$-edge for topological drawings, and $\leq \leq k$-edges are of interest in their own. We will investigate their usefulness for related problems in future work. For example, they might also play a central role to approach the crossing number problem for general drawings of complete and complete bipartite graphs.

Acknowledgments O. Aichholzer is partially supported by the ESF EUROCORES programme EuroGIGA, CRP ComPoSe, under grant FWF [Austrian Fonds zur Förderung der Wissenschaftlichen Forschung] I648-N18. P. Ramos is partially supported by MEC grant MTM2011-22792 and by the ESF EUROCORES programme EuroGIGA, CRP ComPoSe, under grant EUI-EURC-2011-4306. G. Salazar is supported by CONACYT grant 106432. This work was initiated during the workshop Crossing Numbers Turn Useful, held at the Banff International Research Station (BIRS). The authors thank the BIRS authorities and staff for their support. The authors would also like to thank the referees for their invaluable comments and suggestions.

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[^1]:    ${ }^{1}$ We say that $\alpha$ and $\beta$ intersect transversally (or tangentially, respectively) at $x$ if there exists a disk $D$ such that the two connected components of $(D \cap \beta) \backslash\{x\}$ are in different (the same, respectively) connected components of $D \backslash \alpha$.

